PROGRESS IN APPLIED SCIENCE AND TECHNOLOGY
Faculty of Science and Technology
Rajamangala University of Technology Thanyaburi
Vol.11 No.1 (2021): 1–6
ISSN (Print): 2730-3012
ISSN (Online): 2730-3020
https://ph02.tci-thaijo.org/index.php/past

Received 03rd June 2020,
Revised 08th December 2020,
Accepted 15th December 2020

DOI: 10.14456/past.2021.2

Almost Type of Simulation Functions Results

Paiwan Wongsasinchai
Department of Mathematics, Faculty of Science and Technology, Rambhai Barni Rajabhat University, Chanthaburi, Thailand.
E-mail: paiwan252653@gmail.com

Abstract

In this paper, we introduce common point theorems for almost type simulation functions and a complete metric framework. We investigated both the existence and uniqueness of common fixed points of such mappings. We used an example to illustrate the main result observed. Our main results cover several existing results in the corresponding literature.

Keywords: almost type, common point theorems, simulation functions

1. Introduction

The fixed problem can be considered a simple equation $\Upsilon\eta = \eta$. In almost all scientific disciplines, most of the issues can be converted into fixed point equations. The first fixed point theorem was announced in Banach's thesis in 1922 (1), in the setting of complete normed space, which can be described as the abstraction of the successive approximation method.

Theorem 1.1 (1) Let (Λ, d) be a complete metric space and Υ be a self-mapping on the set Λ such that there exists $\rho \in [0,1)$,

$$d(\Upsilon\eta, \Upsilon\sigma) \leq \rho d(\eta, \sigma) \text{ for all } \eta, \sigma \in \Lambda. \quad (1.1)$$

Then, Υ has a unique fixed point in Λ.

Theorem 1.2 (3) Let (Λ, d) be a complete metric space and a self-mapping Υ on the set Λ be an almost contraction, that is, a mapping for which there exist $\delta \in [0,1)$ and there exist $L \geq 0$such that

$$d(\Upsilon\eta, \Upsilon\sigma) \leq \delta d(\eta, \sigma) + Ld(\sigma, \Upsilon\eta), \forall \eta, \sigma \in \Lambda. \quad (1.2)$$

Then,

(i) $Fix(\Upsilon) \neq \emptyset$, where $Fix(\Upsilon) = \{\eta \in \Lambda: \Upsilon\eta = \eta\}$;

(ii) For any $\eta_0 \in \Lambda$, the Picard iteration $\{\eta_n\}$ given by $\eta_{n+1} = \Upsilon\eta_n$ for each $n \geq 0$ converges to some $\eta * \in Fix(\Upsilon)$;

(iii) The following estimate holds

$d(\eta_{n+i-1}, \eta *) \leq \frac{\delta^i}{1-\delta} d(\eta_n, \eta_{n-1}), \forall n \geq 0, i \geq 1.$

Babu et al. [5] defined the class of mappings satisfying condition (B) as follows:

Definition 1.3 (5) Let (Λ, d) be a metric space and a self-mapping Υ on Λ is said to satisfy condition (B) if there exist a constant $\delta \in (0,1)$ and there exists $L \geq 0$ such that

$$d(\Upsilon\eta, \Upsilon\sigma) \leq \delta d(\eta, \sigma) + LK(\eta, \sigma), \forall \eta, \sigma \in \Lambda, \quad (1.3)$$

where

$K(\eta, \sigma) = min\{ d(\eta, \Upsilon\eta), d(\sigma, \Upsilon\sigma), d(\eta, \Upsilon\sigma), d(\sigma, \Upsilon\eta)\}.$

They proved a fixed point theorem for such mappings in complete metric spaces. They also discussed quasi-contraction, almost contraction, and mappings class that satisfy condition (B) in detail.

Khojasteh et al. (6) originated the notion of Z -contractions using a specific family of functions called simulation functions. Subsequently, many researchers generalized this idea in many ways (8-24) and proved many interesting results in the arena of fixed point theory. Recently, Heidary et al. (7) proposed a new notion, the ψ -simulation function. The notion of the Z_ψ-contraction covers several distinct types of contraction, including the Z -contraction that was defined in (6). We denote $\Psi := \{\psi: \mathbb{R}_0^+ \to \mathbb{R}_0^+ | \psi$ is continuous and nondecreasing, and $\psi(r) = 0 \Leftrightarrow r = 0\}$.

