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Small PPQ-injective Modules

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Abstract

Let M be a right $R-$module. A right $R-$module N is called *small pseudo principally* $M-$*injective* (briefly, *small* $PP-M-$*injective*) if, every $R-$monomorphism from a small and principal submodule of M to N can be extended to an $R-$homomorphism from M to N. In this paper, we give some characterizations and properties of *small* $PP-Q-$*injective* modules.

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1. Introduction

Let R be a ring. A right $R-$module M is called *principally injective* (or $P-$*injective*) [5], if every $R-$homomorphism from a principal right ideal of R to M can be extended to an $R-$homomorphism from R to M. Equivalently, $l_M r_R(a) = Ma$ for all $a \in R$. Following [7], a right $R-$module M is called *principally quasi-injective*, if each $R-$homomorphism from a principal submodule of M to M can be extended to an endomorphism of M. In [9], a right $R-$module M is called *small principally injective* (or $SP-$

injective) if, every $R-$homomorphism from a small and principal right ideal aR to M can be extended to an $R-$homomorphism from R to M. A right $R-$ module M is called $PPQ-$*injective* [13] if, every $R-$monomorphism from a principal submodule of M to M extends to an endomorphism of M. In this note we introduce the definition of small $PPQ-$injective modules and give some interesting results on these modules.

Throughout this paper, R will be an associative ring with identity and all modules are unitary right $R-$modules. For right $R-$modules M and N, $Hom_R(M,N)$ denotes the set of all $R-$homomorphisms from M to N and $S=End_R(M)$ denotes the endomorphism ring of M. If X is a subset of M the right (resp. left) annihilator of X in R (resp. S) is denoted by $r_R(X)$ (resp. $l_S(X)$). By notations, $N \subset^{\oplus} M$, $N \subset^{e} M$, and $N \ll M$ we mean that N is a direct summand, an essential submodule and a superfluous submodule of M, respectively. We denote the Jacobson radical of M by $J(M)$.

Following [1], a submodule K of a right $R-$module M is *superfluous* (or *small*) in M, abbreviated $K \ll M$, in case for every submodule L of M, $K+L=M$ implies $L=M$. It is clear that $kR \ll R$ if and only if $k \in J(R)$.

2. Small PPQ-injective Modules

Definition 2.1. Let M be a right $R-$module. A right $R-$module N is called *small pseudo principally* $M-$ *injective* (briefly, *small* $PP-M-$ *injective*) if, every $R-$monomorphism from a small and principal submodule of M to N can be extended to an $R-$homomorphism from M to N. M is called *small pseudo principally quasi-injective* (briefly, *small* $PPQ-$*injective*) if, it is small $PP-M-$injective.

Lemma 2.2.

(1) If N is small $PP-M-$injective, then N is small $PP-X-$injective for any submodule X of M.

(2) Every direct summand of small $PP-M-$injective is also small $PP-M-$injective.

Proof. (1) Let $m \in X$ with $mR << X$ and let $\alpha : mR \to N$ be an $R-$monomorphism. Since $mR \ll M$ [1, Lemma 5.18], there exists an $R-$homomorphism $\hat{\alpha} : M \to N$ such that $\alpha = \hat{\alpha}\iota_2\iota_1$ where $\iota_1 : mR \to X$ and $\iota_2 : X \to M$ are the inclusion maps. Then $\hat{\alpha}\iota_2$ extends α.

(2) Let N be a small $PP-M-$injective module, $X \subset^{\oplus} N$, $m \in M$ with $mR \ll M$ and let $\alpha : mR \to X$ be an $R-$monomorphism. Let $\varphi : X \to N$ be the injection map. Since $\varphi\alpha$ is monic, there exists an $R-$homomorphism $\beta : M \to N$ such that $\varphi\alpha = \beta\iota$ where $\iota : mR \to M$ is the inclusion map. Then $\pi\beta$ extends α where $\pi : N \to X$ is the projection map.

□

Example 2.3. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is a field, $M_R = R_R$ and $N_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Then

(1) N is small $PP-M-$injective.

