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On the *k*-Jacobsthal numbers by matrix methods,

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Abstract

In this paper, we define the k-Jacobsthal S-matrix and k-Jacobsthal W-matrix. After, by using this matrix representation, we obtain some identities and the Binet's formula for k-Jacobsthal numbers.

Keywords: Jacobsthal numbers, Jacobsthal-Lucas numbers, k-Jacobsthal numbers, k-Jacobsthal-Lucas numbers, Binet's formula, matrix methods.

1. Introduction

In the recent years, several recurrence sequences of positive integers have been object of study for many researchers. The most prominent examples are the Fibonacci sequences and Lucas sequences. The Fibonacci sequences and Lucas sequences are famous for possessing wonderful and amazing properties.

The Fibonacci sequences is represented by $\{F_n\}$ and defined by the following recurrence:

$$F_n = F_{n-1} + F_{n-2} \text{ , for } n \geq 2 \quad (1)$$

with initial conditions $F_0 = 0$ and $F_1 = 1$.

Similarly, the classical Lucas sequences is represented with $\{L_n\}$ and defined by

$$L_n = L_{n-1} + L_{n-2} \text{ , for } n \geq 2 \quad (2)$$

with initial conditions $L_0 = 2$ and $L_1 = 1$. Terms of the Fibonacci sequences and Lucas sequences are called Fibonacci numbers and Lucas numbers respectively.

On the other hand, other recurrence sequences of positive integers that also important

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are the sequences of Jacobsthal and Jacobsthal-Lucas that first studied by Horadam [3]. The Jacobsthal sequences represented by $\{J_n\}$ and defined by the following recurrence:

$$J_n = J_{n-1} + 2J_{n-2} \text{ ,for } n \geq 2 \quad (3)$$

with initial conditions $J_0 = 0$ and $J_1 = 1$. Similarly, the Jacobsthal-Lucas sequences is represented with $\{j_n\}$ and defined by the following recurrence:

$$j_n = j_{n-1} + 2j_{n-2} \text{ ,for } n \geq 2 \quad (4)$$

with initial conditions $j_0 = 2$ and $j_1 = 1$.

These sequences have been studied and some it's basic properties are known; (see [3-4]). There are a lot of identities of Fibonacci, Lucas, Jacobthal and Jacobthal-Lucas numbers have been presented in the literatures; (see [10-11, 14]).

It is well-known that many identities concerning Fibonacci, Lucas, Jacobsthal and Jacobsthal-Lucas numbers can be proved by using Binet's formula, induction and matrix.

In 1960, Charles H. King studied on the following Fibonacci Q-matrix

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

He showed that $\det(Q) = -1.$ and

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, \text{ for } n \geq 1.$$

This property provides an alternates proof of Cassini's Fibonacci formula:

$$F_{n+1}F_{n-1} - F_n^{\ 2} = (-1)^n. \quad (5)$$

The above equalities demonstrate that there is very close link between the matrices and Fibonacci numbers [10]

From the idea of Charles H. King, in 2008, Koken and Bozkurt [9] defined the Jacobsthal *F*-matrix and Jacobsthal *M*-matrix as follows:

$$F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \text{ and } M = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \text{ respectively.}$$

and they also have demonstrated a very close link between these matrices and Jacobsthal numbers.

More generally, there are some relations between the integer sequences and matrices [6-10, 12-13].

Very recently, Fibonacci, Lucas, Jacobsthal and Jacobsthal-Lucas sequences were generalized for any positive real number *k*. Also the study of the *k*-Fibonacci sequences, the *k*-Jacobsthal sequences and the *k*-Jacobsthal-Lucas sequences appeared (see [1-2, 5]).

Let *k* be any positive real numbers.

The k-Jacobsthal sequence [5], is defined by

$$J_{k,n} = kJ_{k,n-1} + 2J_{k,n-2}, \text{ for } n \geq 2 \quad (6)$$

with $J_{k,0} = 0$, $J_{k,1} = 1$.

