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Viscosity Approximation Methods for Split Equilibrium Problem and Fixed Point Problem for Finite Family of Nonexpansive Mappings in Hilbert Spaces

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Abstract

In this paper, we present a new iterative scheme bases on the hybrid viscosity approximation method and the hybrid steepest-descent method for finding a common element of the set of common fixed points of a finite family of nonexpansive mappings and the split equilibrium problem in Hilbert spaces.

Keywords: Split equilibrium problem, Variational inequality, Fixed point, Nonexpansive mapping

1. Introduction

Throughout the paper unless otherwise stated, let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$. Let C and Q be nonempty closed convex subset of real Hilbert spaces H_1 and H_2. Let $\{x_n\}$ be a sequence in H_1, then $x_n \to x$ (respectively, $x_n \xrightarrow{w} x$) denotes strong (respectively, weak) convergence of $\{x_n\}$ to a point $x \in H_1$.

A mapping $S : C \to C$ is called nonexpansive,

$$\|Sx - Sy\| \le \|x - y\|, \forall x, y \in C. \tag{1.1}$$

The fixed point problem (in short, FPP) for the mapping $S : C \to C$ is to find $x \in C$ such that

$$Sx = x. \tag{1.2}$$

The solution set of FPP (1.2) is denoted by *Fix(S)*.

In 1967, Halpern (1) considered the following explicit iterative process:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Sx_n, \forall n \ge 0, \tag{1.3}$$

where u is a given point and $S : C \to C$ is nonexpansive. He prove that strong convergence of $\{x_n\}$ to a fixed point of S provide that $\alpha_n = n^{-\theta}$ with $\theta \in (0,1)$.

In 2003, Xu (2) introduced the following iterative process:

$$x_{n+1} = \alpha_n u + (1-\alpha_n D)Sx_n, \forall n \ge 0, \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. He prove the above sequence $\{x_n\}$ converges strongly to the unique solution of the minimization problem with $C = Fix(S)$: $\min_{x\in C} \frac{1}{2}\langle Dx, x\rangle - \langle x,u\rangle$, where D is a strongly positive bounded linear operator on H.

In 2006, Marino and Xu (3) considered the following viscosity iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Sx_n, \forall n \ge 0, \tag{1.5}$$

where f is a contraction on H. They proved the above sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (A-\gamma f)x^*, x-x^* \rangle \geq 0, \forall x \in Fix(S). \quad (1.6)$$

In 2001, Yamada et al. (4) considered the following hybrid steepest-descent iterative method:

$$x_{n+1} = Sx_n - \mu\lambda_n F(Sx_n), \quad (1.7)$$

where F is $k-$Lipschitzian continuous and $\eta-$strongly monotone operator with $k>0, \eta>0$ and $0<\mu<\frac{2\eta}{k^2}$. Under some appropriate conditions, the above sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle F(x^*), x-x^* \rangle \geq 0, \forall x \in Fix(S). \quad (1.8)$$

In (5), Tian considered the following general viscosity type iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I-\mu\alpha_n F)Sx_n, \forall n \geq 0. \quad (1.9)$$

Under certain approximate conditions, the above sequence $\{x_n\}$ converges strongly to a fixed point of S, which solve the variational inequality

$$\langle (\gamma f-\mu F)x^*, x-x^* \rangle \leq 0, \forall x \in Fix(S). \quad (1.10)$$

In (6), Zhou and Wang proposed a simple explicit iterative algorithm for finding a solution of variational inequality over the set of common fixed points of a finite family nonexpansive mappings. They introduced an explicit scheme as follows:

Theorem 1.1. *Let H be a real Hilbert space and $F:H\to H$ be an $k-$Lipschitzian continuous and $\eta-$strongly monotone mapping with $k>0$ and $\eta>0$. Let $\{S_i\}_{i=1}^N$ be N nonexpansive of H such that $C=\bigcap_{i=1}^N Fix(S_i)\neq\phi$. For any point $x_0\in H$, define a sequence $\{x_n\}$ as follows:*

$$x_{n+1} = (1-\lambda_n\mu F)S_N^n S_{N-1}^n,\ldots,S_1^N x_n, \forall n \geq 0, \quad (1.11)$$

where $\mu\in\left(0,\frac{2\eta}{k^2}\right)$ and $S_i^n=(1-\sigma_n^i)I+\sigma_n^i S_i$ for $i=1,2,\ldots,N$. When the parameters satisfy appropriate conditions, the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle Ax^*, y-x^* \rangle \geq 0, \forall y \in C. \quad (1.12)$$

where $A:H\to H$ is a nonlinear mapping.

Recently, Zhang and Yang (7) proposed an explicit iterative algorithm based on the viscosity method for finding a solution for a class of variational inequalities over the common fixed points set of a finite family of nonexpansive mappings as follows:

Theorem 1.2. *Let H be a real Hilbert space and $F:H\to H$ be an $k-$Lipschitzian continuous and $\eta-$strongly monotone mapping with $k>0$ and $\eta>0$. Let $\{S_i\}_{i=1}^N$ be N nonexpansive self-mappings of H such that $C=\bigcap_{i=1}^N Fix(S_i)\neq\phi$ and V be an $\rho-$ Lipschitzian on H with $\rho>0$. For any point $x_0\in H$, define a sequence $\{x_n\}$ as follows manner:*

$$x_{n+1} = \alpha_n\gamma V(x_n) + (I-\lambda_n\mu F)_N^n S_{N-1}^n,\ldots,S_1^N x_n, \quad (1.13)$$

where $0<\gamma\rho<\tau$ with $\tau=\mu(2\eta-\mu k^2), 0<\mu<\frac{2\eta}{k^2}$, $S_i^n=(1-\sigma_n^i)I+\sigma_n^i S_i$, for $i=1,2,\ldots,N$ and *$\sigma_n^i\in(\varsigma_1,\varsigma_2)$ for some $\varsigma_1,\varsigma_2\in(0,1)$.*

When the parameters satisfy appropriate conditions, the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (\mu F-\gamma V)x^*, x-x^* \rangle \geq 0, \forall x \in \bigcap_{i=1}^N Fix(S_i). \quad (1.14)$$

Let $F:C\times C\to\mathbb{R}$ be a bifunction. The equilibrium problem for F is to find such that $z\in C$ such that

$$F(z,y)\geq 0, \forall y\in C. \quad (1.15)$$

The set of all solutions of (1.15) is denoted by *EP(F)* i.e.,

$$EP(F)=\{z\in C: F(z,y)\geq 0, \forall y\in C\}. \quad (1.16)$$

Many problems in physics, optimization, and economics can be reduced to find the solution of (1.16); see (9-12).

