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# Quasi-small P-injective Modules

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**Abstract**

Let $M$ be a right $R-$module. A right $R$-module $N$ is called *$M$-small principally injective* (briefly, *$M$-small $P$-injective*) if, every $R$-homomorphism from an $M$-cyclic small submodule of $M$ to $N$ can be extended to an $R$-homomorphism from $M$ to $N$. In this paper we give some characterizations and properties of quasi-small principally injective modules.

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## 1 Introduction

Throughout this paper, $R$ will be an associative ring with identity and all modules are unitary right $R$-modules. For right $R$-modules $M$ and $N$, $\text{Hom}_R(M, N)$ denotes the set of all $R$-homomorphisms from $M$ to $N$ and $S = End_R(M)$ denotes the endomorphism ring of $M$. A submodule $X$ of $M$ is said to be $M$-cyclic submodule of $M$ if it is the image of an element of $S$. If $X$ is a subset of $M$ the right (resp. left) annihilator of $X$ in $R$ (resp. S) is denoted by $r_R(X)$ (resp. $l_S(X)$). By notations, $N \subset^{\oplus} M$, $N \subset^{e} M$, and $N \ll M$ we mean that $N$ is a direct summand, an essential submodule and a superfluous submodule of $M$, respectively. We denote the Jacobson radical of $M$ by $J(M)$.

Let $R$ be a ring. A right $R$-module $M$ is called *principally injective* (or *$P$-injective*), if every $R$-homomorphism from a principal right ideal of $R$ to $M$ can be extended to an $R$-homomorphism from $R$ to $M$. Equivalently, $l_M r_R(a) = Ma$ for all $a \in R$. This notion was introduced by Camillo [2] for commutative rings. In [5], Nicholson and Yousif studied the structure of principally injective rings and gave some applications. In [9], L.V. Thuyet, and T.C. Quynh, introduced a small principally injective module, a right $R$- module $M$ is called *small principally injective* (briefly, *$SP-$injective*) if, every $R$-homomorphism from a small and principal right ideal $aR$ to $M$ can be extended to an $R$-homomorphism from $R$ to $M$. If $R_R$ is an $SP$-injective module, then we call $R$ is a *right $SP$-injective ring.* In this note we introduce the definition of quasi-small principally injective modules and give some interesting results on these modules.

## 2 Quasi-small P-injective Modules

Recall that a submodule $K$ of a right $R$-module $M$ is *superfluous* (or *small*) in $M$, abbreviated $K \ll M$, in case for every submodule $L$ of $M$, $K + L = M$ implies $L = M$.

**Definition 2.1.** Let $M$ be a right $R-$module. A right $R$-module $N$ is called *$M$-small principally injective* (briefly, *$M$-small $P$-injective*) if, every $R$-homomorphism from an $M$-cyclic small submodule of $M$ to $N$ can be extended to $M$. Equivalently, for any endomorphism $s$ of $M$ with $s(M) \ll M$, every $R$-homomorphism from $s(M)$ to $N$ can be extended to an $R$-homomorphism from $M$ to $N$.

A right $R-$module $M$ is called *quasi-small principally injective* (briefly, *quasi-small $P$-injective*) if it is $M$-small $P$-injective.

**Lemma 2.2.** *Let $M$ and $N_i$ $(1 \leq i \leq n)$ be right $R$-modules. Then $\oplus_{i=1}^{n} N_i$ is $M$-small $P$-injective if and only if $N_i$ is $M$-small $P$-injective for each $i = 1, 2, ..., n$.*

**Proof.** The necessity is trivial. For the sufficiency, it is enough to prove the result for $n = 2$, let $s \in S$ with $s(M) \ll M$, and $\alpha : s(M) \rightarrow N_1 \bigoplus N_2$ be an $R-$homomorphism. Since $N_1$ and $N_2$ are $M$-small $P$-injective, there exists $R$-homomorphisms $\alpha_1 : M \rightarrow N_1$ and $\alpha_2 : M \rightarrow N_2$ such that $\alpha_1 \iota = \pi_1 \alpha$ and $\alpha_2 \iota = \pi_2 \alpha$ where $\pi_1$ and $\pi_2$ are the projection maps from $N_1 \bigoplus N_2$ to $N_1$ and $N_2$, respectively, and $\iota : s(M) \rightarrow M$ is the inclusion map. Set $\widehat{\alpha} = \iota_1 \alpha_1 + \iota_2 \alpha_2 : M \rightarrow N_1 \bigoplus N_2$. Thus it is clear that $\widehat{\alpha}$ extends $\alpha$. □

