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Polynomials with only real zeros [1]

K. Srimud

Department of Mathematics, Faculty of Science and Technology,
Rajamangala University of Technology Thanyaburi, Thailand
e-mail : kulprapa_s@hotmail.com

S. Tadee

Department of Mathematics,
Chulalongkorn University, Bangkok 10330, Thailand
e-mail : s.tadee@hotmail.com

V. Laohakosol

Department of Mathematics, Kasetsart University, Bangkok 10900, Thailand
e-mail : fscivil@ku.ac.th

Abstract

Conditions which ensure that a combination of real polynomials, which have real interlacing zeros, continues to have only real zeros are derived. This gives a generalization of a result of Haglund and is proved using a unified method of Liu-Wang-Yeh.

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1 Introduction

Let

$$RZ = \{P(x) \in \mathbb{R}[x];\ P(x) \text{ has only real zeros}\}$$
$$PF = \{P(x) \in RZ;\ \text{all coefficients of } P(x) \in \mathbb{R}_{\geq 0}\}.$$

Let $f, g \in RZ$ and let $\{r_i\}$ and $\{s_j\}$ be all respective zeros of f and g in non-increasing order. Following Wagner [6], we say that g *alternates* f if $\deg f = \deg g = n$ and

$$s_n \leq r_n \leq s_{n-1} \leq \ldots \leq s_2 \leq r_2 \leq s_1 \leq r_1; \tag{1.1}$$

we say that g interlaces f if $\deg f = \deg g + 1 = n$ and

$$r_n \leq s_{n-1} \leq \ldots \leq s_2 \leq r_2 \leq s_1 \leq r_1. \tag{1.2}$$

The notation $g \preceq f$ denotes *either g alternates f or g interlaces f*. If no equality sign occurs in (1.1) (respectively (1.2)), then we say that g strictly alternates f (respectively g strictly interlaces f). Let $g \prec f$ denote *either g strictly alternates f or g strictly interlaces f*.

Polynomials with only real zeros arise often in combinatorics and other branches of mathematics (see [1],[4]). Let $a_0, a_1, \ldots$ be a sequence of nonnegative real numbers. It is *unimodal* if $a_0 \leq a_1 \leq \ldots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \ldots$ for some k. It is *log-concave* (LC), if $a_{i-1}a_{i+1} \leq a_i^2$ for all $i > 0$. Log-concavity implies unimodality. Unimodal and log-concave sequences occur naturally in combinatorics, algebra, analysis, geometry, computer science, probability and statistics.

One classical approach to unimodality and log-concavity of a finite sequence is to use Newton's inequality : if the polynomial $\sum_{i=0}^{n} a_i x^i$ with positive coefficients has only real zeros, then

$$a_i^2 \geq a_{i-1}a_{i+1}(1 + \frac{1}{i})(1 + \frac{1}{n-i})$$

for $1 \leq i \leq n-1$, and the sequence $a_0, a_1, ..., a_n$ is therefore unimodal and log-concave. Such a sequence of positive numbers whose generating function has only real zeros is called a *Pólya-frequency* (or PF) sequence. A deeper results is the following theorem which provides the basic link between finite Pólya-frequency sequence and polynomials having only real zeros.

Aissen-Schoenberg-Whitney Theorem ([5]).

A finite sequence $a_0, \ldots, a_n$ of nonnegative number is Polya-frequency sequence if and only if its generating function $\sum_{i=0}^{n} a_i x^i$ has only real zeros.

In 2005, Wang and Yeh, [5], proved the following results.

Theorem 1.1. *([5, Theorem 1],[3, Theorem 1.1])Let f and g be real polynomials whose leading coefficients have the same sign. Suppose that $f, g \in RZ$ and $g \preceq f$. If $ad \leq bc$, then $(ax+b)f(x) + (cx+d)g(x) \in RZ$.*

Corollary 1.2. *([5, Corollary 1],[3, Corollary 1.2]) Suppose that f , $g \in PF$ and g interlaces f. If $ad \geq bc$, then $(ax+b)f(x) + x(cx+d)g(x) \in RZ$.*

In 2007, Liu and Wang, [3], gave the following sufficient conditions for a sequence of polynomials to have only real zeros based on the method of interlacing zeros:

Theorem 1.3. *([3, Theorem 2.1]) Let F, f, g be three real polynomials satisfying the following conditions;*
(a) $F(x) = a(x)f(x) + b(x)g(x)$ where $a(x), b(x)$ are two real polynomials, such that $\deg F = \deg f$ or $\deg f + 1$,
(b) $f, g \in RZ$ and $g \preceq f$,
(c) F and g have leading coefficients of the same sign,
(d) $\forall\ r \in \mathbb{R}, f(r) = 0 \Rightarrow b(r) \leq 0$.
Then $F \in RZ$ and $f \preceq F$. In particular, if $g \prec f$ and $b(r) < 0$ whenever $f(r) = 0$, then $f \prec F$.