In 1997, Combettes and Hirstoaga (13) introduced an iterative scheme of finding the solution (1.15) under the assumption that $EP(F)$ is non-empty. Later on, many iterative algorithms are considered to find the element of $Fix(S)\cap EP(F)$; see (14-16).

Recently, some new problems called split variational inequality problems are considered by some authors. Censor et al. (17) initially studied this class of split variational inequality problem. Let H_1 and H_2 be two real Hilbert spaces. Given the operators $f: H_1 \to H_1$ and $g: H_2 \to H_2$, bounded linear operator $A: H_1 \to H_2$, and nonempty closed convex subsets $C \subset H_1$ and $Q \subset H_2$, the split variational inequality problem is formulated as follows:

find a point $x^* \in C$ *such that*

$$\left\langle f(x^*), x - x^* \right\rangle \geq 0, \forall x \in C \quad (1.17)$$

and such that

$$y^* = Ax^* \in Q \text{ solve } \left\langle g(y^*), y - y^* \right\rangle \geq 0, \forall y \in Q. \quad (1.18)$$

After investigating the algorithm of Censor et al. (17), Moudafi (21) introduced a new iterative scheme to solve the following split monotone variational inclusion:

find a point $x^* \in H_1$ *such that*

$$0 \in f\left(x^*\right) + B_1\left(x^*\right) \quad (1.19)$$

and such that

$$y^* = Ax^* \in H_2 \text{ solve } \left\langle g\left(y^*\right) + B_2\left(y^*\right)\right\rangle \quad (1.20)$$

where $B: H_i \to 2^{H_i}$ is a set-valued mappings for $i = 1, 2$.

In 2013, Kazmi and Rizvi (22) considered a new class of split problem called split equilibrium problem. Let $F_1: C \times C \to \mathbb{R}$ and $F_2: Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and $A: H_1 \to H_2$ be a bounded linear operator, then the split equilibrium problem (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \forall x \in C, \quad (1.21)$$

and such that

$$y^* = Ax^* \in Q \text{ solve } F_2(y^*, y) \geq 0, \forall y \in Q. \quad (1.22)$$

When looked separately, (1.21) is the classical equilibrium problem (EP) (1.15), and we denoted its solution set by $EP(F_1)$. The SEP (1.21) and (1.22) constitutes a pair of equilibrium problems which have to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator A, of the solution x^* of the EP (1.21) in H_1 is the solution of another EP (1.22) by $EP(F_2)$.

The solution set SEP (1.21) and (1.22) is denoted by $\overline{\Gamma} = \left\{x^* \in EP(F_1) : Ax^* \in EP(F_2)\right\}$.

In 2012, He (8) proposed the new algorithm for solving split equilibrium problem and investigated the convergence behavior in several ways including both weak and strong convergence. Moreover, they gave some examples and mentioned that there exist many SEPs and the new methods for solving it further need to be explored in the future.

Later, in 2013, Kazmi and Rizvi (22) considered the iterative method to compute the common approximaten solution of a split equilibrium problem, a variational inequality problem and a fixed point problem for a nonexpansive mapping in the framework of real Hilbert spaces. They generated the sequence iteratively as follows:

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n), \\ y_n = P_C(u_n - \lambda_n Du_n), \\ x_{n+1} = \alpha_n v + \beta_n x_n + \gamma S y_n, \end{cases} \quad (1.23)$$

In this paper, motivated by above works, we introduced a new iterative algorithm like viscosity approximation and investigated fixed points of nonexpansive mappings and solutions of split equilibrium problem (1.21) and (1.22) by the following modified iterative scheme;

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n), \\ x_{n+1} = \alpha_n \gamma V(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n \mu D] \times \\ K_N^n K_{N-1}^n, \ldots, K_1^n u_n, \forall n \geq 1, \end{cases} \quad (1.24)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. Strong convergence theorem for common elements are established in Hilbert spaces.

2. Preliminaries

Let H_1 be a real Hilbert space. Then

$$\|x-y\|^2=\|x\|^2-\|y\|^2-2\langle x-y,y\rangle, \tag{2.1}$$

$$\|x-y\|^2\le\|x\|^2+2\langle y,x+y\rangle, \tag{2.2}$$

$$\|\lambda x-(1-\lambda)y\|^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2 -\lambda(1-\lambda)\|x-y\|^2, \tag{2.3}$$

for all $x,y\in H_1$ and $y\in[0,1]$.

We recall some concepts and results which are needed in sequel. A mapping P_C is said to be *metric projection* of H_1 onto C if for every point $x\in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x-P_C x\|\le\|x-y\|,\forall y\in C. \tag{2.4}$$

It is well known that P_C is a nonexpansive mapping and is characterized by the following property:

$$\|P_C x-P_C y\|^2\le\langle x-y,P_C x-P_C y\rangle,\forall x,y\in H_1. \tag{2.5}$$

Moreover, $P_C x$ is characterized by the following properties:

$$\langle x-P_C x,y-P_C y\rangle\le 0, \tag{2.6}$$

$$\|x-y\|^2\ge\|x-P_C x\|^2+\|y-P_C x\|^2, \quad \forall x\in H_1, y\in C, \tag{2.7}$$

$$\|(x-y)-(P_C x-P_C y)\|^2 \ge\|x-y\|^2-\|P_C x-P_C y\|^2,\forall x,y\in H_1, \tag{2.8}$$

It is known that every nonexpansive operator $S:H_1\to H_1$ satisfies, for all $(x,y)\in H_1\times H_1$, the inequality

$$\langle (x-Sx)-(y-Sy),Sy-Sx\rangle \le\frac{1}{2}\|(Sx-x)-(Sy-y)\|^2, \tag{2.9}$$

and therefore, we get, for all $(x,y)\in H_1\times Fix(S)$,

$$\langle x-Sx,y-Sx\rangle\le\frac{1}{2}\|Sx-x\|^2, \tag{2.10}$$

(see, e.g., Theorem 3 in (24) and Theorem 1 in (25)).

