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Some Properties of (p,q) - Fibonacci Numbers

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บทคัดย่อ

การวิจัยในครั้งนี้เราพิจารณาลำดับฟีโบนักชีทั่วไป นั่นคือลำดับ $(p,q)-$ ฟีโบนักชี โดยเราใช้รูปแบบของไบเนทในการแสดงสมบัติบางประการของลำดับ $(p,q)-$ ฟีโบนักชี ทั้งนี้เราได้เอกลักษณ์บางประการสำหรับลำดับ $(p,q)-$ ฟีโบนักชี

คำสำคัญ: ลำดับฟีโบนักชี, จำนวนฟีโบนักชี, ลำดับ $(p,q)-$ ฟีโบนักชี, จำนวน $(p,q)-$ ฟีโบนักชี, รูปแบบของไบเนท

Abstract

In this paper, we consider the generalized Fibonacci sequence which is $(p,q)-$ Fibonacci sequence. We used the Binet's formula to show some properties of $(p,q)-$ Fibonacci number. We get some generalized identities of $(p,q)-$ Fibonacci number.

Keywords: Fibonacci sequence, Fibonacci number, $(p,q)-$ Fibonacci sequence, $(p,q)-$ Fibonacci number, Binet's formula

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1. Introduction

Fibonacci numbers cover a wide range of interest in modern mathematics as they appear in the comprehensive works of Koshy [4] and Vajda [5]. The Fibonacci numbers F_n are the terms of the sequence where each term is the sum of the two previous terms beginning with the initial values $F_0 = 0,\ F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$.

Falcon and Plaza [1] introduced the k- Fibonacci sequence $\{F_{k,n}\}$ which is defined as $F_{k,0} = 0,\ F_{k,1} = 1$ and $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$, $k \geq 1$. If $k = 1$, we get the classical Fibonacci sequence $\{0,1,1,2,3,5,8,13,\ldots\}$. If $k = 2$, we get the Pell sequence $\{0,1,1,2,5,12,29,70,\ldots\}$.

The well-known Binet's formulas for k- Fibonacci numbers, see [1], are given by $F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$ where $r_1 = \frac{k + \sqrt{k^2+4}}{2}$ and $r_2 = \frac{k - \sqrt{k^2+4}}{2}$ are roots of the characteristic equation $r^2 - kr - 1 = 0$.

In 2007, Falco and Plaza [2] studied the k- Fibonacci sequence and the Pascal 2- triangle. Next, they considered the 3- dimensional k- Fibonacci spiral in [3]. In this paper, we find some properties of the (p,q)– Fibonacci numbers.

2. The (p,q)– Fibonacci Number

The (p,q)– Fibonacci sequence $\{F_{p,q,n}\}$ is defined as $F_{p,q,0} = 0, F_{p,q,1} = 1$ and $F_{p,q,n} = pF_{p,q,n-1} + qF_{p,q,n-2}$ for $p \geq 1,\ q \geq 1$ and $n \geq 2$.

The Binet's formulas for (p,q)– Fibonacci numbers are given by $F_{p,q,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$ where $r_1 = \frac{p + \sqrt{p^2+4q}}{2}$ and $r_2 = \frac{p - \sqrt{p^2+4q}}{2}$ are roots of the characteristic equation $r^2 - pr - q = 0$. We note that $r_1 + r_2 = p,$ $r_1 r_2 = -q$ and $r_1 - r_2 = \sqrt{p^2+4q}.$

3. Main Results

Theorem 3.1. Let p, q and n be positive integers. Then

$$F_{p,q,n+1}F_{p,q,n-1} - F_{p,q,n}^2 = (-1)^n q^{n-1}.$$

Proof. Let p, q and n be positive integers. We have

$$\begin{aligned}
&F_{p,q,n+1}F_{p,q,n-1} - F_{p,q,n}^2 \\
&= \left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}\right)\left(\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}\right) - \left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right)^2 \\
&= \frac{\left(r_1^{2n} + r_2^{2n} - r_1^{n-1}r_2^{n+1} - r_1^{n+1}r_2^{n-1}\right)}{(r_1 - r_2)^2} - \frac{\left(r_1^{2n} - 2r_1^n r_2^n + r_2^{2n}\right)}{(r_1 - r_2)^2} \\
&= \frac{-r_1^{n-1} + r_2^{n-1} - r_1^{n-1}r_2^{n+1} + 2r_1^n r_2^n}{(r_1 - r_2)^2} \\
&= \frac{r_1^{n-1}r_2^{n-1}(-1)\left(r_1^2 - 2r_1 r_2 + r_2^2\right)}{(r_1 - r_2)^2} \\
&= \frac{(-1)^n q^{n-1}(r_1 - r_2)^2}{(r_1 - r_2)^2} \\
&= (-1)^n q^{n-1}.
\end{aligned}$$