The first few terms of $\{J_{k,n}\}_{n \geq 0}$ are $0, 1, k, k^2 + 2, k^3 + 4k$ and so on. Terms of this sequence are called k-Jacobsthal numbers.

If $k = 1$, then the classical Jacobsthal sequence is obtained. $J_0 = 0,\ J_1 = 1$ and $J_n = J_{n-1} + 2J_{n-2}$ for $n \geq 2$

$$\{J_n\}_{n \geq 0} = \{0, 1, 1, 3, 5, 11, \ldots\}$$

Binet's formula for the n^{th} *k*-Jacobsthal numbers is defined by

$$J_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \quad (7)$$

where r_1, r_2 are the roots of the characteristic equation:

$x^2 = kx + 2$ and $r_1 > r_2$; $r_1 = \frac{k+\sqrt{k^2+8}}{2}$,

$r_2 = \frac{k-\sqrt{k^2+8}}{2}$ which gives

$$r_1 + r_2 = k,\ r_1 r_2 = -2, r_1 - r_2 = \sqrt{k^2+8} \quad (8)$$

The *k*-Jacobsthal-Lucas sequence [1], is defined by

$$j_{k,n} = kj_{k,n-1} + 2j_{k,n-2}, \text{ for } n \geq 2 \quad (9)$$

with $j_{k,0} = 2$, $j_{k,1} = k$.

The first few terms of $\{j_{k,n}\}_{n\geq 0}$ are

$2, k, k^2+4, k^3+6k$ and so on. Terms of this sequence are called *k*-Jacobsthal-Lucas numbers.

If $k = 1$, then the classical Jacobsthal-Lucas sequence is obtained.

$j_0 = 2,\ j_1 = 1$ and $j_n = j_{n-1} + 2j_{n-2}$ for $n \geq 2$

$\{j_n\}_{n\geq 0} = \{2, 1, 5, 7, 17, ...\}$.

Binet's formula for the n^{th} *k*-Jacobsthal-Lucas numbers is defined by

$$j_{k,n} = r_1^n + r_2^n \quad (10)$$

where r_1, r_2 are the roots of the characteristic equation as in (8).

In [1] and [5] Cassini formulas of *k*-Jacobsthal and *k*-Jacobsthal-Lucas numbers are given by

$$J_{k,n+1}J_{k,n-1} - J_{k,n}^2 = (-1)^n 2^{n-1}, \quad (11)$$

$$j_{k,n+1}j_{k,n-1} - j_{k,n}^2 = (-2)^{n-1}(k^2+8). \quad (12)$$

Motivated by Koken and Bozkurt [9] and the research going on in this direction, in this paper, we defined *k*-Jacobsthal *S*-matrix by

$$S = \begin{bmatrix} k & 2 \\ 1 & 0 \end{bmatrix} \quad (13)$$

It is easy to see that, it can be written

$$\begin{bmatrix} J_{k,n+1} \\ J_{k,n} \end{bmatrix} = S \begin{bmatrix} J_{k,n} \\ J_{k,n-1} \end{bmatrix},$$

and

$$\begin{bmatrix} j_{k,n+1} \\ j_{k,n} \end{bmatrix} = S \begin{bmatrix} j_{k,n} \\ j_{k,n-1} \end{bmatrix},$$

where $J_{k,n}$ and $j_{k,n}$ are the n^{th} *k*-Jacobsthal and *k*-Jacobsthal-Lucas numbers, respectively.

Moreover, we also defined *k*-Jacobsthal *W*-matrix by

$$W = \begin{bmatrix} k^2+1 & 2k \\ k & 2 \end{bmatrix}. \quad (14)$$

By using matrices representation (13) and (14), we obtain some identities and the Binet's formula for k-Jacobsthal numbers. The results presented in this paper, extend some previous results in the literature.