Lemma 2.1. (18) *Let $F:C\times C\to\mathbb{R}$ be a bifunction satisfying the following assumptions:*

(i) $F(x,y)\ge 0,\forall x\in C$;

(ii) F is monotone. i.e., $F(x,y)+F(y,x)\le 0,\forall x\in C$;

(iii) F is upper hemicontinuous, i.e., for each $x,y,z\in C$,

$$\limsup_{t\to 0} F(tz+(1-t)x,y)\le F(x,y); \tag{2.11}$$

(iv) For each $x\in C$ fixed, the function $y\to F(x,y)$ is convex and lower semicontinuous;

(v) Fixed $r>0$ and $z\in C$, there exists a nonempty compact convex subset K of H_1 and $x\in C\cap K$ such that

$$F(y,x)+\frac{1}{r}\langle y-x,x-z\rangle\le 0,\forall y\in C\setminus K. \tag{2.12}$$

Lemma 2.2. (13) *Assume that the bifunctions $F:C\times C\to\mathbb{R}$ satisfying Lemma 2.1. For $r>0$ and for all $x\in H_1$, define a mapping $T_r^{F_1}:H_1\to C$ as follows:*

$$T_r^{F_1}(x)=\left\{\begin{array}{l} z\in C:F_1(z,y) \\ +\frac{1}{r}\langle y-z,z-x\rangle\ge 0,\forall y\in C\end{array}\right\}. \tag{2.13}$$

Then, the following hold:

(i) $T_r^{F_1}$ is nonempty and single-valued.

(ii) $T_r^{F_1}$ is firmly nonexpansive, i.e.,

$$\|T_r^{F_1}x-T_r^{F_1}y\|^2 \le\langle T_r^{F_1}x-T_r^{F_1}y,x-y\rangle,\forall x,y\in H_1. \tag{2.14}$$

(iii) $Fix(T_r^{F_1})=EP(F_1)$.

(iv) $EP(F_1)$ is closed and convex.

Lemma 2.3. *(23) Let $F:C\times C\to\mathbb{R}$ be a bifunction satisfying Lemma 2.1 hold and let $T_r^{F_1}$ be defined as in Lemma 2.2 for $r>0$.*

Let $x,y\in H_1$, and $r_1,r_2>0$. Then

$$\|T_{r_2}^{F_1}y-T_{r_1}^{F_1}x\|^2\le\|y-x\|+\left|\frac{r_2-r_1}{r_2}\right|\|T_{r_2}^{F_1}y-y\|. \tag{2.15}$$

Lemma 2.4. *(19)*
(i) The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{K_i\}_{i=1}^{N}$ *is averaged, then so is the composite* $K_1,\ldots,K_N$. *In particular, if* K_1 *is* α_1*- averaged and* K_2 *is* α_2*-averaged, where* $\alpha_1,\alpha_2\in(0,1)$*, then both* K_1K_2 *and* K_2K_1 *are* $\alpha_n=\alpha_1+\alpha_2-\alpha_1\alpha_2$.
(ii) If the mappings $\{K_i\}_{i=1}^{N}$ *are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} Fix(K_i)=Fix(K_1,\ldots,K_N).$$

In particular, if $N=2$, *we have*

$$Fix(K_1)\cap Fix(K_2)=$$
$$Fix(K_1K_2)=Fix(K_2K_1).$$

Lemma 2.5. (20) *Let* λ *be a number in* $(0,1]$ *and let* $\mu>0$. *Let* $F:H\to H$ *be a* k*- Lipschitzian continuous and* η*-strongly monotone mapping with* $k>0$ *and* $\eta>0$. *Associating with a nonexpansive mapping* $K:H\to H$, *define the mapping* $K^{\lambda}:H\to H$ *by*

$$K^{\lambda}x=Kx-\lambda\mu F(Kx),\forall x\in H.$$

Then K^{λ} *is a contraction provide* $\mu<\frac{2\eta}{k^2}$, *that is*

$$\left\|K^{\lambda}x-K^{\lambda}y\right\|\le(1-\lambda\tau)\left\|x-y\right\|,\forall x,y\in H,$$

where $\tau=1-\sqrt{1-\mu(2\eta-\mu k^2)}\in(0,1)$.

Lemma 2.6. (27) *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with*

$$0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1.$$

Suppose $x_{n+1}=(1-\beta_n)z_n+\beta_n x_n$ *for all integers* $n\ge 0$ *and*

$$\limsup_{n\to\infty}\left(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\right)\le 0.$$

Then, $\lim_{n\to\infty}\|z_n-x_n\|=0$.

Lemma 2.7. (26) *Let* E *be an inner product space. Then, for any* $x,y,z\in E$ *and* $\alpha,\beta,\gamma\in[0,1]$ *with* $\alpha+\beta+\gamma=1$, *we have*

$$\|\alpha x+\beta y+\gamma z\|^2=\alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2$$
$$-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2.$$

Lemma 2.8. (20) *Let* H *be a Hilbert space,* C *be a closed convex subset of* H *and* $S:C\to C$ *be a nonexpansive mapping with* $Fix(S)\ne\phi$. *If* $\{x_n\}$ *is a sequence in* C *weakly converging to* $x\in C$ *and* $\{(I-S)x_n\}$ *converges strongly to* $y\in C$, *then* $(I-S)x=y$.

Lemma 2.9. *(20) Let* $\{a_n\}$ *be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1}\le(1-\delta_n)a_n+\delta_n\sigma_n,\forall n\ge 1,$$

where $\{\delta_n\},\{\sigma_n\}$ *are sequences of real numbers such that*
(i) $\{\delta_n\}\subset[0,1]$ *and* $\sum_{n=1}^{\infty}\delta_n=\infty$, *or equivalently,*
$\prod_{n=1}^{\infty}(1-\delta_n)=\lim_{n\to\infty}\prod_{k=1}^{\infty}(1-\delta_k)=0;$
(ii) $\limsup_{n\to\infty}\delta_n\le 0$ *or* $\sum_{n=1}^{\infty}\delta_n\sigma_n$ *is convergent.*
Then, $\lim_{n\to\infty}a_n=0$.

