Science and Technology RMUTT Journal

Vol.5 No.1 (2015) : 77–82

www.sci.rmutt.ac.th/stj

Online ISSN 2229-1547

# Generalized identities involving common factors of *k*-Generalized Fibonacci, *k*-Jacobsthal and *k*-Jacobsthal-Lucas numbers

Wanna Sriprad[1*], Waratchaya Cherdchanpipat[1], Kamonwan Thaweesuksatien[1], Passakorn Yordsorn[2]

[1]Devision of Mathematics, Faculty of Science and Technology,
Rajamangala University of Technology Thanyaburi,
Pathum Thani 12110, Thailand

[2]Department of Mathematics and Applied Statistics, Faculty of Science and Technology,
Rajabhat Rajanagarindra University, Chachoengsao 24000, Thailand

*E-mail: wanna_sriprad@rmutt.ac.th

## Abstract

In this paper, we present generalized identities involving common factors of *k*-generalized Fibonacci, *k*-Jacobsthal and *k*-Jacobsthal-Lucas numbers and related identities. Binet's formula will employ to obtain the identities.

**Keywords:** generalized Fibonacci numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers, *k*-generalized Fibonacci numbers, *k*-Jacobsthal numbers, *k*-Jacobsthal-Lucas numbers**,** Binet's formula

## 1. Introduction

It is well-known that the Fibonacci sequence is the most prominent example of recurrence sequences of positive integers. It has been studied by many researchers for a long time to get intrinsic theory and applications of this numbers in many research areas as Physics, Engineering, Architecture, Nature and Art.

On the other hand, other recurrence sequences of positive integers that also important are the sequences of Jacobthal and Jacobthal-

Received: May 09, 2015
Revised: Jun 27, 2015
Accepted: Jun 27, 2015

Lucas. These sequences have been studied and some its basic properties are known; (see [3-4]).

There are a lot of identities of Fibonacci, Lucas, Jacobthal and Jacobthal-Lucas numbers have been presented in the literatures; (see [6-7, 9-11]).

More recently, Fibonacci, Lucas, Jacobsthal and Jacobsthal-Lucas sequences were generalized for any positive real number *k*. Also the study of the *k*-Fibonacci sequences, the *k*-Jacobsthal sequences and the *k*-Jacobsthal-Lucas sequences appeared (see [1-2, 5, 7]).

Motivated by the research going on in this direction, in this paper, we present generalized identities involving common factors of $k$-generalized Fibonacci, $k$-Jacobsthal and $k$-Jacobsthal-Lucas numbers and Binet's formula will employ to obtain the identities.

## 2. Preliminaries

In this section, we will introduce some known results and notations that will be used in our main results.

Throughout this paper, let *k* be any positive real number.

The *k*-generalized Fibonacci sequence [8] is defined by

$$U_{k,n} = kU_{k,n-1} + 2U_{k,n-2} \text{ , for } n \geq 2 \qquad (2.1)$$

with $U_{k,0} = 2,\ U_{k,1} = 0$.

The first few terms of $\{U_{k,n}\}_{n\geq 0}$ are $2, 0, 4, 4k, 4k^2 + 8$ and so on. Terms of this sequence are called *k*-generalized Fibonacci numbers.

If $k = 1$, then the classical generalized Fibonacci sequence is obtained. $U_0 = 2,\ U_1 = 0$ and

$$U_n = U_{n-1} + 2U_{n-2} \text{ , for } n \geq 2 .$$

$$\{U_n\}_{n\geq 0} = \{2, 0, 4, 4, 12, 20, ...\} .$$

Binet's formula for the $n^{th}$ *k*-generalized Fibonacci numbers is defined by

$$U_{k,n} = 4\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}, \qquad (2.2)$$

where $r_1$, $r_2$ are the roots of the characteristic equation

$x^2 = kx + 2$ and $r_1 > r_2$ ; $r_1 = \dfrac{k + \sqrt{k^2+8}}{2}$, $r_2 = \dfrac{k - \sqrt{k^2+8}}{2}$ which gives

$$r_1 + r_2 = k,\ r_1 r_2 = -2, r_1 - r_2 = \sqrt{k^2+8} \qquad (2.3)$$

The $k$ -Jacobsthal sequence [5], is defined by

$$J_{k,n} = kJ_{k,n-1} + 2J_{k,n-2} \text{ , for } n \geq 2 \qquad (2.4)$$

with $J_{k,0} = 0$ , $J_{k,1} = 1$.