Definition 1.4 (7) We say that $\zeta: \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ is a ψ -simulation function, if there exists $\psi \in \Psi$ such that:

$(\zeta_1) \zeta(p, q) < \psi(q) - \psi(p)$ for all $p, q > 0$;

(ζ_2) if $\{p_n\}, \{q_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n > 0$ implies $\lim \sup_{n\to\infty} \zeta(p_n, q_n) < 0$.

Let Z_ψ be the set of all ψ -simulation functions. Note that if we take ψ as an identity mapping, then ψ -simulation function becomes a simulation function in the sense of (6).

Example 1.5. (7) Let $\psi \in \Psi$

(i) $\zeta_1(p,q) = k\psi(q) - \psi(p)$ for all $p,q \in [0,\infty)$, where $k \in [0,1)$.

(ii) $\zeta_2(p,q) = \varphi(\psi(q)) - \psi(p)$ for all $p,q \in [0,\infty)$, where φ is a self-mapping from $[0,\infty)$ to $[0,\infty)$ so that $\varphi(0) = 0$ and for each $q > 0, \varphi(q) < q$,

$$\lim_{p\to q} sup\varphi(q) < q.$$

(iii) $\zeta_3(p,q) = \psi(q) - \varphi(q) - \psi(p)$ for all $p,q \in [0,\infty)$ where $\psi: [0,\infty) \to [0,\infty)$ is a mapping such that, for each $q > 0$,

$$\lim_{p\to q} inf \varphi(p) < 0.$$

It is clear that $\zeta_1, \zeta_2, \zeta_3 \in Z_\psi$.

Motivated and inspired by Babu et al.(5), Khojasteh et al. (6) and Heidary et al. (7), we define an Almost type simulation functions in metric spaces.

2. Main Results

Firstly, we present the following definition, which will be used in our main results.

Definition 2.1. Let (Λ, d) be a metric space. We say that a pair of mappings $\Upsilon, \Phi: \Lambda \to \Lambda$ is an almost type simulation functions whenever there is a constant $L \geq 0$, for all $\eta, \sigma \in \Lambda$, such that

$\frac{1}{2} min\{ d(\eta, \Upsilon\eta), d(\sigma, \Phi\sigma)\} \leq d(\eta,\sigma)$ implies
$$\zeta(d(\Upsilon\eta, \Phi\sigma), D(\eta,\sigma) + LK(\eta,\sigma)) \geq 0, \tag{2.1}$$

where
$D(\eta,\sigma) = max\left\{d(\eta,\sigma), \frac{[1+d(\eta,\Upsilon\eta)]d(\sigma,\Phi\sigma)}{1+d(\eta,\sigma)}\right\}$ and
$K(\eta,\sigma) = min\{d(\eta,\Upsilon\eta), d(\sigma,\Phi\sigma), d(\eta,\Phi\sigma), d(\sigma,\Upsilon\eta)\}$.

Lemma 2.2. Let (Λ, d) be a metric space and $\Upsilon, \Phi: \Lambda \to \Lambda$ be two selfmaps. Assume that the pair (Υ, Φ) is an almost type of simulation functions. Then u is a fixed point of Υ if and only if u is a fixed point of Φ. In that case, u is a common fixed point of Υ and Φ, and u is unique.

Proof. Let u be a fixed point of Υ, i.e., $\Upsilon u = u$. Now, we prove that u is a fixed point of Φ. Assume that $d(u,\Phi u) > 0$ Thus from (2.1) we have

$$0 = \frac{1}{2} min\{ d(u,\Upsilon u), d(u,\Phi u)\} \leq d(u,u). \tag{2.2}$$

This implies

$$\zeta(d(\Upsilon u, \Phi u), D(u,u) + LK(u,u)) \geq 0. \tag{2.3}$$

By (ζ_1), we have
$\psi(d(\Upsilon u, \Phi u)) < \psi(D(u,u) + LK(u,u))$.
Therefore,

$$d(\Upsilon u, \Phi u) < D(u,u) + LK(u,u), \tag{2.4}$$

where
$$D(u,u) = max\left\{d(u,u), \frac{[1 + d(u,\Upsilon u)]d(u,\Phi u)}{1 + d(u,u)}\right\} = d(u,\Phi u)$$
and
$K(u,u) = min\{d(u,\Upsilon u), d(u,\Phi u), d(u,\Phi u), d(u,\Upsilon u)\} = 0.$
Hence, using the values of $D(u,u)$ and $K(u,u)$ in (2.4), we obtain $d(u,\Phi u) < d(u,\Phi u) + L0$, which is a contradiction. Therefore $u = \Phi u$. So, u is a fixed point of Φ. Same the way, it is easy to see that if u is a fixed point of Φ then u is a fixed point of Υ also. Next, we prove that u is a unique common fixed point of Υ and Φ. Let u and v be two common fixed points of Υ and Φ such that $d(u,v) > 0$. Thus from (2.1), we have

