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BASIC MATRIX OPERATIONS ON LEFTAND RIGHT CIRCULANT MATRICESVIA REPRESENTER POLYNOMIAL

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Abstract

In this paper, the use of representer polynomial to compute the sum and product and product left and right circulant matrices was presented. Some examples were also provided to illustrate the concept.

Keywords and phrases: left circulant matrix, matrix addition, matrix multiplication, representer polynomial, right circulant matrix

1. Introduction

Wyn-jones [2] introduced the concept of the representer polynomial in his work “Circulants”. Using the said polynomial, he illustrated how to compute the sum and the product of right circulant matrices.

The following are established properties of right circulant matrices from [1]:

a. The sum of two right circulant matrices is right circulant.
b. The product of two right circulantmatrix is right circulant.

Definition 1.1A matrix $RCIRC_n(\vec{c}) \in M_{nxn}(\mathbb{R})$ is said to be a right circulant matrix if it is of the form

$$RCIRC_n(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & & c_{n-3} & c_{n-2} \\ \vdots & & & \ddots & & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & & c_{n-1} & c_0 \end{pmatrix}$$

The matrix $RCIRC_n(\vec{c})$ has the following structure:

1. Each row is a right cyclic shift of the row above it.
2. $c_{i,j} = c_{k,l}$ whenever j – i = l – k (mod n)

Definition 1.2A matrix $LCIRC_n(\vec{c}) \in M_{nxn}(\mathbb{R})$ is said to be a left circulant matrix if it is of the form

$$LCIRC_n(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & c_3 & & c_{n-1} & c_0 \\ \vdots & & & \ddots & & \vdots \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ c_{n-1} & c_0 & c_1 & & c_{n-3} & c_{n-2} \end{pmatrix}$$

The matrix $LCIRC_n(\vec{c})$ has the following structure:

1. Each row is a left cyclic shift of the row above it
2. $LCIRC_n(\vec{c})$is symmetric, that is $LCIRC_n(\vec{c})^T = LCIRC_n(\vec{c})$.

Definition 1.3 Define $circ_n(\mathbb{R}) = \{\vec{c} | \vec{c} \in \mathbb{R}^n\}$. That is $circ_n(\mathbb{R})$ is the set of all circulant vectors in $\mathbb{R}^n$. Let $\vec{u} = (0\ 1\ 0 \ldots 0) \epsilon circ_n(\mathbb{R})$ and form $RCIRC_n(\vec{u})$. Then through convolution with powers of $\vec{u}$ in mod n, the set $\{1, \vec{u}, \vec{u}^2, \ldots, \vec{u}^{n-1}\}$ is the standard orthonormal basis for $circ_n(\mathbb{R})$ and $\{I, RCIRC_n(\vec{u}), RCIRC_n^2(\vec{u}), \ldots, RCIRC_n^{n-1}(\vec{u})\}$ is the standard orthonormal basis for $RCIRC_n(\mathbb{R})$. From these relations any $circ_n(\vec{c})$ and $RCIRC_n(\vec{c})$ can be written as a polynomial. That is $circ_n(\vec{c}) = \sum_{k=0}^{n-1} c_k\, \vec{u}^k$and $RCIRC_n(\vec{c}) = \sum_{k=0}^{n-1} c_k\, RCIRC_n^k(\vec{u})$

The polynomial $A^r(x) = \sum_{k=0}^{n-1} c_k\, x^k$ is called the **representer polynomial**. It evaluates $circ_n(\vec{c})$ at $x = \vec{u}$ and $RCIRC_n(\vec{c})$at $x = RCIRC_n(\vec{u})$.

The Relationship of the Left and Right Circulant Matrices

$$\boldsymbol{LCIRC_n(\vec{c})} = \boldsymbol{\Pi} \boldsymbol{RCIRC_n(\vec{c})}\ [\boldsymbol{Eq.1}]$$

where $\boldsymbol{\Pi} = \begin{pmatrix} \mathbf{1} & \mathbb{O}_{\mathbf{1}} \\ \mathbb{O}_{\mathbf{2}} & \boldsymbol{\tilde{I}_{n-1}} \end{pmatrix}$,$\tilde{I}_{n-1} = \begin{pmatrix} 0 & & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & & 1 & 0 & 0 \\ \vdots & \ddots & & \vdots & \\ 1 & \cdots & 0 & 0 & 0 \end{pmatrix}$,

$\mathbb{O}_1 = (0 \quad 0 \quad 0 \cdots 0)and\mathbb{O}_2 = {\mathbb{O}_1}^t$

Remarks:

$$\boldsymbol{\Pi} = \boldsymbol{\Pi^t} = \boldsymbol{\Pi^{-1}}[\boldsymbol{Eq.2}]$$

$$\boldsymbol{RCIRC_n(\vec{c})\Pi} = \boldsymbol{LCIRC_n(\vec{\gamma})}\ [\boldsymbol{Eq.3}]$$

where$\vec{\gamma} = (c_{0\,,} c_{n-1}, c_{n-2}, \dots, c_1)$

The effect of $\boldsymbol{\Pi}$ in multiplication is as follows.