**Theorem 2.3.** *The following conditions are equivalent for a projective module $M$:*

(1) *Every $M$-cyclic small submodule of $M$ is projective.*

(2) *Every factor module of an $M$-small $P$-injective module is $M$-small $P$-injective.*

(3) *Every factor module of an injective $R$-module is $M$-small $P$-injective.*

**Proof.** (1) $\Rightarrow$ (2) Let $N$ be an $M$-small $P$-injective module, $X$ a submodule of $N$, $s(M) \ll M$, and let $\alpha : s(M) \to N/X$ be an $R$-homomorphism. Then by (1), there exists an $R$-homomorphism $\varphi : s(M) \to N$ such that $\alpha = \eta\varphi$ where $\eta : N \to N/X$ is the natural $R$-epimorphism. Since $N$ is $M$-small $P$-injective, $\varphi$ can be extended to an $R$-homomorphism $\beta : M \to N$. Then $\eta\beta$ is an extension of $\alpha$ to $M$.

(2)$\Rightarrow$ (3) is clear.

(3)$\Rightarrow$(1) Let $s(M) \ll M$, $\gamma : A \to B$ be an $R$-epimorphism, and let $\varphi : s(M) \to B$ be an $R$-homomorphism. Embed $A$ in an injective module $E$ [1, 18.6]. Then $B \simeq A/Ker(\gamma)$ is a submodule of $E/Ker(\gamma)$ so by hypothesis, $\varphi$ can be extended to $\widehat{\varphi} : M \to E/Ker(\gamma)$. Since $M$ is projective, $\widehat{\varphi}$ can be lifted to $\beta : M \to E$. It is clear that $\beta(s(M)) \subset A$. Therefore we have lifted $\varphi$. □

**Theorem 2.4.** *Let $M$ be a right $R$-module and $S = End_R(M)$. Then the following conditions are equivalent:*

(1) *$M$ is quasi-small $P$-injective.*

(2) *$l_S(Ker(s)) = Ss$ for all $s \in S$ with $s(M) \ll M$.*

(3) *$Ker(s) \subset Ker(t)$,where $s, t \in S$ with $s(M) \ll M$, implies $St \subset Ss$.*

(4) *$l_S(Ker(s) \cap Im(t)) = l_S(Im(t)) + Ss$ for all $s, t \in S$ with $s(M) \ll M$.*

(5) *If $\alpha : s(M) \to M$, $s \in S$ with $s(M) \ll M$, then $\alpha s \in Ss$.*

**Proof.** (1) $\Rightarrow$ (2) Let $s \in S$ with $s(M) \ll M$ and let $f \in l_S(Ker(s))$. Then $Ker(s) \subset Ker(f)$, so there exists an $R$-homomorphism $\varphi : s(M) \to N$ such that $\varphi s = f$. Since $s(M) \ll M$ and $M$ is quasi-small $P$-injective, there exists an $R$-homomorphism $\widehat{\varphi} : M \to M$ such that $\widehat{\varphi}\iota = \varphi$, where $\iota : s(M) \to M$ is the inclusion map. Therefore $f = \widehat{\varphi}s \in Ss$. The other inclusion is clear.

(2) $\Rightarrow$ (1) Let $s \in S$ with $s(M) \ll M$, and $\varphi : s(M) \to M$ be an $R$-homomorphism. Then $\varphi s \in S$ and $\varphi s \in l_S(Ker(s))$. By assumption, $\varphi s = us$ for some $u \in S$. This shows that $M$ is quasi-small $P$-injective.

(2)$\Rightarrow$ (3) If $Ker(s) \subset Ker(t)$, where $s, t \in S$ with $s(M) \ll M$, then $l_S(Ker(t)) \subset l_S(Ker(s))$. Since $St \subset l_S(Ker(t))$ and by (2), we have $l_S((Ker(s)) = Ss$ so $St \subset Ss$.