$$0 = \frac{1}{2} min\{ d(u,\Upsilon u), d(v,\Phi v)\} \leq d(u,v). \tag{2.5}$$

This implies

$$\zeta(d(\Upsilon u, \Phi v), D(u,v) + LK(u,v)) \geq 0. \tag{2.6}$$

By (ζ_1) , we have
$\psi(d(\Upsilon u, \Phi v)) < \psi(D(u,v) + LK(u,v))$.
Therefore,

$$d(\Upsilon u, \Phi v) < D(u,v) + LK(u,v), \tag{2.7}$$

where
$$D(u,v) = max\left\{d(u,v), \frac{[1 + d(u,\Upsilon u)]d(v,\Phi v)}{1 + d(u,v)}\right\} = d(u,v)$$
and
$K(u,v) = min\{d(u,\Upsilon u), d(v,\Phi v), d(u,\Phi v), d(v,\Upsilon u)\} = 0.$
Hence, using the values of $D(u,v)$ and $K(u,v)$ in (2.7), we obtain

$$d(\Upsilon u, v) < d(u,v) + L0 \ ,$$

which is a contradiction. We conclude that $d(u,v) = 0$, i.e., $u = v$ and the theorem is proven.

Theorem 2.3. Let (Λ, d) be a complete metric space and $\Upsilon, \Phi: \Lambda \to \Lambda$ be two selfmaps. Assume that the pair (Υ, Φ) is an almost type of simulation functions. Then u is a unique common fixed point of Υ and Φ.

Proof. Let $u_0 \in \Lambda$ be an arbitrary point. We define a sequence $\{u_n\} \subset \Lambda$ by $u_{2n+1} = \Upsilon u_{2n}$ and $u_{2n+2} = \Phi u_{2n+1}$ for $n = 0,1,2,\ldots$
We note that
$\frac{1}{2} min\{ d(\eta, \Upsilon\eta), d(\sigma, \Phi\sigma)\} \leq d(\eta,\sigma)$
if and only if either
$\frac{1}{2} d(\eta, \Upsilon\eta) \leq d(\eta,\sigma)\}$ or $\frac{1}{2} d(\sigma, \Phi\sigma) \leq d(\eta,\sigma)\}$.
Assume that $u_{2n} = u_{2n+1}$ for some n, then $u_{2n} = \Upsilon u_{2n}$. So, u_{2n} is a fixed point of Υ. Thus, by **Lemma 2.2**, we have u_{2n} is a fixed point of Φ also u_{2n} is a common fixed point of Υ and Φ. Same the way, if $u_{2n+1} = u_{2n+2}$ then u_{2n+1} is a fixed point of Φ. Using **Lemma 2.2**, we have u_{2n+1} is a common fixed point of Υ and Φ. Without loss

of generality, we can assume that $d(u_n, u_{n+1}) > 0$ for $n = 0,1,2,\ldots$ From (2.1), we get
$\frac{1}{2}min\{ d(u_{2n}, \Upsilon u_{2n}), d(u_{2n+1}, \Phi u_{2n+1})\} \leq d(u_{2n}, u_{2n+1})$.
This implies
$\zeta(d(\Upsilon u_{2n}, \Phi u_{2n+1}), D(u_{2n}, u_{2n+1}) + LK(u_{2n}, u_{2n+1})) \geq 0$.
By (ζ_1), we have

$$\psi(d(\Upsilon u_{2n}, \Phi u_{2n+1})) < \psi(D(u_{2n}, u_{2n+1}) + LK(u_{2n}, u_{2n+1})). \quad (2.8)$$