2. The Matrix Representation

In this section, we present two different matrix representation of k-Jacobsthal numbers which is called k-Jacobsthal S-matrix and k-Jacobsthal W-matrix. By using this matrix representations we obtain the determinants and elements of S^n and W^n, also we get Cassini formula for k-Jacobsthal numbers. After, we calculate the generalized characteristic roots and Binet's formula of the matrix S^n. Finally, we get some identities for the *k*-Jacobsthal numbers by using these matrices.

Throughout this paper, $J_{k,n}$ and $j_{k,n}$ denote the n^{th} *k*-Jacobsthal and *k*-Jacobsthal-Lucas numbers.

Theorem 2.1 *Let S be a matrix as in (13). Then*

$$S^n = \begin{bmatrix} J_{k,n+1} & 2J_{k,n} \\ J_{k,n} & 2J_{k,n-1} \end{bmatrix}, \ for\ n \geq 1. \quad (15)$$

Proof. We will use the principle of mathematical induction. Since $S^1 = \begin{bmatrix} k & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} J_{k,2} & 2J_{k,1} \\ J_{k,1} & 2J_{k,0} \end{bmatrix},$ we obtain that the results is true for $n = 1$. Next, we assume that it is true for any positive integer $n = m,$ that is

$$S^m = \begin{bmatrix} J_{k,m+1} & 2J_{k,m} \\ J_{k,m} & 2J_{k,m-1} \end{bmatrix}.$$

Now, we show that it is true for $n = m+1$. Then we can write

$$\begin{aligned} S^{m+1} &= S^m S \\ &= \begin{bmatrix} J_{k,m+1} & 2J_{k,m} \\ J_{k,m} & 2J_{k,m-1} \end{bmatrix} \begin{bmatrix} k & 2 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} kJ_{k,m+1} + 2J_{k,m} & 2J_{k,m+1} \\ kJ_{k,m} + 2J_{k,m-1} & 2J_{k,m} \end{bmatrix} \\ &= \begin{bmatrix} J_{k,m+2} & 2J_{k,m+1} \\ J_{k,m+1} & 2J_{k,m} \end{bmatrix}, \end{aligned}$$

and the results follows. □

Corollary 2.2 *For all positive integers* n, *the following equalities hold:*

(i) $\det(S^n) = (-2)^n$,

(ii) $J_{k,n+1}J_{k,n-1} - J_{k,n}^2 = (-1)^n 2^{n-1}$.

Proof. Since $\det(S) = -2$, we get that $\det(S^n) = (\det(S))^n = (-2)^n$. It follows from the determinant of S^n in (15) and (i) that

$$2J_{k,n+1}J_{k,n-1} - 2J_{k,n}^2 = (-2)^n.$$

It implies that

$$J_{k,n+1}J_{k,n-1} - J_{k,n}^2 = (-1)^n 2^{n-1}.$$ □

Theorem 2.3 *Let* n *be an integer and* $n \geq 0$. *Then the well-known Binet's formula of the k-Jacobsthal number is*

$$J_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2},$$

where $r_1 = \dfrac{k + \sqrt{k^2 + 8}}{2}$ *and* $r_2 = \dfrac{k - \sqrt{k^2 + 8}}{2}$.

Proof. Let S be a matrix as in (13). Then the characteristic polynomial of the matrix S is

$$\det(S - \lambda I) = \begin{vmatrix} k - \lambda & 2 \\ 1 & -\lambda \end{vmatrix},$$

which yields the two eigenvalues $\lambda_1 = r_1$, $\lambda_2 = r_2$, where $r_1 = \dfrac{k + \sqrt{k^2 + 8}}{2}$ and $r_2 = \dfrac{k - \sqrt{k^2 + 8}}{2}$.