Corollary 1.4. *([3, Corollary 2.2],[2, Lemma 3.6]) Let f and g be two real polynomials with positive leading coefficients α and β respectively. Suppose that the following conditions are satisfied;*
(a) f , $g \in RZ$ and g interlaces f,
(b) $F(x) = (ax+b)f(x) + x(x+d)g(x)$ where $a, b, d \in \mathbb{R}$ with $d \geq 0, d \geq b/a$ and either $a > 0$ or $a < -\beta/\alpha$,
(c) all zeros of f are nonpositive if $a > 0$ and nonnegative if $a < -\beta/\alpha$.
Then $F \in RZ$. In addition, if each zero r of f satisfies $-d \leq r \leq 0$, then f interlaces F.

Haglund in [2] used Corollary 1.4 to prove facts about rook polynomials in graph theory.
Here we prove a generalization of Corollary 1.4 and give an example.

2 Result

Our main result is

Theorem 2.1. *Let f and g be two real polynomials with both positive or both negative leading coefficients α and β respectively. Suppose that the following conditions are satisfied;*
(a) f , $g \in RZ$ and g interlaces f,

(b) $F(x) = (ax+b)f(x) + x(cx+d)g(x)$ *where* $a, b, c, d \in \mathbb{R}$ *with* $a \neq 0$ *and* $d \geq bc/a$,
(c) if $a > 0$, *then all zeros of* f *are nonpositive,*
(d) if $a < 0$, *then all zeros of* f *are nonnegative.*
Then $F \in RZ$. *In addition, if* $c > 0$ *and* $-d/c \leq r \leq 0$ *for each zero* r *of* f, *then* f *interlaces* F. *If* $c = 0$ *and* $r \leq 0$ *for each zero* r *of* f, *then* f *interlaces* F.

Proof. Let $n \in \mathbb{N}$,

$$f(x) := \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x + \alpha_0 \in \mathbb{R}[x],$$

and $\alpha = \alpha_n$. Since g interlaces f, then $\deg g = n-1$. We distinguish two possibilities.

First, assume α and β are positive.
If $a > 0$, by (c) we have $r \leq 0$ for each zero of f. Thus, all coefficients of f are nonnegative, i.e., $f \in PF$. Since g interlaces f, we have $g \in PF$. By Corollary 1.2 and $ad \geq bc$, we deduce that $F \in RZ$.
If $a < 0$, by (d) we have $r \geq 0$ for each zero of f. By (a), all zeros of g are nonnegative. Thus, all coefficients of f and g are alternating in sign. Since α is positive and all coefficients of f are alternating in sign, we see that $(-1)^i\alpha_{n-i} \geq 0$ $(0 \leq i \leq n)$. Let

$$f_1(x) := (-1)^n f(-x), \quad g_1(x) := (-1)^{n-1}g(-x) \quad \text{and} \quad F_1(x) := (-1)^n F(-x).$$

We have

$$\begin{aligned} f_1(x) = (-1)^n f(-x) &= (-1)^n[\alpha_n(-x)^n + \alpha_{n-1}(-x)^{n-1} + \cdots + \alpha_1(-x) + \alpha_0] \\ &= \alpha_n x^n + (-1)\alpha_{n-1}x^{n-1} + \cdots + (-1)^{n-1}\alpha_1 x + (-1)^n\alpha_0, \end{aligned}$$

and so all coefficients of $f_1(x)$ are nonnegative, i.e., $f_1 \in PF$. Similarly, $g_1 \in PF$. Since g interlaces f, the polynomial g_1 also interlaces f_1. From

$$\begin{aligned} F_1(x) = (-1)^n F(-x) &= (-1)^n[(a(-x)+b)f(-x) - x(c(-x)+d)g(-x)] \\ &= (-ax+b)(-1)^n f(-x) + x(-cx+d)(-1)^{n-1}g(-x) \\ &= (-ax+b)f_1(x) + x(-cx+d)g_1(x), \end{aligned}$$

since $a < 0$ and $d \geq bc/a$, we get $ad \leq bc$, and so $(-a)d \geq b(-c)$. By Corollary 1.2, $F_1(x) \in RZ$ yielding $F(x) \in RZ$.