Therefore,

$$d(\Upsilon u_{2n}, \Phi u_{2n+1}) < D(u_{2n}, u_{2n+1}) + LK(u_{2n}, u_{2n+1}), \quad (2.9)$$

where

$$\begin{aligned} D(u_{2n}, u_{2n+1}) &= max\left\{d(u_{2n}, u_{2n+1}), \frac{[1+d(u_{2n},\Upsilon u_{2n})]d(u_{2n+1},\Phi u_{2n+1})}{1+d(u_{2n},u_{2n+1})}\right\} \\ &= max\left\{d(u_{2n}, u_{2n+1}), \frac{[1+d(u_{2n},u_{2n+1})]d(u_{2n+1},u_{2n+2})}{1+d(u_{2n},u_{2n+1})}\right\} \\ &= max\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+2})\}, \end{aligned}$$

so that

$$D(u_{2n}, u_{2n+1}) = max\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+2})\}$$

and

$$\begin{aligned} K(u_{2n}, u_{2n+1}) &= min\left\{\begin{matrix} d(u_{2n}, \Upsilon u_{2n}), d(u_{2n+1}, \Phi u_{2n+1}), \\ d(u_{2n}, \Phi u_{2n+1}), d(u_{2n+1}, \Upsilon u_{2n}) \end{matrix}\right\} \\ &= min\left\{\begin{matrix} d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+2}), \\ d(u_{2n}, u_{2n+2}), d(u_{2n+1}, u_{2n+1}) \end{matrix}\right\} \\ &= 0. \end{aligned}$$

Hence, using the values of $D(u_{2n}, u_{2n+1})$ and$K(u_{2n}, u_{2n+1})$ in (2.9), we obtain

$$d(u_{2n+1}, u_{2n+2}) < \max\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+2})\} + L0. \quad (2.10)$$

If $d(u_{2n}, u_{2n+1}) < d(u_{2n+1}, u_{2n+2})$ for some n, then from (2.10), we have

$$d(u_{2n+1}, u_{2n+2}) < d(u_{2n+1}, u_{2n+2})$$

which is a contradiction. Therefore,

$$d(u_{2n+1}, u_{2n+2}) < d(u_{2n}, u_{2n+1})$$

and

$$D(u_{2n}, u_{2n+1}) = d(u_{2n}, u_{2n+1})$$

Hence, from (2.8), we have

$$\psi(d(u_{2n+1}, u_{2n+2})) < \psi(d(u_{2n}, u_{2n+1})). \quad (2.11)$$

Same the way, we have

$$\psi(d(u_{2n+2}, u_{2n+3})) < \psi(d(u_{2n+1}, u_{2n+2})). \quad (2.12)$$

Therefore from (2.11) and (2.12), we have
$\psi(d(u_{n+1}, u_{n+2})) < \psi(d(u_n, u_{n+1}))$ for $n = 0,1,2,\ldots$
which implies that

$$d(u_{n+1}, u_{n+2}) \leq d(u_n, u_{n+1}) \text{ for } n = 0,1,2,\ldots, n. \quad (2.13)$$

Hence, the sequence $\{d(u_n, u_{n+1})\}$ is a non-increasing and bounded below. Then it is convergent and there exists a real number $\alpha \geq 0$ such that

$$\alpha = \lim_{n\to\infty} d(u_n, u_{n+1}). \quad (2.14)$$

To prove that $\alpha = 0$, assume $\alpha > 0$. For $n \geq 0$, we consider

$$\begin{aligned} &\frac{1}{2}min\{d(u_{2n}, \Upsilon u_{2n}), d(u_{2n+1}, \Phi u_{2n+1})\} \\ &= \frac{1}{2}min\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+2})\} \\ &\leq d(u_{2n}, u_{2n+1}). \end{aligned}$$

This implies

$$\zeta(d(u_{2n+1}, u_{2n+2}), d(u_{2n}, u_{2n+1})) \geq 0.$$

Hence,

$$\lim_{n\to\infty} sup\zeta(d(u_{2n+1}, u_{2n+2}), d(u_{2n}, u_{2n+1})) \geq 0.$$

Using (2.14), we obtain

$$\lim_{n\to\infty} d(u_{2n+1}, u_{2n+2}) = \alpha > 0. \quad (2.15)$$

Using (ζ_2) with $p_n = d(u_{2n+1}, u_{2n+2})$ and $q_n = d(u_{2n}, u_{2n+1})$, we have

$$\lim_{n\to\infty} sup\zeta(d(u_{2n+1}, u_{2n+2}), d(u_{2n}, u_{2n+1})) < 0,$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = 0. \quad (2.16)$$

We know prove that $\{u_n\}$is a Cauchy sequence. On the contrary suppose that $\{u_n\}$ is not Cauchy. Then there is an $\varepsilon > 0$ and sequences of integers $\{2m_k\}$and $\{2n_k\}$ with $m_k > n_k > k$ such that

$$d(u_{2m_k}, u_{2n_k}) \geq \varepsilon \text{ and } d(u_{2m_k-2}, u_{2n_k}) < \varepsilon. \quad (2.17)$$