If we calculate the eigenvectors of matrix S corresponding to the eigenvalues λ_1 , λ_2 , we obtain $v_1 = (r_1, 1)$ and $v_2 = (r_2, 1)$, respectively. Then we can diagonalizable of matrix S by

$$D = Q^{-1}SQ,$$

where $Q = (v_1^T, v_2^T) = \begin{bmatrix} r_1 & r_2 \\ 1 & 1 \end{bmatrix},$

and then we have

$$D = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}.$$

Furthermore, we obtain

$$S = QDQ^{-1}.$$

From properties of similar matrices, we can write

$$S^n = QD^nQ^{-1}. \quad (16)$$

where n is any positive integer.

By (15) and (16), we get

$$\begin{bmatrix} J_{k,n+1} & 2J_{k,n} \\ J_{k,n} & 2J_{k,n-1} \end{bmatrix} = \begin{bmatrix} \dfrac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2} & 2\dfrac{r_1^n - r_2^n}{r_1 - r_2} \\ \dfrac{r_1^n - r_2^n}{r_1 - r_2} & 2\dfrac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2} \end{bmatrix}.$$

Thus, the proof is completed. □

Consequently, limiting ratio of the successive k-Jacobsthal number is

$$\lim_{n\to\infty}\frac{J_{k,n+1}}{J_{k,n}}=\lim_{n\to\infty}\frac{r_1^{n+1}-r_2^{n+1}}{r_1^n-r_2^n}=r_1.$$

Theorem 2.4 *The generalized characteristic roots of S^n are*

$$\lambda_1,\lambda_2=\frac{j_{k,n}\pm\sqrt{k^2+8}J_{k,n}}{2}$$

where λ_1 and λ_2 denote the characteristic root of S^n.

Then, $J_{k,n}=\dfrac{r_1^n-r_2^n}{r_1-r_2}$ *and* $j_{k,n}=r_1^n+r_2^n$, *where* $r_1=\dfrac{k+\sqrt{k^2+8}}{2}$ *and* $r_2=\dfrac{k-\sqrt{k^2+8}}{2}$.

Proof. Form (15), we get that the characteristic polynomial of S^n is

$$\det(S^n-\lambda I)=\lambda^2-(J_{k,n+1}+2J_{k,n-1})\lambda+2(J_{k,n+1}J_{k,n-1}-J_{k,n}^2).$$

Since $J_{k,n+1}+2J_{k,n-1}=j_{k,n}$ and $J_{k,n+1}J_{k,n-1}-J_{k,n}^2=(-1)^n2^{n-1}$, we get

$$\det(S^n-\lambda I)=\lambda^2-j_{k,n}\lambda+2(-1)^n2^{n-1}=\lambda^2-j_{k,n}\lambda+(-2)^n.$$

Thus, the characteristic equation of S^n is

$$\lambda^2-j_{k,n}\lambda+(-2)^n=0 \tag{17}$$

and we get the characteristic roots as following :

$$\lambda_1,\lambda_2=\frac{j_{k,n}\pm\sqrt{j_{k,n}^2-4(-2)^n}}{2}.$$

Since $j_{k,n}^2-4(-2)^n=(k^2+8)J_{k,n}^2$, we obtain

$$\lambda_1,\lambda_2=\frac{j_{k,n}\pm\sqrt{k^2+8}J_{k,n}}{2}.$$

Consequently,

$$r_1^n=\frac{j_{k,n}+\sqrt{k^2+8}J_{k,n}}{2}\text{ and } r_2^n=\frac{j_{k,n}-\sqrt{k^2+8}J_{k,n}}{2},$$

where r_1,r_2 are eigenvalues of matrix S. Then we have

$$J_{k,n}=\frac{r_1^n-r_2^n}{r_1-r_2}\text{ and } j_{k,n}=r_1^n+r_2^n.\quad\square$$

From matrix equation (15), we can write,

$$\frac{S^n}{J_{k,n-1}}=\begin{bmatrix}\frac{J_{k,n+1}}{J_{k,n-1}} & \frac{2J_{k,n}}{J_{k,n-1}}\\ \frac{J_{k,n}}{J_{k,n-1}} & \frac{2J_{k,n-1}}{J_{k,n-1}}\end{bmatrix}.$$