The first few terms of $\{J_{k,n}\}_{n\geq 0}$ are $0, 1, k, k^2 + 2, k^3 + 4k$ and so on. Terms of this sequence are called $k$ -Jacobsthal numbers.

If $k = 1$, then the classical Jacobsthal sequence is obtained.

$$J_0 = 0,\ J_1 = 1 \text{ and } J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2$$

$$\{J_n\}_{n\geq 0} = \{0, 1, 1, 3, 5, 11, ...\}$$

Binet's formula for the $n^{th}$ *k*-Jacobsthal numbers is defined by

$$J_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \qquad (2.5)$$

where $r_1$, $r_2$ are the roots of the characteristic equation as in (2.3).

The *k*-Jacobsthal-Lucas sequence [1], is defined by

$$j_{k,n} = kj_{k,n-1} + 2j_{k,n-2} \text{ , for } n \geq 2 \qquad (2.6)$$

with $j_{k,0} = 2$ , $j_{k,1} = k$ .

The first few terms of $\{j_{k,n}\}_{n\geq 0}$ are

$2, k, k^2+4, k^3+6k$ and so on. Terms of this sequence are called *k*-Jacobsthal-Lucas numbers.

If $k=1$, then the classical Jacobsthal-Lucas sequence is obtained.

$j_0=2,\ j_1=1$ and $j_n=j_{n-1}+2j_{n-2}$ for $n\geq 2$

$\{j_n\}_{n\geq 0}=\{2,1,5,7,17,...\}$.

Binet's formula for the $n^{th}$ *k*-Jacobsthal-Lucas numbers is defined by

$$j_{k,n}=r_1^{\,n}+r_2^{\,n} \quad (2.7)$$

where $r_1$, $r_2$ are the roots of the characteristic equation as in (2.3).

## 3. Main Results

In this section we present generalized identities involving common factors of *k*-generalized Fibonacci, *k*-Jacobsthal and *k*-Jacobsthal-Lucas numbers. We shall use the Binet's formula for the *k*-generalized Fibonacci, *k*-Jacobsthal and *k*-Jacobsthal-Lucas numbers for derivation.

**Theorem 3.1**

$$U_{k,2n+p}\,j_{k,2n}=U_{k,4n+p}+4^nU_{k,p}, \quad (3.1)$$

*where* $n\geq 0$ *and* $p\geq 0$.

***Proof:***

$$U_{k,2n+p}\,j_{k,2n}=4\frac{r_1^{2n+p-1}-r_2^{2n+p-1}}{r_1-r_2}\left(r_1^{2n}+r_2^{2n}\right)$$

$$=4\left(\frac{r_1^{4n+p-1}-r_2^{4n+p-1}}{r_1-r_2}\right)+4(r_1r_2)^{2n}\left(\frac{r_1^{p-1}-r_2^{p-1}}{r_1-r_2}\right)$$

$$=4\left(\frac{r_1^{4n+p-1}-r_2^{4n+p-1}}{r_1-r_2}\right)+(-2)^{2n}4\left(\frac{r_1^{p-1}-r_2^{p-1}}{r_1-r_2}\right)$$

$$=U_{k,4n+p}+(-2)^{2n}U_{k,p}$$

$$=U_{k,4n+p}+4^nU_{k,p}.$$

**Corollary 3.2**

$$U_{k,2n+p}\,j_{k,2n}=4\left(J_{k,4n+p-1}+4^nJ_{k,p-1}\right), \quad (3.2)$$

*where* $n\geq 0$ *and* $p\geq 1$.

**Theorem 3.3**

$$U_{k,2n+p}\,j_{k,2n+1}=U_{k,4n+p+1}-2^{2n+1}U_{k,p-1}, , \quad (3.3)$$

*where* $n\geq 0$ *and* $p\geq 1$.