Case(i): We will prove $\lim_{k\to\infty} d(u_{2m_k}, u_{2n_k}) = \varepsilon$. From (2.17), we have $\varepsilon \leq d(u_{2m_k}, u_{2n_k})$. Taking limit infimum as $k \to \infty$, we obtain

$$\varepsilon \leq \lim_{k\to\infty} inf d(u_{2m_k}, u_{2n_k}). \quad (2.18)$$

We consider

$$\begin{aligned} d(u_{2m_k}, u_{2n_k}) &\leq d(u_{2m_k}, u_{2m_k-2}) + d(u_{2m_k-2}, u_{2n_k}) \\ &< d(u_{2m_k}, u_{2m_k-2}) + \varepsilon. \end{aligned}$$

Taking limit superior as $k \to \infty$, we obtain

$$\lim_{k\to\infty} sup d(u_{2m_k}, u_{2n_k}) \leq \varepsilon. \quad (2.19)$$

Using (2.18) and (2.19), we obtain

$$\lim_{k\to\infty} d(u_{2m_k}, u_{2n_k}) = \varepsilon.$$

Same the way we prove the following:
Case(ii): $\lim_{k\to\infty} d(u_{2m_k}, u_{2n_k+1}) = \varepsilon$.
Case(iii): $\lim_{k\to\infty} d(u_{2m_k+1}, u_{2n_k}) = \varepsilon$.
Case(iv): $\lim_{k\to\infty} d(u_{2m_k+1}, u_{2n_k+1}) = \varepsilon$.

Case(v): $\lim_{k\to\infty} d(u_{2m_k+1}, u_{2n_k+2}) = \varepsilon$.

Now, from the definition of $D(\eta,\sigma)$, we have

$$\lim_{k\to\infty} D(u_{2m_k}, u_{2n_k-1})$$
$$= \lim_{k\to\infty} max \left\{ \begin{array}{c} d(u_{2m_k}, u_{2n_k-1}), \\ \frac{[1 + d(u_{2m_k}, \Upsilon u_{2m_k})] d(u_{2n_k-1}, \Phi u_{2n_k-1})}{1 + d(u_{2m_k}, u_{2n_k-1})} \end{array} \right\}$$
$$= \lim_{k\to\infty} max \left\{ \begin{array}{c} d(u_{2m_k}, u_{2n_k-1}), \\ \frac{[1 + d(u_{2m_k}, u_{2m_k+1})] d(u_{2n_k-1}, u_{2n_k-1})}{1 + d(u_{2m_k}, u_{2n_k-1})} \end{array} \right\}$$
$$= \varepsilon$$

and

$$\lim_{k\to\infty} K(u_{2m_k}, u_{2n_k-1})$$
$$= \lim_{k\to\infty} min \left\{ \begin{array}{c} d(u_{2m_k}, \Upsilon u_{2m_k}), d(u_{2n_k-1}, \Phi u_{2n_k-1}), \\ d(u_{2m_k}, \Phi u_{2n_k-1}), d(u_{2n_k-1}, \Upsilon u_{2m_k}) \end{array} \right\}$$
$$= \lim_{k\to\infty} min \left\{ \begin{array}{c} d(u_{2m_k}, u_{2m_k+1}), d(u_{2n_k-1}, u_{2n_k}), \\ d(u_{2m_k}, u_{2n_k}), d(u_{2n_k-1}, u_{2m_k+1}) \end{array} \right\}$$
$$= 0.$$

Taking k sufficiently large with $m_k > n_k > k$ and since $\{d(u_n, u_{n+1})\}$is a non-increasing,

$$\begin{aligned} d(u_{2m_k}, \Upsilon u_{2m_k}) &= d(u_{2m_k}, \Upsilon u_{2m_k+1}) \\ &\le d(u_{2n_k}, u_{2n_k+1}) \\ &\le d(u_{2n_k-1}, u_{2n_k}) \\ &\le d(u_{2n_k-1}, \Phi u_{2n_k-1}). \end{aligned}$$

So,

$$\begin{aligned} &\frac{1}{2} min\{d(u_{2m_k}, \Upsilon u_{2m_k}), d(u_{2n_k-1}, \Phi u_{2n_k-1})\} \\ &\quad = \tfrac{1}{2} d(u_{2m_k}, \Upsilon u_{2m_k}) \\ &\quad \le d(u_{2m_k}, u_{2m_k+1}). \end{aligned} \tag{2.20}$$