3. Main Results

Theorem 3.1. *Let* H_1 *and* H_2 *be two real Hilbert spaces and let* $C\subset H_1$ *and* $Q\subset H_1$ *be nonempty closed convex subsets. Let* $A:H_1\to H_2$ *be a bounded linear operator. Let* $F_1,h_1:C\times C\to\mathbb{R}$ *and* $F_2,h_2:Q\times Q\to\mathbb{R}$ *are bifunctions satisfying Lemma 2.1 and* F_2 *is upper semicontinuous. Let* $\{K_i\}_{i=1}^{N}$ *be a finite family of nonexpansive mappings on* H_1 *such that* $\bigcap_{i=1}^{N}Fix(K_i)\cap\overline{\Gamma}\ne\phi$.
Let $D:H_1\to H_1$ *be a* k *- Lipschitzian continuous and* η *- strongly monotone mapping with* $k>0$ *and* $\eta>0$, *and* $V:H_1\to H_1$ *be a* ρ*- Lipschitzian continuous mapping with* $\rho>0$. *Let* $0<\mu<\frac{2\eta}{k^2}$ *and* $0<\gamma\rho<\tau$, *where* $\tau=1-\sqrt{1-\mu(2\eta-\mu k^2)}$. *Suppose that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two sequences in* $(0,1)$. *For a given* $x_0\in C$ *arbitrarily, let the iterative sequences* $\{u_n\}$ *and* $\{x_n\}$ *be generated by iterative algorithm:*

$$\begin{cases}u_n=T_{r_n}^{F_1}(x_n+\xi A^*(T_{r_n}^{F_2}-I)Ax_n),\\ x_{n+1}=\alpha_n\gamma V(x_n)+\beta_n x_n\\ +[(1-\beta_n)I-\alpha_n\mu D]K_N^nK_{N-1}^n,\ldots,K_1^nu_n,\forall n\ge 1,\end{cases}\qquad(3.1)$$

where $K_i=(1-\sigma_n^i)I+\sigma_n^iK_i$ *for* $i=1,2,\ldots,N$ *and* $\sigma_n^i\in(\xi_1,\xi_2)$ *for some* $\xi_1,\xi_2\in(0,1),\{r_n\}\subset(0,\infty)$, $\xi\in\left(0,\frac{1}{L}\right)$, L *is the spectral radius of the operator* A^*A *and* A^* *is the adjoint of* A. *Assume that the following conditions are satisfied*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty;$

(C2) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1;$

(C3) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0;$

(C4) $\lim_{n\to\infty} |\sigma_{n+1}^i - \sigma_n| = 0$ *for* $i = 1, 2, \ldots, N.$

Then, the sequence $\{x_n\}$ *generated by (3.1) converges strongly to* $x^* \in \bigcap_{i=1}^{N} Fix(K_i) \cap \overline{\Gamma}$, *where*

$$x^* = P_{\bigcap_{i=1}^{N} Fix(K_i)\cap\overline{\Gamma}}(I - \mu D + \gamma V)x^*$$

is the unique solution of the variational inequality $\langle (\mu D - \gamma V) x^*, x - x^* \rangle \ge 0, \forall x \in \bigcap_{i=1}^{N} Fix(K_i) \cap \overline{\Gamma}.$

Proof We prove Theorem 3.1 for $N = 2$, since our methods easily deduce the general case. Since $D : H_1 \to H_1$ is a k - Lipschitzian continuous and η - strongly monotone mapping, and $V : H_1 \to H_1$ is a ρ - Lipschitzian continuous mapping, we have

$$\begin{aligned}
&\|(I - \mu D)x - (I - \mu D)y\|^2 \\
&= \|x - y\|^2 - 2\mu \langle x - y, Dx - Dy\rangle \\
&+ \mu^2 \|Dx - Dy\|^2 \\
&\le (1 - 2\mu\eta + \mu^2 k^2)\|x - y\|^2 \\
&\le (1 - \tau)^2 \|x - y\|^2,
\end{aligned}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$ and hence

$$\begin{aligned}
&\left\| P_{\bigcap_{i=1}^{N} Fix(K_i)\cap\overline{\Gamma}}(I - \mu D + \gamma V)x - P_{\bigcap_{i=1}^{N} Fix(K_i)\cap\overline{\Gamma}}(I - \mu D + \gamma V)y \right\| \\
&\le \|(I - \mu D + \gamma V)x - (I - \mu D + \gamma V)y\| \\
&\le (1 - \tau)\|x - y\| + \gamma\rho\|x - y\| \\
&= (1 - (\tau - \gamma\rho))\|x - y\|,
\end{aligned}$$

for all $x, y \in H_1$. Therefore, $P_{\bigcap_{i=1}^{N} Fix(K_i)\cap\overline{\Gamma}}(I - \mu D + \gamma V)$ is a contraction of H_1 into itself, which implies that there exists a unique element $x^* \in H_1$ such that

$$x^* = P_{\bigcap_{i=1}^{N} Fix(K_i)\cap\overline{\Gamma}}(I - \mu D + \gamma V)x^*.$$

Step 1. First we will prove that $\{x_n\}$ is bounded. Since $p \in \bigcap_{i=1}^{N} Fix(K_i) \cap \overline{\Gamma}$, i.e., $p \in \cap\overline{\Gamma}$, and we have $p = T_{r_n}^{F_1} p$ and $Ap = T_{r_n}^{F_2} Ap$. We estimate

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}^{F_1}(x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n) - p\|^2 \\
&\le \|x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n - p\|^2 \\
&\le \|x_n - p\|^2 + \xi^2 \|A^*(T_{r_n}^{F_2} - I)Ax_n\|^2 \\
&+ 2\xi \langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle.
\end{aligned} \tag{3.2}$$

Thus, we have

$$\begin{aligned}
\|u_n - p\|^2 &\le \|x_n - p\|^2 \\
&+ \xi^2 \langle (T_{r_n}^{F_2} - I)Ax_n, AA^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\
&+ 2\xi \langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle.
\end{aligned} \tag{3.3}$$

Now, we have

$$\begin{aligned}
&\xi^2 \langle (T_{r_n}^{F_2} - I)Ax_n, AA^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\
&\le L\xi^2 \langle (T_{r_n}^{F_2} - I)Ax_n, (T_{r_n}^{F_2} - I)Ax_n\rangle \\
&= L\xi^2 \|(T_{r_n}^{F_2} - I)Ax_n\|^2.
\end{aligned} \tag{3.4}$$

Denoting $\Lambda := 2\xi \langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle$ and using (2.10), we have

$$\begin{aligned}
\Lambda &= 2\xi \langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\
&= 2\xi \langle A(x_n - p), A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\
&= 2\xi \left\langle \begin{array}{l} A(x_n - p) + (T_{r_n}^{F_2} - I)Ax_n \\ -(T_{r_n}^{F_2} - I)Ax_n, A^*(T_{r_n}^{F_2} - I)Ax_n \end{array}\right\rangle \\
&= 2\xi \left\{ \langle T_{r_n}^{F_2} Ax_n - Ap, (T_{r_n}^{F_2} - I)Ax_n\rangle - \|(T_{r_n}^{F_2} - I)Ax_n\|^2 \right\} \\
&\le 2\xi \left\{ \frac{1}{2}\|(T_{r_n}^{F_2} - I)Ax_n\|^2 - \|(T_{r_n}^{F_2} - I)Ax_n\|^2 \right\} \\
&\le -\xi \|(T_{r_n}^{F_2} - I)Ax_n\|^2.
\end{aligned} \tag{3.5}$$

Using (3.3), (3.4) and (3.5), we obtain

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + \xi(L\xi - 1)\|(T_{r_n}^{F_2} - I)Ax_n\|^2. \tag{3.6}$$

Since $\xi \in \left(0, \frac{1}{L}\right)$, we obtain

$$\|u_n - p\|^2 \le \|x_n - p\|^2. \tag{3.7}$$

Indeed, taking into account the control conditions (C1) and (C2), we may assume, without loss generality, that $\alpha_2 \le 1 - \beta_n$ for all $n \ge 1$.