***Proof.***

$$U_{k,2n+p}\,j_{k,2n+1}=4\frac{r_1^{2n+p-1}-r_2^{2n+p-1}}{r_1-r_2}\left(r_1^{2n+1}+r_2^{2n+1}\right)$$

$$=4\left(\frac{r_1^{4n+p}-r_2^{4n+p}}{r_1-r_2}\right)+4(r_1r_2)^{2n+1}\left(\frac{r_1^{p-2}-r_2^{p-2}}{r_1-r_2}\right)$$

$$=4\left(\frac{r_1^{4n+p}-r_2^{4n+p}}{r_1-r_2}\right)+(-2)^{2n+1}4\left(\frac{r_1^{p-2}-r_2^{p-2}}{r_1-r_2}\right)$$

$$=U_{k,4n+p+1}+(-2)^{2n+1}U_{k,p-1}$$

$$=U_{k,4n+p+1}-2^{2n+1}U_{k,p-1}.$$

**Corollary 3.4**

$$U_{k,2n+p}\,j_{k,2n+1}=4\left(J_{k,4n+p}-2^{2n+1}J_{k,p-2}\right), \quad (3.4)$$

*where* $n\geq 0$ *and* $p\geq 2$.

**Theorem 3.5**

$$U_{k,2n}\,j_{k,2n+p}=U_{k,4n+p}+2^{2n-1}U_{k,p+2}, \quad (3.5)$$

*where* $n\geq 0$ *and* $p\geq 0$.

***Proof:***

$$U_{k,2n}\,j_{k,2n+p}=4\frac{r_1^{2n-1}-r_2^{2n-1}}{r_1-r_2}\left(r_1^{2n+p}+r_2^{2n+p}\right)$$

$$=4\left(\frac{r_1^{4n+p-1}-r_2^{4n+p-1}}{r_1-r_2}\right)-4(r_1r_2)^{2n-1}\left(\frac{r_1^{p+1}-r_2^{p+1}}{r_1-r_2}\right)$$

$$=4\left(\frac{r_1^{4n+p-1}-r_2^{4n+p-1}}{r_1-r_2}\right)-(-2)^{2n-1}4\left(\frac{r_1^{p+1}-r_2^{p+1}}{r_1-r_2}\right)$$

$$=U_{k,4n+p}-(-2)^{2n-1}U_{k,p+2}$$

$$=U_{k,4n+p}+2^{2n-1}U_{k,p+2}.$$

**Corollary 3.6**

$$U_{k,2n}\,j_{k,2n+p}=4\left(J_{k,4n+p-1}+2^{2n-1}J_{k,p+1}\right), \quad (3.6)$$

*where* $n\geq 0$ *and* $p\geq 0$.

**Theorem 3.7**

$$U_{k,2n+1}j_{k,2n+p} = U_{k,4n+p+1} - 4^n U_{k,p+1}, \qquad (3.7)$$

*where* $n \geq 0$ *and* $p \geq 0$.

***Proof:***

$$U_{k,2n+1}j_{k,2n+p} = 4\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2}\left(r_1^{2n+p} + r_2^{2n+p}\right)$$

$$= 4\left(\frac{r_1^{4n+p} - r_2^{4n+p}}{r_1 - r_2}\right) - 4(r_1r_2)^{2n}\left(\frac{r_1^{p} - r_2^{p}}{r_1 - r_2}\right)$$

$$= 4\left(\frac{r_1^{4n+p} - r_2^{4n+p}}{r_1 - r_2}\right) - (-2)^{2n}4\left(\frac{r_1^{p} - r_2^{p}}{r_1 - r_2}\right)$$

$$= U_{k,4n+p+1} - (-2)^{2n}U_{k,p+1}$$

$$= U_{k,4n+p+1} - 4^n U_{k,p+1}.$$

**Corollary 3.8**

$$U_{k,2n+1}j_{k,2n+p} = 4\left(J_{k,4n+p} - 4^n J_{k,p}\right), \qquad (3.8)$$

*where* $n \geq 0$ *and* $p \geq 0$.

**Theorem 3.9**

$$U_{k,2n+p}j_{k,n} = U_{k,3n+p} + (-2)^n U_{k,n+p}, \qquad (3.9)$$

*where* $n \geq 0$ *and* $p \geq 0$.