Using (2.16), there exists $k_1 \in \mathbb{N}$ such that for any$k > k_1$,

$$d(u_{2m_k}, u_{2m_k+1}) < \frac{\varepsilon}{2}.$$

Also, there exists $k_2 \in \mathbb{N}$such that for any $k > k_2$,

$$d(u_{2n_k-1}, u_{2n_k}) < \frac{\varepsilon}{2}.$$

Thus, for any $k > max\{k_1, k_2\}$and $m_k > n_k > k$, we have

$$\begin{aligned} \varepsilon &\le d(u_{2n_k}, u_{2m_k}) \\ &\le d(u_{2n_k}, u_{2n_k-1}) + d(u_{2n_k-1}, u_{2m_k}) \\ &\le \tfrac{\varepsilon}{2} + d(u_{2n_k-1}, u_{2m_k}), \end{aligned}$$

which implies that

$$\frac{\varepsilon}{2} \le d(u_{2n_k-1}, u_{2m_k}).$$

For any $k > max\{k_1, k_2\}$ k and $m_k > n_k > k$,

$$\begin{aligned} d(u_{2m_k}, u_{2m_k+1}) &< \frac{\varepsilon}{2} \\ &\le d(u_{2n_k-1}, u_{2m_k}). \end{aligned}$$

Therefore, from (2.20), we have

$$\begin{aligned} &\frac{1}{2} min\{d(u_{2m_k}, \Upsilon u_{2m_k}), d(u_{2n_k-1}, \Phi u_{2n_k-1})\} \\ &\quad = \frac{1}{2} d(u_{2m_k}, u_{2m_k+1}) \\ &\quad \le d(u_{2n_k-1}, u_{2m_k}). \end{aligned}$$

This implies

$$\zeta \left(\begin{array}{c} d(\Upsilon u_{2m_k}, \Phi u_{2n_k-1}), \\ D(u_{2m_k}, u_{2n_k-1}) + LK(u_{2m_k}, u_{2n_k-1}) \end{array} \right) \ge 0.$$

By (ζ_1) , we have
$\psi(d(\Upsilon u_{2m_k}, \Phi u_{2n_k-1})) < \psi(D(u_{2m_k}, u_{2n_k-1}) + LK(u_{2m_k}, u_{2n_k-1}))$.

Since ψ is nondecreasing, we get
$d(\Upsilon u_{2m_k}, \Phi u_{2n_k-1}) < D(u_{2m_k}, u_{2n_k-1}) + LK(u_{2m_k}, u_{2n_k-1}))$.

Thus,

$$\limsup_{k\to\infty} d(u_{2m_k+1}, u_{2n_k})$$
$$< \limsup_{k\to\infty} D(u_{2m_k}, u_{2n_k-1}) + L\limsup_{k\to\infty} K(u_{2m_k}, u_{2n_k-1})$$

and then $\varepsilon < \varepsilon$, which is a contradiction. Therefore, $\{u_n\}$ is a Cauchy sequence in Λ. Since (Λ, d) is a complete metric space, we have $\{u_n\}$ is convergent to some point u in Λ. Therefore

$u = \lim_{n\to\infty} u_{2n+1} = \lim_{n\to\infty} \Upsilon u_{2n}$ and
$u = \lim_{n\to\infty} u_{2n+2} = \lim_{n\to\infty} \Phi u_{2n+1}$.

Hence,

$$\lim_{n\to\infty} \Upsilon u_{2n} = u = \lim_{n\to\infty} \Phi u_{2n+1}.$$

We assume that Υ is continuous. Since $u_{2n} \to u$ as $n \to \infty$, we have $\Upsilon u_{2n} \to \Upsilon u$ as $n \to \infty$. Hence $0 \le d(u, \Upsilon u) \le (d(u, \Upsilon u_{2n}) + d(\Upsilon u_{2n}, \Upsilon u)) \to 0$ as $n \to \infty$, so that $d(u, \Upsilon u) = 0$. Thus, u is a fixed point of Υ. Now by **Lemma 2.2**, we have u is a unique common fixed point of Υ and Φ. Same the way, we can prove that u is a unique common fixed point of Υ and Φ whenever Φ is continuous.