Now, by Lemma 2.4, we have $p \in Fix\left(K_2K_1\right)$. From (3.1), (3.7) and Lemma 2.5, then

$$\begin{aligned}
&\|x_{n+1}-p\| \\
&=\|\alpha_n\gamma V(x_n)+\beta_n x_n+[(1-\beta_n)I-\alpha_n\mu D]K_2^nK_1^nu_n-K_2^nK_1^np\| \\
&=\|\alpha_n[\gamma V(x_n)-\mu D(p)]+\beta_n(x_n-p) \\
&+[(1-\beta_n)I-\alpha_n\mu D]K_2^nK_1^nu_n-[(1-\beta_n)I-\alpha_n\mu D]K_2^nK_1^np\| \\
&\le \alpha_n\|\gamma V(x_n)-\mu D(p)\|+\beta_n\|x_n-p\| \\
&+(1-\beta_n)\left\|\begin{matrix}\left(I-\frac{\alpha_n}{1-\beta_n}\mu D\right)K_2^nK_1^nu_n \\ -\left(I-\frac{\alpha_n}{1-\beta_n}\mu D\right)K_2^nK_1^np\end{matrix}\right\| \\
&\le (1-\beta_n)\left(1-\frac{\alpha_n\tau}{1-\beta_n}\right)\|u_n-p\|+\beta_n\|x_n-p\| \\
&+\alpha_n\|\gamma V(x_n)-\mu D(p)\| \qquad (3.8)\\
&\le (1-\beta_n-\alpha_n\tau)\|x_n-p\|+\beta_n\|x_n-p\| \\
&+\alpha_n\|\gamma V(x_n)-\mu D(p)\| \\
&\le (1-\alpha_n\tau)\|x_n-p\|+\alpha_n\gamma\|V(x_n)-V(p)\| \\
&+\alpha_n\|\gamma V(p)-\mu D(p)\| \\
&\le (1-\alpha_n\tau)\|x_n-p\|+\alpha_n\gamma\rho\|x_n-p\|+\alpha_n\|\gamma V(p)-\mu D(p)\| \\
&=[1-\alpha_n(\tau-\gamma\rho)]\|x_n-p\|+\alpha_n(\tau-\gamma\rho)\frac{\|\gamma V(p)-\mu D(p)\|}{\tau-\gamma\rho} \\
&\le \max\left\{\|x_n-p\|,\frac{\|\gamma V(p)-\mu D(p)\|}{\tau-\gamma\rho}\right\}.
\end{aligned}$$

It follows from (3.8) and mathematics induction that

$$\|x_{n+1}-p\|\le \max\left\{\|x_1-p\|,\frac{\|\gamma V(p)-\mu D(p)\|}{\tau-\gamma\rho}\right\}, \forall n=1,2,\ldots.$$

Therefore $\{x_n\}$ is bounded. We also obtain that $\{u_n\}$, $\{V(x_n)\}, \{K_1u_n\}$ are all bounded.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$

Indeed, set $x_{n+1}=\beta_nx_n+(1-\beta_n)z_n, \forall n\ge 1.$

Then from the definition of z_n, we obtain

$$z_n=\frac{x_{n+1}-\beta_nx_n}{1-\beta_n}$$

Observe that

$$\begin{aligned}
z_{n+1}-z_n&=\frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_nx_n}{1-\beta_n} \\
&=\frac{\alpha_{n+1}\gamma V(x_{n+1})+[(1-\beta_{n+1})I-\alpha_{n+1}\mu D]K_2^{n+1}K_1^{n+1}u_{n+1}}{1-\beta_{n+1}} \\
&-\frac{\alpha_n\gamma V(x_n)+[(1-\beta_n)I-\alpha_n\mu D]K_2^nK_1^nu_n}{1-\beta_n} \\
&=\frac{\alpha_{n+1}}{1-\beta_{n+1}}[\gamma V(x_{n+1})-\mu DK_2^{n+1}K_1^{n+1}u_{n+1}] \\
&+\frac{\alpha_n}{1-\beta_n}[\mu DK_2^nK_1^nu_n-\gamma V(x_n)] \\
&+K_2^{n+1}K_1^{n+1}u_{n+1}-K_2^{n+1}K_1^{n+1}u_n+K_2^nK_1^nu_n-K_2^{n+1}K_1^{n+1}u_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left[\gamma\|V(x_{n+1})\|+\mu\|DK_2^{n+1}K_1^{n+1}u_{n+1}\|\right] \\
&+\frac{\alpha_n}{1-\beta_n}\left[\mu\|DK_2^nK_1^nu_n\|+\gamma\|V(x_n)\|\right] \qquad (3.9)\\
&+\|K_2^{n+1}K_1^{n+1}u_n-K_2^nK_1^nu_n\|+\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\|.
\end{aligned}$$

Note that

$$\begin{aligned}
&\|K_2^{n+1}K_1^{n+1}u_n-K_2^nK_1^nu_n\| \\
&\le \|K_2^{n+1}K_1^{n+1}u_n-K_2^{n+1}K_1^nu_n\|+\|K_2^{n+1}K_1^nu_n-K_2^nK_1^nu_n\| \qquad (3.10)\\
&\le \|K_1^{n+1}u_n-K_1^nu_n\|+\|K_2^{n+1}K_1^nu_n-K_2^nK_1^nu_n\|
\end{aligned}$$

and

$$\begin{aligned}
&\|K_1^{n+1}u_n-K_1^nu_n\| \\
&=\|(1-\sigma_{n+1}^1)u_n+\sigma_{n+1}^1K_1u_n-(1-\sigma_n^1)u_n-\sigma_n^1K_1u_n\| \\
&=\|(-\sigma_{n+1}^1+\sigma_n^1)u_n+(\sigma_{n+1}^1u_n-\sigma_n^1)K_1u_n\| \\
&\le \left|\sigma_{n+1}^1+\sigma_n^1\right|\left(\|u_n\|+\|K_1u_n\|\right).
\end{aligned}$$