***Proof.***

$$U_{k,2n+p}j_{k,n} = 4\frac{r_1^{2n+p-1} - r_2^{2n+p-1}}{r_1 - r_2}(r_1^n + r_2^n)$$

$$= 4\frac{r_1^{3n+p-1} - r_2^{3n+p-1}}{r_1 - r_2} + 4(r_1r_2)^n\left(\frac{r_1^{n+p-1} - r_2^{n+p-1}}{r_1 - r_2}\right)$$

$$= 4\frac{r_1^{3n+p-1} - r_2^{3n+p-1}}{r_1 - r_2} + 4(-2)^n\left(\frac{r_1^{n+p-1} - r_2^{n+p-1}}{r_1 - r_2}\right)$$

$$= U_{k,3n+p} + (-2)^n U_{n+p}.$$

**Corollary 3.10**

$$U_{k,2n+p}j_{k,n} = 4\left(J_{k,3n+p-1} + (-2)^n J_{k,n+p-1}\right), \qquad (3.10)$$

*where* $n \geq 0$ *and* $p \geq 1$.

**Theorem 3.11**

$$U_{k,n}j_{k,2n+p} = U_{k,3n+p} - (-2)^{n-1}U_{k,n+p+2}, \qquad (3.11)$$

*where* $n \geq 0$ *and* $p \geq 0$.

***Proof.***

$$U_{k,n}j_{k,2n+p} = 4\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}(r_1^{2n+p} + r_2^{2n+p})$$

$$= 4\frac{r_1^{3n+p-1} - r_2^{3n+p-1}}{r_1 - r_2} - 4(r_1r_2)^{n-1}\frac{(r_1^{n+p+1} - r_2^{n+p+1})}{r_1 - r_2}$$

$$= 4\frac{r_1^{3n+p-1} - r_2^{3n+p-1}}{r_1 - r_2} - (-2)^{n-1}4\frac{(r_1^{n+p+1} - r_2^{n+p+1})}{r_1 - r_2}$$

$$= U_{k,3n+p} - (-2)^{n-1}U_{k,n+p+2}.$$

**Corollary 3.12**

$$U_{k,n}j_{k,2n+p} = 4\left(J_{k,3n+p-1} - (-2)^{n-1}J_{k,n+p+1}\right), \qquad (3.12)$$

*where* $n \geq 0$ *and* $p \geq 0$.

**Theorem 3.13**

$$4j_{k,4n+p-1} - 4^{n+1}j_{k,p-1} = (r_1 - r_2)^2 J_{k,2n}U_{k,2n+p}, \qquad (3.13)$$

*where* $n \geq 0$ *and* $p \geq 1$.

***Proof:***

$$(r_1 - r_2)^2 J_{k,2n}U_{k,2n+p}$$

$$= (r_1 - r_2)^2\left(\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2}\right)\left(4\frac{r_1^{2n+p-1} - r_2^{2n+p-1}}{r_1 - r_2}\right)$$

$$= 4\left(r_1^{4n+p-1} + r_2^{4n+p-1} - (r_1r_2)^{2n}(r_1^{p-1} + r_2^{p-1})\right)$$

$$= 4(r_1^{4n+p-1} + r_2^{4n+p-1} - (-2)^{2n}(r_1^{p-1} + r_2^{p-1}))$$

$$= 4(j_{k,4n+p-1} - 4^n j_{k,p-1})$$

$$= 4j_{k,4n+p-1} - 4^{n+1}j_{k,p-1}.$$

**Theorem 3.14**

$$4j_{k,4n+p-1} + 2^{2n+1}j_{k,p+1} = (r_1 - r_2)^2 J_{k,2n+p}U_{k,2n}, \qquad (3.14)$$

*where* $n \geq 0$ *and* $p \geq 1$.

***Proof:***

$$(r_1-r_2)^2 J_{k,2n+p}U_{k,2n}$$
$$=(r_1-r_2)^2\left(\frac{r_1^{2n+p}-r_2^{2n+p}}{r_1-r_2}\right)\left(4\frac{r_1^{2n-1}-r_2^{2n-1}}{r_1-r_2}\right)$$
$$=4\left(r_1^{4n+p-1}+r_2^{4n+p-1}-(r_1r_2)^{2n-1}(r_1^{p+1}+r_2^{p+1})\right)$$
$$=4(r_1^{4n+p-1}+r_2^{4n+p-1}-(-2)^{2n-1}(r_1^{p+1}+r_2^{p+1}))$$
$$=4(j_{k,4n+p-1}+2^{2n-1}j_{k,p+1})$$
$$=4j_{k,4n+p-1}+2^{2n+1}j_{k,p+1}.$$

**Theorem 3.15**

$$4j_{k,4n+p-1}-4^{n+1}j_{k,p-1}=(r_1-r_2)^2 J_{k,n}j_{k,n}U_{k,2n+p}, \quad (3.15)$$

*where* $n\geq 0$ *and* $p\geq 1$.