Example Let $X = [0,2]$ be endowed with the usual metric. Define a mapping $\Upsilon, \Phi: X \to X$ as $\Upsilon, \Phi = 2 - \eta$ for all $\eta \in X$. Then, Υ and Φ are not a Z -contraction with respect to ζ where for all $t, s \in [0, \infty)$

$$\zeta(t,s) = \alpha s - t, \alpha \in [0,1).$$

For all $\eta \ne \sigma$, we get

$$\begin{aligned} &\frac{1}{2} min\{ d(\eta, \Upsilon\eta), d(\sigma, \Phi\sigma)\} \\ &= \frac{1}{2} min\{|\eta - (2 - \eta)|, |\sigma - (2 - \sigma)|\} \\ &= \frac{1}{2} min\{|\eta - 2 + \eta|, |\sigma - 2 + \sigma|\} \\ &= \frac{1}{2} min\{|2\eta - 2|, |2\sigma - 2|\} \\ &\le |\eta - \sigma| \\ &= d(\eta, \sigma). \end{aligned}$$

And

$$\begin{aligned} &\zeta(d(\Upsilon\eta, \Phi\sigma), d(\eta, \sigma)) \\ &= \alpha|\eta - \sigma| - |2 - \eta - (2 - \sigma)| \\ &= \alpha|\eta - \sigma| - |\eta - \sigma| \\ &< |\eta - \sigma| - |\eta - \sigma| \\ &= 0. \end{aligned}$$

Now, we show that Υ and Φ are a modified almost Z -contraction with respect to ζ.

$$\begin{aligned} &\zeta(d(\Upsilon\eta, \Phi\sigma), D(\eta, \sigma) + LK(\eta, \sigma)) \\ &= \alpha[|D(\eta, \sigma) + LK(\eta, \sigma)|] - |2 - \eta - (2 - \sigma)| \\ &= \alpha[|D(\eta, \sigma) + LK(\eta, \sigma)|] - |\eta - \sigma|, \end{aligned}$$

where

$$D(\eta,\sigma) = max\left\{|\eta-\sigma|, \frac{[1+|\eta-(2-\eta)|]|\sigma-(2-\sigma)|}{1+|\eta-\sigma|}\right\} = max\left\{|\eta-\sigma|, \frac{[1+|2\phi-2|]|2\eta-2|}{1+|\eta-\sigma|}\right\}$$

and

$$D(\eta,\sigma) = min\left\{\begin{matrix}|\eta-(2-\eta)|, |\sigma-(2-\sigma)|,\\ |\eta-(2-\sigma)|, |\sigma-(2-\eta)|\end{matrix}\right\} = min\{|2\eta-2|, |2\sigma-2|, |2\eta-2|, |\eta+\sigma-2|\} = min\{|2\eta-2|, |2\sigma-2|, |\eta+\sigma-2|\}.$$

We deduce that

$$\zeta\big(d(\Upsilon\eta,\Phi\sigma), D(\eta,\sigma)+LK(\eta,\sigma)\big) = \alpha\left[max\left\{|\eta-\sigma|, \frac{[1+|2\eta-2|]|2\sigma-2|}{1+|\eta-\sigma|}\right\} + K\,min\left\{\begin{matrix}|2\eta-2|, |2\sigma-2|,\\ |\eta+\sigma-2|\end{matrix}\right\}\right] - |\eta-\sigma|.$$

Thus, we get two cases :

Cases(i): If $\eta=\sigma$, then

$$\zeta(d(\Upsilon\eta,\Phi\sigma), D(\eta,\sigma)+LK(\eta,\sigma)) = \alpha\left[\begin{matrix}[1+|2\eta-2|]\\ |2\eta-2|+L|2\eta-2|\end{matrix}\right] \geq 0.$$

Cases(ii): Suppose that $\eta>\sigma$.

Then

$$\zeta(d(\Upsilon\eta,\Phi\sigma), D(\eta,\sigma)+LK(\eta,\sigma)) = \alpha\frac{[1+|2\eta-2|]|2\sigma-2|}{1+|\eta-\sigma|} + \alpha L|2\sigma-2| - |\eta-\sigma|.$$

We choose $\alpha=\frac{1}{2}$ and $L=6$, then we get

$$\zeta(d(\Upsilon\eta,\Phi\sigma), D(\eta,\sigma)+LK(\eta,\sigma)) = \frac{1}{2}\frac{[1+|2\eta-2|]|2\sigma-2|}{1+|\eta-\sigma|} + 3|2\sigma-2| - |\eta-\sigma|.$$

Thus, all of the conditions of Theorem 2.3 are satisfied. Thus, Υ and Φ have a unique common fixed point $u=1$.