Since $\lim_{n\to\infty}\left|\sigma_{n+1}^i-\sigma_n\right|=0$ for $i=1,2$ and $\{u_n\},\{K_1u_n\}$ are bounded, we see that

$$\lim_{n\to\infty}\|K_1^{n+1}u_n-K_1^nu_n\|=0. \qquad (3.11)$$

Similarly, we get

$$\begin{aligned}
&\|K_2^{n+1}K_1^nu_n-K_2^nK_1^nu_n\| \\
&=\|(-\sigma_{n+1}^2+\sigma_n^2)K_1^nu_n+(\sigma_{n+1}^2u_n-\sigma_n^2)K_2K_1^nu_n\| \\
&\le \left|\sigma_{n+1}^2+\sigma_n^2\right|\left(\|K_1^nu_n\|+\|K_2K_1^nu_n\|\right)
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|K_2^{n+1}K_1^nu_n-K_2^nK_1^nu_n\|=0. \qquad (3.12)$$

Since $u_n=T_{r_n}^{F_1}(x_n+\xi A^*(T_{r_n}^{F_2}-I)Ax_n)$ and $u_{n+1}=T_{r_{n+1}}^{F_1}(x_{n+1}+\xi A^*(T_{r_{n+1}}^{F_2}-I)Ax_{n+1})$, it follows from Lemma 2.3 that

$$
\begin{aligned}
&\|u_{n+1}-u_n\|\\
&\le \left\|x_{n+1}-x_n+\xi[A^*(T_{r_{n+1}}^{F_2}-I)Ax_{n+1}-A^*(T_{r_{n+1}}^{F_2}-I)Ax_n]\right\|\\
&+\left|1-\frac{r_n}{r_{n+1}}\right|\left\|\begin{array}{l}T_{r_{n+1}}^{F_1}(x_{n+1}+\xi A^*(T_{r_{n+1}}^{F_2}-I)Ax_{n+1})\\ -(x_{n+1}+\xi A^*(T_{r_{n+1}}^{F_2}-I)Ax_{n+1})\end{array}\right\|\\
&\le \|x_{n+1}-x_n+\xi A^*A(x_{n+1}-x_n)\|\\
&+\xi\|A\|\left\|T_{r_{n+1}}^{F_2}Ax_{n+1}-T_{r_n}^{F_2}Ax_n\right\|+\varsigma_n \qquad (3.13)\\
&\le \left\{\|x_{n+1}-x_n\|^2-2\xi\|Ax_{n+1}-Ax_n\|^2+\xi^2\|A\|^4\|x_{n+1}-x_n\|^2\right\}^{\frac{1}{2}}\\
&+\xi\|A\|\left\{\|Ax_{n+1}-Ax_n\|+\left|1-\frac{r_n}{r_{n+1}}\right|\left\|T_{r_{n+1}}^{F_1}Ax_{n+1}-Ax_{n+1}\right\|\right\}+\varsigma_n\\
&\le (1-2\xi\|A\|^2+\xi^2\|A\|^4)^{\frac{1}{2}}\|x_{n+1}-x_n\|+\xi\|A\|^2\|x_{n+1}-x_n\|\\
&+\xi\|A\|\sigma_n+\varsigma_n\\
&=(1-\xi\|A\|^2)\|x_{n+1}-x_n\|+\xi\|A\|^2\|x_{n+1}-x_n\|+\xi\|A\|\sigma_n+\varsigma_n\\
&\le \|x_{n+1}-x_n\|+\xi\|A\|\sigma_n+\varsigma_n,
\end{aligned}
$$

where

$$\sigma_n := \left|1-\frac{r_n}{r_{n+1}}\right|\left\|T_{r_{n+1}}^{F_2}Ax_{n+1}-Ax_{n+1}\right\|$$

and

$$\varsigma_n := \left|1-\frac{r_n}{r_{n+1}}\right|\left\|\begin{array}{l}T_{r_{n+1}}^{F_1}(x_{n+1}+\xi A^*(T_{r_{n+1}}^{F_2}-I)Ax_{n+1})\\ -(x_{n+1}+\xi A^*(T_{r_{n+1}}^{F_2}-I)Ax_{n+1})\end{array}\right\|.$$

Using (3.11), (3.12) and (3.13), we get

$$
\begin{aligned}
&\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left[\gamma\|V(x_{n+1})\|+\mu\|DK_2^{n+1}K_1^{n+1}u_{n+1}\|\right]\\
&+\frac{\alpha_n}{1-\beta_n}\left[\mu\|DK_2^nK_1^nu_n\|+\gamma\|V(x_n)\|\right] \qquad (3.14)\\
&+\|K_2^{n+1}K_1^nu_n-K_2^nK_1^nu_n\|+\xi\|A\|\sigma_n+\varsigma_n.
\end{aligned}
$$

From conditions (C1), (C2), (C3) and (3.12), we get

$$
\begin{aligned}
&\limsup_{n\to\infty}\left(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\right)\\
&\le \limsup_{n\to\infty}\left\{\frac{\alpha_{n+1}}{1-\beta_{n+1}}\left[\gamma\|V(x_{n+1})\|+\mu\|DK_2^{n+1}K_1^{n+1}u_{n+1}\|\right]\right.\\
&+\frac{\alpha_n}{1-\beta_n}\left[\mu\|DK_2^nK_1^nu_n\|+\gamma\|V(x_n)\|\right]\\
&\left.+\|K_2^{n+1}K_1^nu_n-K_2^nK_1^nu_n\|+\xi\|A\|\sigma_n+\varsigma_n\right\}=0.
\end{aligned}
$$

