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On two diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$

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Abstract

In this paper, we show that diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no solution in non-negative integer.

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1 Introduction

In 2002, J. Sandor studied two diophantine equations $3^x + 3^y = 6^z$ and $4^x + 18^y = 22^z$. After that D. Acu (2007) studied the diophantine equation $2^x +$

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$5^y = z^2$. He found that this equation has exactly two solutions in non-negative integer $(x, y, z) \in \{(3, 0, 3), (2, 1, 3)\}$. Inspired by [1], we study two diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$, where x, y and z are non-negative integers.

2 Main Results

In this study, we use Catalan's conjecture (see [4]). It is proved there that the only solution in integers $a > 1$, $b > 1$, $x > 1$ and $y > 1$ of the equation $a^x + b^y = 1$ is $a = y = 3$ and $b = x = 2$. Now we have the followings.

Theorem 2.1. *The diophantine equation* $4^x + 7^y = z^2$ *has on solution in non-negative integers.*

Proof. From the diophantine equation $4^x + 7^y = z^2$, we consider in 3 cases.

Case 1: $x = 0$. We have $7^y = z^2 - 1 = (z - 1)(z + 1)$. Then there are non-negative integers a and b such that $7^a = z - 1$, $7^b = z + 1$, $a < b$ and $a + b = y$. So we have $7^a(7^{b-a} - 1) = 7^b - 7^a = (z + 1) - (z - 1) = 2$. Therefore, $7^a = 1$ or $a = 0$. It follows that $z = 0$ and $7^b = z + 1 = 3$. This is impossible.

Case 2: $y = 0$. We have $2^{2x} = 4^x = z^2 - 1 = (z - 1)(z + 1)$. Then there are non-negative integers a and b such that $2^a = z - 1$, $2^b = z + 1$, $a < b$ and $a + b = 2x$. Therefore, $2^a \cdot (2^{b-a} - 1) = 2^b - 2^a = (z + 1) - (z - 1) = 2$. It follows that $2^a = 1$ or $2^a = 2$. That is $a = 0$ or $a = 1$.
If $a = 0$ then $z = 2$ and $2^b = 3$. This is impossible. Thus $a = 1$. This implies that $z = 3$ and $b = 2$. Thus $2x = a + b = 3$. That is, x is not integer which is a contradiction.

Case 3: $x > 0$ and $y > 0$. We have $7^y = z^2 - 4^x = (z - 2^x)(z + 2^x)$. Then there are non-negative integers a and b such that $7^a = z - 2^x$, $7^b = z + 2^x$, $a < b$ and $a + b = y$. Therefore, $7^a(7^{b-a} - 1) = 7^b - 7^a = (z + 2^x) - (z - 2^x) = 2^{x+1}$. It follows that $7^a = 1$ and $7^{b-a} - 2^{x+1} = 1$. By Catalan's conjecture, we can conclude that this diophantine equation has no solution. The theorem is proved. □

Theorem 2.2. *The diophantine equation* $4^x + 11^y = z^2$ *has on solution in non-negative integers.*

Proof. From the diophantine equation $4^x + 11^y = z^2$, we consider in 3 cases.

Case 1: $x = 0$. We have $11^y = z^2 - 1 = (z - 1)(z + 1)$. Then there are non-negative integers a and b such that $11^a = z - 1$, $11^b = z + 1$, $a < b$ and

$a + b = y$. So we have $11^a(11^{b-a} - 1) = 11^b - 11^a = (z + 1) - (z - 1) = 2$. Therefore, $11^a = 1$ or $a = 0$. It follows that $z = 0$ and $11^b = z + 1 = 3$. This is impossible.

Case 2: $y = 0$. We have $2^{2x} = 4^x = z^2 - 1 = (z - 1)(z + 1)$. Then there are non-negative integers a and b such that $2^a = z - 1$, $2^b = z + 1$, $a < b$ and $a + b = 2x$. Therefore, $2^a \cdot (2^{b-a} - 1) = 2^b - 2^a = (z + 1) - (z - 1) = 2$. It follows that $2^a = 1$ or $2^a = 2$. That is $a = 0$ or $a = 1$.
If $a = 0$ then $z = 2$ and $2^b = 3$. This is impossible. Thus $a = 1$. This implies that $z = 3$ and $b = 2$. Thus $2x = a + b = 3$. That is, x is not integer which is a contradiction.

Case 3: $x > 0$ and $y > 0$. We have $11^y = z^2 - 4^x = (z - 2^x)(z + 2^x)$. Then there are non-negative integers a and b such that $11^a = z - 2^x$, $11^b = z + 2^x$, $a < b$ and $a + b = y$. Therefore, $11^a(11^{b-a} - 1) = 11^b - 11^a = (z + 2^x) - (z - 2^x) = 2^{x+1}$. It follows that $11^a = 1$ and $11^{b-a} - 2^{x+1} = 1$. By Catalan's conjecture, we can conclude that this diophantine equation has no solution. The theorem is proved. □

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