Since $\lim_{n\to\infty}\frac{J_{k,n+1}}{J_{k,n}}=r_1$, it follows that

$$\lim_{n\to\infty}\frac{S^n}{J_{k,n-1}}=\begin{bmatrix}r_1^2 & 2r_1\\ r_1 & 2\end{bmatrix}=\begin{bmatrix}kr_1+2 & 2r_1\\ r_1 & 2\end{bmatrix}.$$

If we compute the determinant of both sides, we reach the characteristic equation of the k-Jacobsthal S-matrix as follows:

$$r_1^2-kr_1-2=0.$$

Theorem 2.5 *Let W be a matrix as in (14). Then*

$$W^n=\begin{bmatrix}J_{k,2n+1} & 2J_{k,2n}\\ J_{k,2n} & 2J_{k,2n-1}\end{bmatrix},$$

for $n\geq 1$.

Proof. It can be show easily by induction on n. $\square$

Corollary 2.6 *For any positive integers n, the following equalities hold:*

(i) $\det(W^n)=2^{2n}$,

(ii) $J_{k,2n+1}J_{k,2n-1}-J_{k,2n}^2=2^{2n-1}$.

Proof. The proof is similar to Corollary 2.2 $\square$

Theorem 2.7 *Let m and n be a positive integer. Then, the following relation between the k-Jacobsthal and k-Jacobsthal-Lucas numbers*

$$j_{k,n+m}=J_{k,m+1,}j_{k,n}+2J_{k,m}j_{k,n-1}$$

is valid.

Proof. It follows from the definition of the k-Jacobsthal-Lucas numbers that

$$\begin{bmatrix} j_{k,n+1} \\ j_{k,n} \end{bmatrix} = S \begin{bmatrix} j_{k,n} \\ j_{k,n-1} \end{bmatrix}. \quad (18)$$

Multiply both side of (18) with S^m , we get

$$S^m \begin{bmatrix} j_{k,n+1} \\ j_{k,n} \end{bmatrix} = S^{m+1} \begin{bmatrix} j_{k,n} \\ j_{k,n-1} \end{bmatrix}.$$

Using (15), we obtain

$$\begin{bmatrix} j_{k,n+m+1} \\ j_{k,n+m} \end{bmatrix} = \begin{bmatrix} J_{k,m+2} j_{k,n} + 2J_{k,m+1} j_{k,n-1} \\ J_{k,m+1} j_{k,n} + 2J_{k,m} j_{k,n-1} \end{bmatrix}$$

which implies that

$$j_{k,n+m} = J_{k,m+1} j_{k,n} + 2J_{k,m} j_{k,n-1},$$

and the proof is completed. □

Theorem 2.8 *Let m and n be positive integers. Then, the following equalities hold:*

(i) $J_{k,m+n} = J_{k,m} J_{k,n+1} + 2J_{k,m-1} J_{k,n}$,

(ii) $J_{k,2n} = J_{k,n} J_{k,n+1} + 2J_{k,n} J_{k,n-1} = J_{k,n} j_{k,n}$,

(iii) $J_{k,2n+1} = J_{k,n+1}^2 + 2J_{k,n}^2$,

(iv) $(-1)^n \cdot 2^{n-1} J_{k,m-n} = J_{k,m} J_{k,n-1} - J_{k,m-1} J_{k,n}$.

Proof. Let S be a matrix as in (13).

Since $S^{m+n} = S^m S^n$, we get

$$\begin{bmatrix} J_{k,m+n+1} & 2J_{k,m+n} \\ J_{k,m+n} & 2J_{k,m+n-1} \end{bmatrix} = \begin{bmatrix} J_{k,m+1}J_{k,n+1} + 2J_{k,m}J_{k,n} & 2(J_{k,m+1}J_{k,n} + 2J_{k,m}J_{k,n-1}) \\ J_{k,m}J_{k,n+1} + 2J_{k,m-1}J_{k,n} & 2(J_{k,m}J_{k,n} + 2J_{k,n-1}J_{k,m-1}) \end{bmatrix}.$$

Thus, equalities (i), (ii) and (iii) are easily seen.