***Proof:***

$$(r_1-r_2)^2 J_{k,n}j_{k,n}U_{k,2n+p}$$
$$=(r_1-r_2)^2\left(\frac{r_1^{n}-r_2^{n}}{r_1-r_2}\right)(r_1^n+r_2^n)\left(4\frac{r_1^{2n+p-1}-r_2^{2n+p-1}}{r_1-r_2}\right)$$
$$=4(r_1^{2n}-r_2^{2n})(r_1^{2n+p-1}-r_2^{2n+p-1})$$
$$=4\left(r_1^{4n+p-1}+r_2^{4n+p-1}-(r_1r_2)^{2n}(r_1^{p-1}+r_2^{p-1})\right)$$
$$=4\left(r_1^{4n+p-1}+r_2^{4n+p-1}-(-2)^{2n}(r_1^{p-1}+r_2^{p-1})\right)$$
$$=4\left(j_{k,4n+p-1}-4^n j_{k,p-1}\right)$$
$$=4j_{k,4n+p-1}-4^{n+1}j_{k,p-1}.$$

For different values of $p$ Theorem 3.1 to Theorem 3.14 can be expressed for even and odd $k$-generalized Fibonacci, $k$-Jacobsthal and $k$-Jacobsthal-Lucas numbers.

## 4. Conclusion

In this paper, by using the Binet's formula we obtained some generalized identities involving common factors of $k$-generalized Fibonacci, $k$-Jacobsthal and $k$-Jacobsthal-Lucas numbers. The results presented in this paper extend some previous results in the literature.

## 5. Acknowledgements

The authors would like to thank the faculty of science and technology, Rajamangala University of Technology Thanyaburi for the financial support.

## 6. References

[1] H. Campos, P. Catarino, A.P. Aires, P. Vasco, and A. Borges. On Some Identities for $k$-Jacobsthal –Lucas Number. *Int Journal of Math. Analysis*. 8(10) (2014): 489-494.

[2] S. Falco´n, Plaza, A. The *k*-Fibonacci sequence and the Pascal 2-triangle. *Chaos, Solitons & Fractals*. 33 (1) (2007): 38-49.

[3] AF. Horadam. Jacobsthal Number Representation. *The Fibonacci Quarterly*. 34(1) (1986): 79-83.

[4] AF. Horadam. Jacobsthal and Pell curves. *The Fibonacci Quarterly*. 26(1) (1988): 79-83.

[5] D. Jhala, K. Sisodiya, and G.P.S.Rathore, On Some Identities for $k$-Jacobsthal Number. *Int Journal of Math. Analysis*. 7(12) (2013): 551-556.

[6] Y. K. Panwar, B. Singh and V. K. Gupta. Generalized Identities Involving Common factors of generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas numbers. *International Journal of Analysis and Application*. 3(1) (2013): 53-59.

[7] Y. K. Panwar, B. Singh and V. K. Gupta. Identities of Common Factors of generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas

numbers. *Applied Mathematics and Physics.* 1(4) (2013): 126-128.

[8] Y. K. Panwer, M. Singh. $k$- Generalized Fibonacci Numbers. *Applied Mathematics and Physics*. 2(1) (2014): 10-12.

[9] B.Singh, P. Bhadouria and O. Sikhwal. Generalized Identities Involving Common Factors of Fibonacci and Lucas Numbers. *International Journal of Algebra*. 5(13) (2011): 637-645.

[10] M. Thongmoon. New Identities for the Even and Odd Fibonacci and Lucas Numbers. *Int. J. Contemp. Math. Sciences*. 4(14) (2009): 671-676.

[11] M. Thongmoon. Identities for the common factors of Fibonacci and Lucas numbers. *International Mathematical Forum*. 4(7) (2009): 303-308.