Left multiplication: fixes the first row and does horizontal flip on the remaining part of the matrix

Right multiplication: fixes the first column and does vertical flip on the remaining part of the matrix

The Representer Polynomial and the Basic Matrix Operations

Let $\vec{a} = \sum_{k=0}^{n-1} a_k \vec{u}^k$ and $\vec{b} = \sum_{k=0}^{n-1} b_k \vec{u}^k$ be circulant vectors then $RCIRC_n(\vec{a}) + RCIRC_n(\vec{b}) = RCIRC_n(\vec{a} + \vec{b})andLCIRC_n(\vec{a}) + LCIRC_n(\vec{b}) = LCIRC_n(\vec{a} + \vec{b})$

On the other hand, on matrix multiplication the calculation will be a little bit different. The calculation is the same as the multiplication of polynomials, but with all the powers of $\vec{u}$ taken modulo n.

As an example, consider the right circulant matrices $RCIRC_5(1, 0, -3, 2, 1)$ and $RCIRC_5(3, -1,2,0,4)$ and their corresponding left circulant matrices $LCIRC_5(1, 0, -3, 2, 1)$and $LCIRC_5(3, -1,2,0,4)$.

Sum:

$$RCIRC_5(1, 0, -3, 2, 1) + RCIRC_5(3, -1,2,0,4) = RCIRC_5(4, -1, -1, 2, 5)$$
$$LCIRC_5(1, 0, -3, 2, 1) + LCIRC_5(3, -1,2,0,4) = LCIRC_5(4, -1, -1, 2, 5)$$

Product:

$RCIRC_5(1, 0, -3, 2, 1) \cdot RCIRC_5(3, -1,2,0,4)$will be determined by the resulting circulant vector.

$(1, 0, -3, 2, 1) = 1 - 3\vec{u}^2 + 2\vec{u}^3 + \vec{u}^4and(3, -1,2,0,4) = 3 - \vec{u} + 2\vec{u}^2 + 4\vec{u}^4$

$$\begin{aligned}(1 - 3\vec{u}^2 + 2\vec{u}^3 + \vec{u}^4)&(3 - \vec{u} + 2\vec{u}^2 + 4\vec{u}^4)\\ &= 3 - \vec{u} - 7\vec{u}^2 + 9\vec{u}^3 - \vec{u}^4 + 3\vec{u}^5 - 10\vec{u}^6 + 8\vec{u}^7 + 4\vec{u}^8\\ &= 6 - 11\vec{u} + \vec{u}^2 + 13\vec{u}^3 - \vec{u}^4\end{aligned}$$

Hence,
$RCIRC_5(1, 0, -3, 2, 1) \cdot RCIRC_5(3, -1,2,0,4) = RCIRC_5(6, -11, 1, 13, -1)$.

The computation for the product of two left circulant matrices and the product of a left circulant matrix and a right circrulant matrix will be based on the following results.

2. Main Results

Theorem 2.1 The product of two left circulant matrices is a right circulant matrix.

Proof. Let $LCIRC_n(\vec{v})$ and $LCIRC_n(\vec{w})$ be left circulant matrices where $\vec{v} = (v_0, v_1, \dots, v_{n-1})$and $\vec{w} = (w_0, w_1, \dots, w_{n-1})$. Then by Eq. 1,
$LCIRC_n(\vec{v}) = \Pi RCIRC_n(\vec{v})$and $LCIRC_n(\vec{w}) = \Pi RCIRC_n(\vec{w})$.

$LCIRC_n(\vec{v}) \cdot LCIRC_n(\vec{w})$
$= \Pi RCIRC_n(\vec{v})\Pi RCIRC_n(\vec{w})$
$= \Pi LCIRC_n(\vec{\rho})RCIRC_n(\vec{w})$ where $\vec{\rho} = (v_0, v_{n-1}, v_{n-2}, \dots, v_1)$[by Eq. 3]
$= \Pi\Pi RCIRC_n(\vec{\rho})RCIRC_n(\vec{w})$ [by Eq. 1]
$= RCIRC_n(\vec{\rho})RCIRC_n(\vec{w})$ [by Eq. 2]

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Theorem 2.2 The product of a left circulant matrix and a right circulant matrix is a left circulant matrix.

Proof. Let $LCIRC_n(\vec{v})$ be left circulant matrix and $RCIRC_n(\vec{w})$ be a right circulant matrix where $\vec{v} = (v_0, v_1, \dots, v_{n-1})$and $\vec{w} = (w_0, w_1, \dots, w_{n-1})$.. Then by Eq. 1,$LCIRC_n(\vec{v}) = \Pi RCIRC_n(\vec{v})$.