Next, we note that

$$S^{-n} = \frac{1}{(-2)^n} \begin{bmatrix} 2J_{k,n-1} & -2J_{k,n} \\ -J_{k,n} & J_{k,n+1} \end{bmatrix}.$$

Since $S^{m-n} = S^m S^{-n}$, we get

$$\begin{bmatrix} J_{k,m-n+1} & 2J_{k,m-n} \\ J_{k,m-n} & 2J_{k,m-n-1} \end{bmatrix} = \frac{(-1)^n}{2^n} \begin{bmatrix} 2(J_{k,m+1}J_{k,n-1} - J_{k,m}J_{k,n}) & -2(J_{k,m+1}J_{k,n} - J_{k,m}J_{k,n+1}) \\ 2(J_{k,m}J_{k,n-1} - J_{k,m-1}J_{k,n}) & -2(J_{k,m}J_{k,n} - J_{k,m-1}J_{k,n+1}) \end{bmatrix}.$$

and (iv) immediately seen. □

3. Conclusion

In this paper, we introduced k-Jacobsthal S-matrix and k-Jacobsthal W-matrix. After, by using this matrix representation, we obtain some identities and the Binet's formula for k-Jacobsthal numbers. The results presented in this paper, extend some previous results in the literature.

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5. References

[1] H. Campos, P. Catarino, A.P. Aires, P. Vasco, and A. Borges. On Some Identities for k-Jacobsthal –Lucas Number. *Int Journal of Math. Analysis*. 8(10) (2014): 489-494.

[2] S. Falco´n, Plaza, A. The *k*-Fibonacci sequence and the Pascal 2-triangle. *Chaos, Solitons & Fractals*. 33 (1) (2007): 38-49.

[3] AF. Horadam. Jacobsthal Number Representation. *The Fibonacci Quarterly*. 34(1) (1986): 79-83.

[4] AF. Horadam. Jacobsthal and Pell curves. *The Fibonacci Quarterly*. 26(1) (1988): 79-83.

[5] D. Jhala, K. Sisodiya, and G.P.S.Rathore, On Some Identities for k-Jacobsthal Number. *Int Journal of Math. Analysis*. 7(12) (2013): 551-556.

[6] D. Kalman. Generalized Fibonacci numbers by matrix methods. *The Fibonacci Quarterly*. 20(1) (1982): 73-76.

[7] E. Karaduman. An application of Fibonacci numbers in matrices. *Appl. Math. and Comp*. 147 (2003): 903-908.

[8] E. Karaduman. On determinants of matrices with general Fibonacci numbers entries. *Appl. Math. and Comp*. 167 (2005) 670–676

[9] F. Koken and D. Bozkurt. On The Jacobsthal Numbers by Matrix Methods. *Int. J. Contemp. Math. Sciences*. 3 (13) (2008): 605-614.

[10] T. Koshy. *Fibonacci and Lucas Numbers with Applications*. New York: John Wiley and Sons; 2001.

[11] Y. K. Panwar, B. Singh and V. K. Gupta. Generalized Identities Involving Common factors of generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas numbers. *International Journal of Analysis and Application*. 3(1) (2013): 53-59.

[12] J.R. Silvester. Fibonacci properties by matrix methods. *Mathematical Gazette*. 63 (1979): 188–191.

[13] A. Stakhov. Fibonacci matrices, a generalization of the 'Cassini formula', and a new coding theory. *Chaos, Solitons & Fractals.* 30 (2006): 56-66.

[14] M. Thongmoon. New Identities for the Even and Odd Fibonacci and Lucas Numbers. *Int. J. Contemp. Math. Sciences*. 4 (14) (2009): 671-676.