(2) N is small $PPQ-$injective.

Proof. (1) Let $0 \neq x \in F$ and $m = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$. It is clear that only $mR = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is the nonzero small and principal submodule of M. Let

$\varphi: mR \to N$ be an $R-$monomorphism. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in mR$, there exists $x_{11}, x_{12} \in F$ such that $\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}$.

Then $\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$

$$= \varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}.$$

It follows that $x_{11} = 0$.

Define $\hat{\varphi}: M \to N$ by $\hat{\varphi}\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} ax_{12} & bx_{12} \\ 0 & 0 \end{pmatrix}$ for every $a, b, c \in F$.

It is clear that $\hat{\varphi}$ is an $R-$homomorphism and $\hat{\varphi}$ extends φ.

Then N is small $PP-M-$injective.

(2) By Lemma 2.2 (1)

□

A right $R-$module N is called *small principally* $M-$*injective* (briefly, $SP-M-$*injective*) [12] if, every $R-$homomorphism from a small and principal submodule of M to N can be extended to an $R-$homomorphism from M to N. Two right $R-$modules M, N are called *mutually* $SP-$*injective* if M is $SP-N-$injective and N is $SP-M-$injective.

Proposition 2.4. If $M_1 \oplus M_2$ is small $PPQ-$injective, then M_1 and M_2 are mutually $SP-$injective.

Proof. Let $a \in M_2$ with $aR << M_2$ and let $\varphi: aR \to M_1$ be an $R-$homomorphism. Define $\alpha: aR \to M_1 \oplus M_2$ by $\alpha(x) = (\varphi(x), x)$ for every $x \in aR$. It is clear that α is an $R-$monomorphism. Since $M_1 \oplus M_2$ is small $PP-M_2-$injective by Lemma 2.2, there exists an $R-$homomorphism

$\beta : M_2 \to M_1 \oplus M_2$ such that $\alpha = \beta\iota$ where $\iota : aR \to M_2$ is the inclusion map. Let $\pi : M_1 \oplus M_2 \to M_1$ be the projection map. Then $\varphi = \pi\alpha = \pi\beta\iota$.

This proves that M_1 is $SP - M_2 -$injective. The same argument shows that M_2 is $SP - M_1 -$injective.

□

Theorem 2.5. Let M be a right $R-$module. If every small and principal submodule of M is projective, then every factor module of a small $PP - M -$injective module is small $PP - M -$injective.

Proof. Let N be a small $PP - M -$injective module, X a submodule of N, $mR << M$, and let $\varphi : aR \to N / X$ be an $R-$monomorphism. Then by assumption, there exists an $R-$homomorphism $\hat{\varphi} : aR \to N$ such that $\varphi = \eta\hat{\varphi}$ where $\eta : N \to N / X$ is the natural $R-$epimorphism. If $x \in Ker(\hat{\varphi})$, then $\varphi(x) = \eta\hat{\varphi}(x) = X$ so $x = 0$ which shows that $\hat{\varphi}$ is monic. Since N is a small $PP - M -$injective, there exists an $R-$homomorphism $\beta : M \to N$ which is an extension of $\hat{\varphi}$ to M. Then $\eta\beta$ is an extension of φ to M.

□

Let M be a right $R-$module with $S = End_R(M)$. Following [7], write

$$W(S) = \{s \in S : \; Ker(s) \subset^e M\}.$$

It is known that $W(S)$ is an ideal of S. A right $R-$module M is called a *principal self-generator* if every element $m \in M$ has the form $m = \gamma(m_1)$ for some $\gamma : M \to mR$.

Lemma 2.6. Let M be a principal, small $PPQ-$injective module. If $Ker(s) = Ker(t)$, where $s, t \in S$ with $s(M) << M$, then $St \subset Ss$.

Proof. Let $Ker(s) = Ker(t)$, where $s, t \in S$ with $s(M) << M$. Define $\varphi : s(M) \to M$ by $\varphi(s(m)) = t(m)$ for every $m \in M$. It is obvious that φ is an $R-$monomorphism. Since $s(M)$ is a small and principal submodule of M

and M is small $PPQ-$injective, let $\hat{\varphi}$ be an extension of φ. Then $t=\varphi s=\hat{\varphi}s\in Ss$ and so $St\subset Ss$.