(3)$\Rightarrow$(4) Let $s, t \in S$ with $s(M) \ll M$ and let $u \in l_S(Ker(s) \cap Im(t))$. Then $u(Ker(s) \cap Im(t)) = 0$ so $Ker(st) \subset Ker(ut)$. Since $st(M) \subset s(M)$, $st(M) \ll M$ and hence $Sut \subset Sst$ by (3). Write $ut = vst$ where $v \in S$. Then $(u - vs)t = 0$, and therefore, $u - vs \in l_S(Im(t))$ it follows that $u \in l_S(Im(t)) + Ss$. The other inclusion is clear.

(4)$\Rightarrow$ (5) Put $t = 1_M$ in (4), then we have $\alpha s \in l_S(Ker(s)) = l_S(Ker(s) \cap Im(1_M)) = l_S(Im(1_M)) + Ss = Ss$.

(5)$\Rightarrow$ (1) Clear. □

**Corollary 2.5.** [9, Lemma 2.2 ] *The following conditions are equivalent for a Ring $R$:*

(1) *$R$ is $SP-$injective.*

(2) *$lr(a) = Ra$ for all $a \in J(R)$.*

(3) *$r(a) \subset r(b)$, where $a \in J(R)$, $b \in R$ implies $Rb \subset Ra$.*

(4) *$l(r(a) \cap bR) = l(b) + Ra$ for all $a \in J(R)$ and $b \in R$.*

(5) *If $\alpha : aR \to R$, $a \in J(R)$, is an $R$-homomorphism, then $\alpha(a) \in Ra$.*

**Example 2.6.** *Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where $F$ is a field, $M_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Then $M$ is quasi-small $P$-injective.*

**Proof.** It is clear that only $N = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is the nonzero small $M$-cyclic submodule of $M$. Let $s \in S = End_R(M)$ such that $s(\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}) \subset \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Since $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $s(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ for some $0 \neq x \in F$. Then for each $a, b \in F$, $s(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}) = s[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}] = s(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $s = 0$ it follows that $l_S(Ker(s)) = 0 = Ss$. This shows that $M$ is quasi-small $P$-injective. □

**Lemma 2.7.** *Any direct summand of a quasi-small $P$-injective module is again quasi-small $P$-injective.*

**Proof.** Let $M$ be a quasi-small $P$-injective module, $M = A \oplus B$, $s \in End_R(A)$ with $s(A) \ll A$ and let $\alpha : s(A) \to A$ be an $R$-homomorphism. Note that $s(A) \ll M$ [1, Lemma 5.18]. Define $s' : M \to M$ by $s'(a + b) = s(a)$ and $\alpha' : s'(M) \to M$ by $\alpha'(s'(a + b)) = \alpha(s(a))$. It is clear that $s'$ and $\alpha'$ are $R$-homomorphisms and $s'(M) \ll M$. Since $M$ quasi-small $P$-injective, there exists an $R$-homomorphism $\varphi : M \to M$ such that $\alpha' = \varphi\iota$ where $\iota : s'(M) \to M$ is the inclusion map. Let $\widehat{\alpha}$ be the restriction of $\varphi$ to $A$. Then $\widehat{\alpha}$ is an extension of $\alpha$. □

A right $R$-module $M$ is called *co-Hopfian* [3] if any injective endomorphism of $M$ is an isomorphism. A right $R$-module $M$ is called weakly *co-Hopfian* if any injective endomorphism $f$ of $M$ is essential; that is, $f(M) \subset^e M$.