□

Remark 3.2. From Theorem 3.1, if $p = 1$ and $q = 1$ then the Cassini's identity is obtained, i.e., $F_{1,1,n+1}F_{1,1,n-1} - F_{1,1,n}^2 = (-1)^n$. It is similarly as $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

Theorem 3.3. Let p, q and n be positive integers and $n \geq 2$. Then

$$F_{p,q,n-2}F_{p,q,n+1} - F_{p,q,n-1}F_{p,q,n} = (-1)^{n-1} pq^{n-2}.$$

Proof. Let p, q and n be positive integers and $n \geq 2$. We have

$$F_{p,q,n-2}F_{p,q,n+1} - F_{p,q,n-1}F_{p,q,n}$$

$$= \left(\frac{r_1^{n-2} - r_2^{n-2}}{r_1 - r_2}\right)\left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}\right) - \left(\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}\right)\left(\frac{r_1^{n} - r_2^{n}}{r_1 - r_2}\right)$$

$$= \frac{\left(r_1^{2n-1} + r_2^{2n-1} - r_1^{n-2}r_2^{n+1} - r_1^{n+1}r_2^{n-2}\right)}{\left(r_1 - r_2\right)^2} - \frac{\left(r_1^{2n-1} + r_2^{2n-1} - r_1^{n}r_2^{n-1} - r_1^{n-1}r_2^{n}\right)}{\left(r_1 - r_2\right)^2}$$

$$= \frac{-r_1^{n-2} + r_2^{n+1} - r_1^{n+1}r_2^{n-2} + r_1^{n}r_2^{n-1} + r_1^{n-1}r_2^{n}}{\left(r_1 - r_2\right)^2}$$

$$= \frac{r_1^{n-2}r_2^{n-2}(-1)\left(r_1^3 + r_2^3 - r_1^2 r_2 - r_1 r_2^2\right)}{\left(r_1 - r_2\right)^2}$$

$$= (-1)^{n-1} q^{n-2}\left(r_1 + r_2\right)$$

$$= (-1)^{n-1} pq^{n-2}.$$

□

Theorem 3.4. Let p, q, m and n be positive integers. Then

$$F_{p,q,m+n} = F_{p,q,m}F_{p,q,n+1} + qF_{p,q,m-1}F_{p,q,n}.$$

Proof. Let p, q are n be positive integers. We have

$$F_{p,q,m}F_{p,q,n+1} + qF_{p,q,m-1}F_{p,q,n}$$

$$= \left(\frac{r_1^{m} - r_2^{m}}{r_1 - r_2}\right)\left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}\right) + \left(-r_1 r_2\right)\left(\frac{r_1^{m-1} - r_2^{m-1}}{r_1 - r_2}\right)\left(\frac{r_1^{n} - r_2^{n}}{r_1 - r_2}\right)$$

$$= \frac{\left(r_1^{m+n+1} + r_2^{m+n+1} - r_1^{n+1}r_2^{m} - r_1^{m}r_2^{n+1}\right)}{\left(r_1 - r_2\right)^2} + \frac{\left(-r_1^{m+n} + r_2 - r_2^{m+n}r_1 + r_1^{n+1}r_2^{m} + r_1^{m}r_2^{n+1}\right)}{\left(r_1 - r_2\right)^2}$$

$$= \frac{r_1^{m+n+1} + r_2^{m+n+1} - r_1^{m+n}r_2 + r_2^{m+n}r_1}{\left(r_1 - r_2\right)^2}$$

$$= \frac{r_1^{m+n}\left(r_1 - r_2\right) - r_1^{m+n}\left(r_1 - r_2\right)}{\left(r_1 - r_2\right)^2}$$

$$= \frac{r_1^{m+n} - r_1^{m+n}}{\left(r_1 - r_2\right)}$$

$$= F_{p,q,m+n}.$$

□

Remark 3.5. From Theorem 3.4, if $p = 1$ and $q = 1$ then the shifting property is obtained, i.e.,

$$F_{1,1,m+n} = F_{1,1,m}F_{1,1,n+1} + F_{1,1,m-1}F_{1,1,n}.$$

It is similarly as $F_{m+n} = F_m F_{n+1} + F_{m-1}F_n$.

Theorem 3.6. Let p, q, m and n be positive integers. If $m \geq n$, then

$$F_{p,q,m}^2 - F_{p,q,m-n}F_{p,q,m+n} = (-1)^{m-n} q^{m-n} F_{p,q,n}^2.$$

Proof. Let p, q, m and n be positive integers.