Acknowledgements

The author thanks Rambhai Barni Rajabhat University for the support.

Declaration of conflicting interests

The author declared that they have no conflicts of interest in the research, authorship, and this article's publication.

References

1. Banach S, Sur les operations dans lesensembles abstraits et leurs applications auxequationsint egrales, Fund Math. 1922;3:133–81.
2. Zamfirescu T, Fixpoint theorems in metric spaces, Arch. Math. (Basel) 1972;23:292–8.
3. Berinde V, Approximating fixed points of weak contractions using the Picard iteration, Nonlinear. Anal. Forum. 2004;1(9):43–53.
4. Berinde V, General constructive fixed point theorems for Ciric-type almost contractions in metric spaces, Carpathian J. Math. 2008;24(2): 10–9.
5. Babu GVR, Sandhya ML, Kameswari MVR, A note on a fixed point theorem of Berinde on weak contractions, Carpathian J. Math. 2008; 24: 8–12.
6. Khojasteh F, Shukla S, Radenovic S, A new approach to the study of fixed point theory for simulation functions, Filomat 2015;29(6):1189–94.
7. Heidary JG, Farajzadeh A, Azhini M, Khojasteh FA, New Common Fixed Point Theorem for Suzuki Type Contractions via Generalized - simulation Functions. Sahand Commun. Math. Anal. (SCMA) 2019; 16:129–48.
8. Radenovic S, Chandok S, Simulation type functions and coincidence points, Filomat 2018;32(1):141– 7.
9. Radenovic S, Vetro F, Vujakovic J, An alternative and easy approach to fixed point results via simulation functions, Demonstr. Math. 2017;50(1):223–30.
10. Roldan-Lopez-de-Hierro AF, Karapnar E, Roldan-Lopez-de-Hierro C, Martınez Moreno J, Coincidence point theorems on metric spaces via simulation functions, J. Comput. Appl. Math. 2015;275:345–55.
11. Roldan-Lopez-de-Hierro AF, Samet B, \varphi - admissibility results via extended simulation functions, J. Fixed Point Theory Appl. 2017;19(3):1997–2015.
12. Chanda A, Damjanovic B, Dey LK, Fixed point results on \psi-metric spaces via simulation functions, Filomat 2017;31(11):3365–75.
13. Komal S, Kumam P, Gopal D, Best proximity point for Z-contraction and Suzuki type Z-contraction mappings with an application to fractional calculus, Appl. Gen. Topol. 2016;17(2):185–98.
14. Kumam P, Gopal D, Budhiya L, A new fixed point theorem under Suzuki type Z-contraction mappings, J. Math. Anal. 2017;8(1):113–19.
15. Mongkolkeha C, Cho YJ, Kumam P, Fixed point theorems for simulation functions in b-metric spaces via the wt-distance, Appl. Gen. Topol. 2017;18(1):91–105.
16. Nastasi A, Vetro P, Fixed point results on metric and partial metric spaces via simulation functions, J. Nonlinear Sci. Appl. 2015;8(6):1059–69.
17. Padcharoen A, Kumam P, Saipara P, Chaipunya P, Generalized Suzuki type Z-contraction in complete metric spaces, Kragujevac Journal of Mathematics. 2018;42(3):419–30.
18. Saipara P, Kumam P, Bunpatcharacharoen P, Some Results for Generalized Suzuki Type Z-Contraction in \theta-Metric Spaces, Thai J. Math. 2018:203–19.
19. Chandok S, Chanda A, Dey LK, Pavlovic M, Radenovic S, Simulation functions and Geraghty type results, Bol. Soc. Paran. Mat. 2018.
20. Aleksic S, Chandok S, Radenovic S, Simulation functions and Boyd-Wong type results, Tbilisi Mathematical Journal. 2019;12(1):105–15.

21. Klangpraphan C, Panyanak B, Fixed point theorems for some generalized multivalued nonexpansive mappings in Hadamard spaces, Thai J. Math. 2019;17(2):543–55.
22. Isik H, Gungor NB, Park C, Jang SY, Fixed point theorems for almost Z-contractions with an application, Mathematics. 2018;37(6).
23. Olgun M, Bicer O, Alyildiz T, A new aspect to Picard operators with simulation functions, Turk. J. Math. 2016;40:832–7.
24. Bunpatcharacharoen P, Saelee S, Saipara P, Modified almost type Z-contraction, Thai Journal of Mathematics, 2020;18(1):252–60.