□

Proposition 2.7. Let M be a principal module which is a principal self-generator and $Soc(M_R)\subset^e M$. If M is small $PPQ-$injective, then $J(S)\subset W(S)$.

Proof. Let $s\in J(S)$. If $Ker(s)\not\subset^e M$, then $Ker(s)\cap N=0$ for some nonzero submodule N of M. Since $Soc(M_R)\subset^e M$, $Soc(M_R)\cap N\neq 0$. Then there exists a simple submodule kR of M such that $kR\subset Soc(M)\cap N$ [1, Corollary 9.10]. As M is a principal self-generator and kR is simple, $kR=t(M)$ for some $t\in S$. It follows that $Ker(st)=Ker(t)$. Since M is a principal module, $J(M)\ll M$ [10, 21.6] and we have $J(S)M\subset J(M)$, it follows that $st(M)$ is a small submodule of M. Since M is small $PPQ-$injective, $St\subset Sst$ by Lemma 2.6. Write $t=gst$ where $g\in S$. It follows that $(1-gs)t=0$ and so $t=(1-gs)^{-1}0=0$, a contradiction.

□

Proposition 2.8. Let M be a principal nonsingular module which is a principal self-generator and $Soc(M_R)\subset^e M$. If M is small $PPQ-$injective, then $J(S)=0$.

Proof. Since $J(S)\subset W(S)$ by Proposition 2.7, we show that $W(S)=0$. Let $s\in W(S)$ and let $m\in M$. Define $\varphi:R\to M$ by $\varphi(r)=mr$ for every $r\in R$. It is clear that φ is an $R-$homomorphism. Thus

$$\begin{aligned} r_R(s(m)) &= \{r\in R:\ s(mr)=0\} \\ &= \{r\in R:\ mr\in Ker(s)\} \\ &= \{r\in R:\ \varphi(r)\in Ker(s)\} \\ &= \varphi^{-1}(Ker(s)). \end{aligned}$$

It follows that $\varphi^{-1}(Ker(s)) \subset^e R$ [3, Lemma 5.8(a)] so $r_R(s(m)) \subset^e R$. Thus $s(m) \in Z(M_R) = 0$ because M is nonsingular. As this is true for all $m \in M$, we have $s = 0$. Hence $W(S) = 0$ as required.

□

Proposition 2.9. Let M be a small $PPQ-$injective module $m \in M$ and $s \in S$.

(1) If mR is a simple and small right $R-$module, then Sm is a simple left $S-$module.

(2) If $s(M)$ is a simple and small right $R-$module, then Ss is a simple left $S-$module.

Proof. (1) If A is a nonzero submodule of Sm and $0 \neq \alpha(m) \in A$, then $S\alpha(m) \subset A$. Note that $\alpha(m)R$ is a nonzero homomorphic image of the simple module mR, then $\alpha(m)R$ is simple. It is clear that $\alpha(m)R \ll M$.

Define $\varphi : \alpha(m)R \to M$ by $\varphi(\alpha(m)r) = s(m)$ for every $r \in R$. Since $Ker(\alpha) \cap mR = 0$, φ is well-defined. It is clear that φ is an $R-$homomorphism. Since $\alpha(m)R$ is simple and φ is nonzero, $Ker(\varphi) = 0$. Then there exists an $R-$homomorphism $\hat{\varphi} \in S$ is an extension of φ. Hence $m = \hat{\varphi}\alpha(m) \in S\alpha(m)$. It follows that $Sm = S\alpha(m)$ so $A = Sm$.

(2) By the similar proof of (1).

□

Proposition 2.10. Let M be a small $PPQ-$injective module. If $Sm_1 \oplus ... \oplus Sm_n$ is direct, $m_i \in M$ with $m_iR \ll M$, $(1 \leq i \leq n)$, then any $R-$monomorphism $\alpha : m_1R + ... + m_nR \to M$ has an extension in S.