Suppose that $m \geq n$, we have

$$F_{p,q,m}^2 - F_{p,q,m-n}F_{p,q,m+n}$$

$$= \left(\frac{r_1^{m} - r_2^{m}}{r_1 - r_2}\right)^2 - \left(\frac{r_1^{m-n} - r_2^{m-n}}{r_1 - r_2}\right)\left(\frac{r_1^{m+n} - r_2^{m+n}}{r_1 - r_2}\right)$$

$$= \frac{\left(r_1^{2m} - 2r_1^{m}r_2^{m} + r_2^{2m}\right)}{\left(r_1 - r_2\right)^2} - \frac{\left(r_1^{2m} + r_2^{2m} - r_1^{m+n}r_2^{m-n} - r_1^{m-n}r_2^{m+n}\right)}{\left(r_1 - r_2\right)^2}$$

$$= \frac{-r_1^{m-n}r_2^{m-n}\left(r_1^{2n} - 2r_1^{n}r_2^{n} + r_2^{2n}\right)}{\left(r_1 - r_2\right)^2}$$

$$= \frac{(-1)^{m-n}q^{m-n}\left(r_n^{n} - r_2^{n}\right)^2}{\left(r_1 - r_2\right)^2}$$

$$= (-1)^{m-n}q^{m-n}F_{p,q,n}^2.$$

□

Remark 3.7. From Theorem 3.6, if $p = 1$ and $q = 1$ then the Catalan's identity is obtained, i.e.,

$$F_{1,1,m}^2 - F_{1,1,m-n}F_{1,1,m+n} = (-1)^{m-n} F_{1,1,n}^2.$$

It is similarly as $F_m^2 - F_{m-n}F_{m+n} = (-1)^{m-n} F_n^2$.

Theorem 3.8. Let p, q, m, n and k be positive integers. Then

$$F_{p,q,m+n}F_{p,q,m+k} - F_{p,q,m}F_{p,q,m+n+k} = (-q)^m F_{p,q,n}F_{p,q,k}.$$

Proof. Let p, q, m, n and k be positive integers.

We have

$F_{p,q,m+n}F_{p,q,m+k} - F_{p,q,m}F_{p,q,m+n+k}$

$$= \left(\frac{r_1^{m+n} - r_2^{m+n}}{r_1 - r_2}\right)\left(\frac{r_1^{m+k} - r_2^{m+k}}{r_1 - r_2}\right) - \left(\frac{r_1^{m} - r_2^{m}}{r_1 - r_2}\right)\left(\frac{r_1^{m+n+k} - r_2^{m+n+k}}{r_1 - r_2}\right)$$

$$= \frac{r_1^m r_2^m \left(-r_1^n r_2^k - r_1^k r_2^n + r_1^{n+k} + r_2^{n+k}\right)}{(r_1 - r_2)^2}$$

$$= \frac{(-q)^m \left(-r_2^k \left(r_1^n - r_2^n\right) + r_1^k \left(r_1^n - r_2^n\right)\right)}{(r_1 - r_2)^2}$$

$$= \frac{(-q)^m \left(r_1^k - r_1^k\right)\left(r_1^n - r_2^n\right)}{(r_1 - r_2)^2}$$

$$= (-q)^m F_{p,q,n}F_{p,q,k}.$$

□

Remark 3.9. From Theorem 3.8, if $p = 1$ and $q = 1$ then the Vajda's identity is obtained, i.e.,

$$F_{1,1,m+n}F_{1,1,m+k} - F_{1,1,m}F_{1,1,m+n+k} = (-1)^m F_{1,1,n}F_{1,1,k}.$$

It is similarly as $F_{m+n}F_{m+k} - F_m F_{m+n+k} = (-1)^m F_n F_k$.

Theorem 3.10.

$$\lim_{n\to\infty} \frac{F_{p,q,n}}{F_{p,q,n-1}} = r_1$$

and

$$\lim_{n\to\infty} \frac{F_{p,q,n-1}}{F_{p,q,n}} = \frac{1}{r_1}.$$

Proof. We have

$$\lim_{n\to\infty} \frac{F_{p,q,n}}{F_{p,q,n-1}} = \lim_{n\to\infty} \frac{\dfrac{r_1^n - r_2^n}{r_1 - r_2}}{\dfrac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}}.$$

Then

$$\lim_{n\to\infty} \frac{F_{p,q,n}}{F_{p,q,n-1}} = \lim_{n\to\infty} \frac{r_1^n - r_2^n}{r_1^{n-1} - r_2^{n-1}}.$$

So,

$$\lim_{n\to\infty} \frac{F_{p,q,n}}{F_{p,q,n-1}} = \lim_{n\to\infty} \frac{1 - \left(\dfrac{r_2}{r_1}\right)^n}{\dfrac{1}{r_1} - \left(\dfrac{r_2}{r_1}\right)^n \left(\dfrac{1}{r_2}\right)}.$$

Since $\left|\frac{r_2}{r_1}\right| < 1$, we obtain that

$$\lim_{n\to\infty} \frac{F_{p,q,n}}{F_{p,q,n-1}} = \frac{1}{\left(\dfrac{1}{r_1}\right)} = r_1.$$

In addition, we have

$$\lim_{n\to\infty} \frac{F_{p,q,n-1}}{F_{p,q,n}} = \frac{1}{r_1}.$$

□

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