By Lemma 2.6, we have

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \qquad (3.15)$$

Consequently, we obtain from

$x_{n+1}=\beta_nx_n+(1-\beta_n)z_n$, we get

$x_{n+1}-x_n=\beta_n(x_n-x_n)+(1-\beta_n)(z_n-x_n)$, then

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0. \qquad (3.16)$$

Step 3. We will show that

$$\lim_{n\to\infty}\|K_2^nK_1^nu_n-u_n\|=0.$$

Since

$x_{n+1}=\alpha_n\gamma V(x_n)+\beta_nx_n+[(1-\beta_n)I-\alpha_n\mu D]K_2^nK_1^nu_n$,

we have

$$
\begin{aligned}
&\|x_n-K_2^nK_1^nu_n\|=\|x_n-x_{n+1}+x_{n+1}-K_2^nK_1^nu_n\|\\
&=\|x_n-x_{n+1}+\alpha_n\gamma V(x_n)+\beta_nx_n\\
&+[(1-\beta_n)I-\alpha_n\mu D]K_2^nK_1^nu_n-K_2^nK_1^nu_n\|\\
&=\|x_n-x_{n+1}+\alpha_n\gamma V(x_n)-\alpha_n\mu DK_2^nK_1^nu_n+\beta_nx_n\\
&+K_2^nK_1^nu_n-\beta_nK_2^nK_1^nu_n-K_2^nK_1^nu_n\|\\
&\le\|x_n-x_{n+1}\|+\alpha_n\|\gamma V(x_n)-\mu DK_2^nK_1^nu_n\|\\
&+\beta_n\|x_n-K_2^nK_1^nu_n\|,
\end{aligned}
$$

that is

$$
\begin{aligned}
\|x_n-K_2^nK_1^nu_n\|&\le\frac{1}{1-\beta_n}\|x_n-x_{n+1}\|\\
&+\frac{\alpha_n}{1-\beta_n}\|\gamma V(x_n)-\mu DK_2^nK_1^nu_n\|.
\end{aligned}
$$

From (3.16) and conditions (C1)-(C2), we get

$$\lim_{n\to\infty}\|x_n-K_2^nK_1^nu_n\|=0. \qquad (3.17)$$

From (2.2), (3.7) and (3.8) we get

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&=\left\|\begin{array}{l}\alpha_n\gamma V(x_n)+\beta_nx_n\\ +[(1-\beta_n)I-\alpha_n\mu D]K_2^nK_1^nu_n-p\end{array}\right\|^2\\
&=\|\alpha_n[\gamma V(x_n)-\mu D(p)]+\beta_n(x_n-K_2^nK_1^nu_n)\\
&+(I-\alpha_n\mu D)K_2^nK_1^nu_n-(I-\alpha_n\mu D)K_2^nK_1^np\|^2\\
&\le\left\|\begin{array}{l}(I-\alpha_n\mu D)K_2^nK_1^nu_n-(I-\alpha_n\mu D)K_2^nK_1^np\\ +\beta_n(x_n-K_2^nK_1^nu_n)\end{array}\right\|^2\\
&+2\alpha_n\langle\gamma V(x_n)-\mu D(p),x_{n+1}-p\rangle\\
&\le\left[\begin{array}{l}\|(I-\alpha_n\mu D)K_2^nK_1^nu_n-(I-\alpha_n\mu D)K_2^nK_1^np\|\\ +\beta_n\|x_n-K_2^nK_1^nu_n\|\end{array}\right]^2\\
&+2\alpha_n\|\gamma V(x_n)-\mu D(p)\|\|x_{n+1}-p\|\\
&\le\left[(1-\alpha_n\tau)]\|u_n-p\|+\beta_n\|x_n-K_2^nK_1^nu_n\|\right]^2\\
&+2\alpha_n\|\gamma V(x_n)-\mu D(p)\|\|x_{n+1}-p\|\\
&\le\left(\|u_n-p\|+\|x_n-K_2^nK_1^nu_n\|\right)^2\\
&+2\alpha_n\|\gamma V(x_n)-\mu D(p)\|\|x_{n+1}-p\|\\
&=\|u_n-p\|^2+\|x_n-K_2^nK_1^nu_n\|^2\\
&+2\|u_n-p\|\|x_n-K_2^nK_1^nu_n\| \qquad (3.18)\\
&+2\alpha_n\|\gamma V(x_n)-\mu D(p)\|\|x_{n+1}-p\|
\end{aligned}
$$

Observe that

$$\|u_n - p\|^2 = \|T_{r_n}^{F_1}(x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n) - T_{r_n}^{F_1}p\|^2$$
$$\leq \langle u_n - p, x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n - p\rangle$$
$$= \frac{1}{2}\{\|u_n - p\|^2 + \|x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n - p\|^2$$
$$-\|(u_n - p) - [x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n - p]\|^2\}$$
$$= \frac{1}{2}\left\{\begin{array}{l}\|u_n - p\|^2 + \|x_n - p\|^2 \\ -\|u_n - x_n - \xi A^*(T_{r_n}^{F_2} - I)Ax_n\|^2\end{array}\right\}$$
$$= \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2$$
$$-[\|u_n - x_n\|^2 + \xi^2\|A^*(T_{r_n}^{F_2} - I)Ax_n\|^2$$
$$-2\xi\langle u_n - x_n, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle]\}.$$

Hence, we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - x_n\|^2$$
$$+2\xi\|A(u_n - x_n)\|\|A^*(T_{r_n}^{F_2} - I)Ax_n\|^2. \quad (3.19)$$

Next, we will show that $\lim_{n\to\infty}\|(T_{r_n}^{F_2} - I)Ax_n\|^2 = 0.$ From (3.6) and (3.18), we obtain

$$\|x_{n+1} - p\|^2$$
$$\leq [(1-\alpha_n\tau)\|u_n - p\| + \beta_n\|x_n - K_2^n K_1^n u_n\|]^2$$
$$+2\alpha_n\|\gamma V(x_n) - \mu D(p)\|\|x_{n+1} - p\|$$
$$\leq [\|u_n - p\| - \alpha_n\tau\|u_n - p\| + (1-\alpha_n)\|x_n - K_2^n K_1^n u_n\|]^2$$
$$+2\alpha_n\|\gamma V(x_n) - \mu D(p)\|\|x_{n+1} - p\|$$
$$\leq (\|u_n - p\| + \|x_n - K_2^n K_1^n u_n\|)^2$$
$$+2\alpha_n\|\gamma V(x_n) - \mu D(p)\|\|x_{n+1} - p\|$$
$$= \|u_n - p\|^2 + 2\|u_n - p\|\|x_n - K_2^n K_1^n u_n\|$$
$$+\|x_n - K_2^n K_1^n u_n\|^2 + 2\alpha_n\|\gamma V(x_n) - \mu D(p)\|\|x_{n+1} - p\|$$
$$\leq \|x_n - p\|^2 + \xi(L\xi - 1)\|(T_{r_n}^{F_2} - I)Ax_n\|^2$$
$$+2\|u_n - p\|\|x_n - K_2^n K_1^n u_n\|$$
$$+\|x_n - K_2^n K_1^n u_n\|^2 + 2\alpha_n\|\gamma V(x_n) - \mu D(p)\|\|x_{n+1} - p\|$$
$$= \|x_n - p\|^2 - \xi(1 - L\xi)\|(T_{r_n}^{F_2} - I)Ax_n\|^2$$
$$+2\|u_n - p\|\|x_n - K_2^n K_1^n u_n\|$$
$$+\|x_n - K_2^n K_1^n u_n\|^2$$
$$+2\alpha_n\|\gamma V(x_n) - \mu D(p)\|\|x_{n+1} - p\|.$$