Case 1: $LCIRC_n(\vec{v}) \cdot RCIRC_n(\vec{w})$
$LCIRC_n(\vec{v}) \cdot RCIRC_n(\vec{w}) = \Pi RCIRC_n(\vec{v})RCIRC_n(\vec{w})$
This is a left circulant matrix because $RCIRC_n(\vec{v})RCIRC_n(\vec{w})$ is right circulant matrix.

Case 2: $RCIRC_n(\vec{w})LCIRC_n(\vec{v})$
$RCIRC_n(\vec{w})LCIRC_n(\vec{v})$
$= RCIRC_n(\vec{w})\Pi RCIRC_n(\vec{v})$ [by Eq.1]
$= LCIRC_n(\vec{\sigma})RCIRC_n(\vec{v})$where $\vec{\sigma} = (w_0, w_{n-1}, w_{n-2}, \dots, w_1)$ [by Eq. 3]
$= \Pi RCIRC_n(\vec{\sigma})RCIRC_n(\vec{v})$ [by Eq. 1]
This is a left circulant matrix using the same argument in the Case 1.

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3. Examples

Consider $LCIRC_5(1,2,-7,0,10), LCIRC_5(-4,\ -8,1,5,3)$ and $RCIRC_5(2,7,1,-3,1)$

$$\begin{aligned}
&LCIRC_5(1,2,-7,0,10)\cdot\ LCIRC_5(-4,-8,1,5,3)\\
&\quad = \mathit{\Pi}RCIRC_5(1,2,-7,0,10)\cdot\ \mathit{\Pi}RCIRC_5(-4,-8,1,5,3)\\
&\quad = \mathit{\Pi}LCIRC_5(1,10,0,-7,2)\cdot RCIRC_5(-4,-8,1,5\ ,3)\\
&\quad = RCIRC_5(1,10,0,-7,2)\cdot RCIRC_5(-4,-8,1,5\ ,3)
\end{aligned}$$

$$\begin{aligned}
&(1,10,0,-7,2) = 1+10\vec{u}-7\vec{u}^3+2\vec{u}^4 \text{and}\\
&\quad (-4,-8,1,5\ ,3) = -4-8\vec{u}+\vec{u}^2+5\vec{u}^3+3\vec{u}^4\\
&(1+10\vec{u}-7\vec{u}^3+2\vec{u}^4)(-4-8\vec{u}+\vec{u}^2+5\vec{u}^3+3\vec{u}^4)\\
&\quad = -4-48\vec{u}-79\vec{u}^2+43\vec{u}^3+101\vec{u}^4+7\vec{u}^5-33\vec{u}^6-11\vec{u}^7+6\vec{u}^8\\
&\quad = -4+7-48\vec{u}-33\vec{u}-79\vec{u}^2-11\vec{u}^2+43\vec{u}^3+6\vec{u}^3+101\vec{u}^4\\
&\quad = 3-81\vec{u}-90\vec{u}^2+49\vec{u}^3+101\vec{u}^4 = (3,-81,-90,49,101)
\end{aligned}$$

$$\begin{aligned}
&\rightarrow LCIRC_5(1,2,-7,0,10)\cdot\ LCIRC_5(-4,-8,1,5,3)\\
&\quad = RCIRC_5(3,-81,-90,49,101)
\end{aligned}$$

$$\begin{aligned}
&LCIRC_5(-4,-8,1,5,3)\cdot RCIRC_5(2,7,1,-3,1)\\
&\qquad\quad = \mathit{\Pi}RCIRC_5(-4,-8,1,5,3)\cdot RCIRC_5(2,7,1,-3,1)
\end{aligned}$$

$$\begin{aligned}
&(-4-8\vec{u}+\vec{u}^2+5\vec{u}^3+3\vec{u}^4)(2+7\vec{u}+\vec{u}^2-3\vec{u}^3+\vec{u}^4)\\
&\quad = -8-44\vec{u}-58\vec{u}^2+21\vec{u}^3+62\vec{u}^4+15\vec{u}^5-11\vec{u}^6-4\vec{u}^7+3\vec{u}^8\\
&\quad = -8+15-44\vec{u}-11\vec{u}-58\vec{u}^2-4\vec{u}^2+21\vec{u}^3+3\vec{u}^3+62\vec{u}^4\\
&\quad = 7-55\vec{u}-62\vec{u}^2+24\vec{u}^3+62\vec{u}^4 = (7,-55,-62,24,62)
\end{aligned}$$

$$\begin{aligned}
&\rightarrow LCIRC_5(-4,-8,1,5,3)\cdot RCIRC_5(2,7,1,-3,1)\\
&\quad = \Pi\mathrm{RCIRC}_5(7,-55,-62,24,62)\\
&\quad = \mathrm{LCIRC}_5(7,-55,-62,24,62)
\end{aligned}$$

4. References

[1] Karner, H., Schneid, J., Ueberhuber, C.,“Spectral decomposition of realcirculant matrices”, Linear Algebra and Its Applications, 367 (2003), 301 – 311

[2] Wyn-jones, A., “Circulants”, www.circulants.org/circ/circall.pdf