The remaining possibility is when α and β are negative.

If $a > 0$, by (c), we have $r \leq 0$ for each zero r of f. Thus, all coefficients of f are nonpositive. Since g interlace f, all coefficients of g are also nonpositive. Let

$$f_2(x) := -f(x), \quad g_2(x) := -g(x).$$

Thus, f_2, $g_2 \in PF$ and g_2 interlace f_2. From

$$-F(x) = (ax+b)(-f(x)) + x(cx+d)(-g(x)) = (ax+b)f_2(x) + x(cx+d)g_2(x),$$

since $a > 0$ and $d \geq bc/a$, Corollary 1.2 implies $-F(x) \in RZ$, and so $F(x) \in RZ$.

If $a < 0$, by (d) all zero r of f are nonnegative. Thus, the coefficients of f and g are alternating in sign. Since α is negative, we get $(-1)^i \alpha_{n-i} \leq 0 \ \ (0 \leq i \leq n)$. Let

$$f_3(x) := (-1)^{n+1} f(-x), \quad g_3(x) := (-1)^n g(-x), \quad F_3(x) := (-1)^{n+1} F(-x).$$

From

$$\begin{aligned} f_3(x) &= (-1)^{n+1} f(-x) = (-1)^{n+1}[\alpha_n(-x)^n + \alpha_{n-1}(-x)^{n-1} + \cdots + \alpha_1(-x) + \alpha_0] \\ &= (-1)\alpha_n x^n + (-1)^2 \alpha_{n-1} x^{n-1} + \cdots + (-1)^n \alpha_1 x + (-1)^{n+1}\alpha_0, \end{aligned}$$

we see that all coefficients of $f_3(x)$ are nonnegative, i.e., $f_3 \in PF$. Similarly, $g_3 \in PF$ and g_3 interlaces f_3. Thus,

$$\begin{aligned} F_3(x) &= (-1)^{n+1}[(-ax+b)f(-x) - x(-cx+d)g(-x)] \\ &= (-ax+b)(-1)^{n+1} f(-x) + x(-cx+d)(-1)^n g(-x) \\ &= (-ax+b)f_3(x) + x(-cx+d)g_3(x). \end{aligned}$$

Since $a < 0$ and $d \geq bc/a$, we have $(-a)d \geq b(-c)$, and so $F_3 \in RZ$ showing that $F \in RZ$.

There remains to check the final two additional assertions.

If $c > 0$ and $-d/c \leq r \leq 0$, for each zero r of f, then $r(cr+d) \leq 0$. If the leading coefficient of f and F have same sign, by Theorem 1.3, f interlaces F. If the leading coefficient of f and F have different sign, then $a < 0$. From (d), each zero r of f is nonnegative. This implies that the zeros of f can only be 0, and so $f(x) = \alpha x^n$, $g(x) = \beta x^{n-1}$. Thus,

$$\begin{aligned} F(x) &= (ax+b)(\alpha x^n) + x(cx+d)(\beta x^n) = (a\alpha + c\beta)x^{n+1} + (b\alpha + d\beta)x^n \\ &= x^n[(a\alpha + c\beta)x + (b\alpha + d\beta)] \end{aligned}$$

showing that f interlace F.

If $c = 0$ and $r \leq 0$ for each zero r of f, we treat two cases separately.

Case $a < 0$. By (d), we have $r \geq 0$ for each zero r of f. We must then have $r = 0$, and so $f(x) = \alpha x^n$ and $g(x) = \beta x^{n-1}$, by (a). From

$$\begin{aligned} F(x) &= (ax+b)f(x) + (dx)g(x) = (ax+b)\alpha x^n + (dx)\beta x^{n-1} \\ &= (\alpha a x + b\alpha + d\beta)x^n, \end{aligned}$$

we conclude that f interlaces F.

Case $a > 0$. By (b), the leading coefficients of F and g have same sign. Since $c = 0$ and $d \geq bc/a$, we get $d \geq 0$ and so $dr \leq 0$ for each zero r of f. By (b), we get $\deg F = \deg f + 1$, and by Theorem 1.3, we conclude that f interlace F. □

We end this note with an example.

Example 2.2. *Let $f(x) = -x^2 - 4x - 3$, $g(x) = -x - 2$, $a = 2$, $b = 3$, $c = 2$ and $d = 8$. We have $F(x) = -4x^3 - 23x^2 - 34x - 9$ such that all zeros of F are ≈ -3.51235, ≈ -1.9006 and ≈ -0.33705. Thus $F \in RZ$ and f interlaces F.*

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