Proof. Since $\alpha|_{m_iR}$ is monic, for each i, there exists an $R-$homomorphism $\varphi_i : M \to M$ such that $\varphi_i(m_i) = \alpha(m_i)$. Since $(\sum_{i=1}^{n} m_i)R \ll M$,

$(\sum_{i=1}^{n} m_i)R \subset \sum_{i=1}^{n} m_i R$ and $\alpha\big|_{(\sum_{i=1}^{n} m_i)R}$ is monic, α can be extended to $\varphi : M \to M$ such that, for any $r \in R$,

$$\varphi(\sum_{i=1}^{n} m_i)r = \alpha(\sum_{i=1}^{n} m_i)r .$$

It follows that $\sum_{i=1}^{n} \varphi(m_i) = \sum_{i=1}^{n} \varphi_i(m_i)$. Since $Sm_1 \oplus ... \oplus Sm_n$ is direct, $\varphi(m_i) = \varphi_i(m_i)$ for all $1 \le i \le n$. Therefore φ is an extension of α.

□

Lemma 2.11. Let M be a small $PPQ-$injective module. If $r_R(m) = r_R(n)$, where $m, n \in M$ with $mR \ll M$, then $Sn \subset Sm$.

Proof. Let $r_R(m) = r_R(n)$, where $m, n \in M$ with $mR \ll M$. Define $\varphi : mR \to M$ by $\varphi(mr) = nr$ for every $r \in R$. It is obvious that φ is an $R-$monomorphism. Since M is small $PPQ-$injective, there exists $\hat{\varphi} \in S$ such that $\hat{\varphi}$ extends φ. Then $n = \varphi(m) = \hat{\varphi}(m) \in Sm$ so $Sn \subset Sm$.

□

Theorem 2.12. Let M be a small $PPQ-$injective module, $m, n \in M$ with $mR \ll M$.

(1) If mR embeds in nR, then Sm is an image of Sn.

(2) If $mR \simeq nR$, then $Sm \simeq Sn$.

Proof. (1) Let $f : mR \to nR$ be an $R-$monomorphism. Since M is small $PPQ-$ injective, there exists $\hat{f} \in S$ such that $\hat{f}$ extends f. Let $\sigma : Sn \to Sm$ defined by $\sigma(s(n)) = s\hat{f}(m)$ for every $s \in S$. Since $\sigma(s(n)) = sf(m) \in s(nR)$, σ is well-defined. It is clear that σ is an $S-$homomorphism. Note that $f(m)R = \hat{f}(m)R \ll M$. Since f is monic, $r_R(f(m)) = r_R(m)$ and hence by Lemma 2.11, $Sm \subset Sf(m)$. Then $m \in Sf(m) \subset \sigma(Sn)$.

(2) Let $f : mR \to nR$ be an $R-$isomorphism. Write $f(ma) = n$, $a \in R$. Since M is small $PPQ-$injective, f can be extended to

$\hat{f}: M \to M$. Define $\sigma: Sn \to Sm$ by $\sigma(s(n)) = s\hat{f}(ma)$ for every $s \in S$. It is clear that σ is an $S-$epimorphism. If $s(n) \in Ker(\sigma)$, then $0 = \sigma(s(n)) = s\hat{f}(ma) = sf(ma) = s(n)$. This shows that σ is monic.

□

Proposition 2.13. Let M be a principal, small $PPQ-$injective module. Then $Soc(M_R) \subset r_M(J(S))$.

Proof. Let mR be a simple submodule of M. Suppose $\alpha(m) \neq 0$ for some $\alpha \in J(S)$. Then $r_R(\alpha(m)) = r_R(m)$ because $r_R(m)$ is maximal. Since M is small $PPQ-$injective and $\alpha(m)R$ is a small and principal submodule of M, $Sm \subset S\alpha(m)$ by Lemma 2.11. Write $m = \beta\alpha(m)$ where $\beta \in S$. Then $(1-\beta\alpha)m = 0$ so $m = (1-\beta\alpha)^{-1}0 = 0,$ a contradiction.

□

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