**Proposition 2.8.** *Let $M$ be a quasi-small $P$-injective module. If $X$ is an $M$-cyclic and essential submodule of $M$ and $M$ is weakly co-Hopfian, then $X$ is weakly co-Hopfian.*

**Proof.** Let $f$ be an injective endomorphism of $X$. Then there exists an $R$-homomorphism $g : M \to M$ such that $\iota f = g\iota$ where $\iota : X \to M$ is the inclusion map. It is obvious that $Ker(g) \cap X = 0$, so $Ker(g) = 0$. Hence $g$ is monic, so $g(X) \subset^e M$ by [3, Corollary 1.2.]. It follows that $f(X) \subset^e X$. Therefore $X$ is weakly co-Hopfian. □

**Proposition 2.9.** *Let $M$ be a quasi-small $P$-injective module and $s_i \in S$ with $s_i(M) \ll M$, $(1 \leq i \leq n)$.*

(1) *If $Ss_1 \oplus ... \oplus Ss_n$ is direct, then any $R$-homomorphism $\alpha : s_1(M) + ... + s_n(M) \to M$ has an extension in $S$.*

(2) *If $s_1(M) \oplus ... \oplus s_n(M)$ is direct, then $S(s_1 + ... + s_n) = Ss_1 + ... + Ss_n$.*

**Proof.** (1) Since $M$ is quasi-small $P$-injective, for each $i$, there exists an $R$-homomorphism $\varphi_i : M \to M$ such that $\varphi_i s_i(m) = \alpha s_i(m)$ for all $m \in M$. Since $(\sum_{i=1}^n s_i)(M) \ll M$ and $(\sum_{i=1}^n s_i)(M) \subset \sum_{i=1}^n s_i(M)$, $\alpha$ can be extended to $\varphi : M \to M$ such that, for any $m \in M$,

$$\varphi(\sum_{i=1}^n s_i)(m) = \alpha(\sum_{i=1}^n s_i)(m).$$

It follows that $\sum_{i=1}^n \varphi s_i = \sum_{i=1}^n \varphi_i s_i$. Since $Ss_1 \oplus ... \oplus Ss_n$ is direct, $\varphi s_i = \varphi_i s_i$ for all $(1 \leq i \leq n)$. Therefore $\varphi$ is an extension of $\alpha$.

(2) Let $\alpha_1 s_1 + ... + \alpha_n s_n \in Ss_1 + ... + Ss_n$. For each $i$, define $\varphi_i : (s_1 + ... + s_n)(M) \to M$ by $\varphi_i((s_1 + ... + s_n)(m)) = s_i(m)$ for every $m \in M$. Since $s_1(M) \oplus ... \oplus s_n(M)$ is direct, $\varphi_i$ is well-defined, so it is clear that $\varphi_i$ is an $R$-homomorphism. By assumption and $(s_1 + ... + s_n)(M) \ll M$, there exists $\widehat{\varphi_i} \in S$ which is an extension of $\varphi_i$. Then

$$s_i = \varphi_i(s_1 + ... + s_n) = \widehat{\varphi_i}(s_1 + ... + s_n) \in S(s_1 + ... + s_n).$$

Consequently, $\alpha_1(s_1)+...+\alpha_n(s_n) \in S(s_1+...+s_n)$. This shows that $Ss_1+...+Ss_n \subset S(s_1 + ... + s_n)$. The other inclusion always holds. □

**Proposition 2.10.** *Let $M$ be a quasi-small $P$-injective module and $s_1(M) \oplus ... \oplus s_n(M)$ a direct sum of small and fully invariant $M$-cyclic submodules of $M$. Then for any fully invariant small submodule $A$ of $M$, we have*

$$A \cap (s_1(M) \oplus ... \oplus s_n(M)) = (A \cap s_1(M)) \oplus ... \oplus (A \cap s_n(M)).$$

**Proof.** Always $(A \cap s_1(M)) \oplus ... \oplus (A \cap s_n(M)) \subset A \cap (s_1(M) \oplus ... \oplus s_n(M))$. Let $a = \sum_{i=1}^{n} s_i(m_i) \in A \cap (s_1(M) \oplus ... \oplus s_n(M))$. Let $\pi_k : \oplus_{i=1}^{n} s_i(M) \to s_k(M)$ be the projection map. Since, for each $i$, $s_i(M)$ is small and fully invariant, $Ss_i(M) \subset s_i(M)$ so $\oplus_{i=1}^{n} Ss_i$ is direct. Then by Proposition 2.9, $\pi_k$ has an extension $\widehat{\pi_k}$ in $S$. It follows that $s_i(m_i) = \widehat{\pi}_i(a) \in A \cap s_i(M)$. Therefore $a \in \oplus_{i=1}^{n}(A \cap s_i(M))$. □

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