Then

$$\xi(1 - L\xi)\|(T_{r_n}^{F_2} - I)Ax_n\|^2$$
$$\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\|u_n - p\|\|x_n - K_2^n K_1^n u_n\|$$
$$+\|x_n - K_2^n K_1^n u_n\|^2 + 2\alpha_n\|\gamma V(x_n) - \mu D(p)\|\|x_{n+1} - p\| \quad (3.20)$$
$$\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\|$$
$$+2\|u_n - p\|\|x_n - K_2^n K_1^n u_n\|$$
$$+\|x_n - K_2^n K_1^n u_n\|^2 + 2\alpha_n\|\gamma V(x_n) - \mu D(p)\|\|x_{n+1} - p\|.$$

Since

$$\xi(1 - L\xi) > 0, \alpha_n \to 0, \|x_{n+1} - x_n\| \to 0,$$
$$\|x_n - K_2^n K_1^n u_n\| \to 0$$

as $n \to \infty$, we obtain

$$\lim_{n\to\infty}\|(T_{r_n}^{F_2} - I)Ax_n\| = 0. \quad (3.21)$$

Next, we will show $\lim_{n\to\infty}\|u_n - x_n\| = 0.$ From (3.18) and (3.19), we get

$$\|x_{n+1} - p\|^2$$
$$\leq \|u_n - p\|^2 + 2\|u_n - p\|\|x_n - K_2^n K_1^n u_n\|$$
$$+\|x_n - K_2^n K_1^n u_n\|^2$$
$$+2\alpha_n\|\gamma V(x_n) - \mu D(p)\|\|x_{n+1} - p\|$$
$$\leq \|x_n - p\|^2 - \|u_n - x_n\|^2$$
$$+2\xi\|A(u_n - x_n)\|\|(T_{r_n}^{F_2} - I)Ax_n\|$$
$$+2\|u_n - p\|\|x_n - K_2^n K_1^n u_n\| + \|x_n - K_2^n K_1^n u_n\|^2$$
$$+2\alpha_n\|\gamma V(x_n) - \mu D(p)\|\|x_{n+1} - p\|.$$

It follows that

$$\|u_n - x_n\|^2$$
$$\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2$$
$$+2\xi\|A(u_n - x_n)\|\|(T_{r_n}^{F_2} - I)Ax_n\|$$
$$+2\|u_n - p\|\|x_n - K_2^n K_1^n u_n\| + \|x_n - K_2^n K_1^n u_n\|^2$$
$$+2\alpha_n\|\gamma V(x_n) - \mu D(p)\|\|x_{n+1} - p\| \quad (3.22)$$
$$\leq (\|x_n - p\| - \|x_{n+1} - p\|)\|x_n - x_{n+1}\|$$
$$+2\xi\|A(u_n - x_n)\|\|(T_{r_n}^{F_2} - I)Ax_n\|$$
$$+2\|u_n - p\|\|x_n - K_2^n K_1^n u_n\| + \|x_n - K_2^n K_1^n u_n\|^2$$
$$+2\alpha_n\|\gamma V(x_n) - \mu D(p)\|\|x_{n+1} - p\|.$$

From condition (C1), (3.16), (3.17) and (3.21), we obtain

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \quad (3.23)$$

Since

$$\|K_2^n K_1^n u_n - u_n\| \le \|K_2^n K_1^n u_n - x_n\| + \|x_n - u_n\|. \quad (3.24)$$

From (3.17) and (3.23), we also have

$$\lim_{n\to\infty}\|K_2^n K_1^n u_n - u_n\| = 0. \quad (3.25)$$

Step 4. We will prove that $w \in \bigcap_{i=1}^{N} Fix(K_i) \cap \overline{\Gamma}.$

Step 4.1 We shall show that $w \in \bigcap_{i=1}^{N} Fix(K_i).$

Since $\{\sigma_k^i\}$ is bounded for $i = 1, 2,$ we can assume that $\sigma_{k_j}^i \to \sigma_\infty^i$ as $j \to \infty,$ where $0 < \xi_1 \le \sigma_\infty^i \le \xi_2 < 1$ for $i = 1, 2.$ Define

$$K_i^\infty = (1-\sigma_\infty^i)I + \sigma_\infty^i K_i, i = 1, 2.$$

Note that

$$\begin{aligned}&\|K_i^{k_j}x - K_i^\infty x\|\\ &= \|(1-\sigma_{k_j}^i)x + \sigma_{k_j}^i K_i x - (1-\sigma_\infty^i)x - \sigma_\infty^i K_i x\|\\ &\le |\sigma_{k_j}^i - \sigma_\infty^i|(\|x\| + \|K_i x\|).\end{aligned}$$

Hence, we deduce that

$$\limsup_{j\to\infty x\in B}\|K_i^{k_j}x - K_i^\infty x\| = 0, \quad (3.26)$$

where B is an arbitrary bounded subset of H.
Since

$$\begin{aligned}&Fix(K_1^\infty) \cap Fix(K_2^\infty)\\ &= Fix(K_1) \cap Fix(K_2) \ne \phi\end{aligned}$$

and

K_i^∞ is σ_∞^i - averaged for $i = 1, 2,$ by Lemma 2.4, we know that

$Fix(K_2^\infty K_1^\infty) = Fix(K_2^\infty) \cap Fix(K_1^\infty).$ Since

$$\begin{aligned}&\|u_{n_j} - K_2^\infty K_1^\infty u_{n_j}\|\\ &\le \|u_{n_j} - K_2^{n_j} K_1^{n_j} u_{n_j}\|\\ &+ \|K_2^{n_j} K_1^{n_j} u_{n_j} - K_2^\infty K_1^{n_j} u_{n_j}\|\\ &+ \|K_2^\infty K_1^{n_j} u_{n_j} - K_2^\infty K_1^\infty u_{n_j}\|\\ &\le \|u_{n_j} - K_2^{n_j} K_1^{n_j} u_{n_j}\| + \sup_{x\in B'}\|K_2^{n_j} u_n - K_2^\infty u_n\|\\ &+ \sup_{x\in B''}\|K_1^n u - K_1^\infty u\|,\end{aligned} \quad (3.27)$$

where B' is a bounded subset including $\{K_1^{n_j} u_{n_j}\}$ and B'' is a bounded subset including $\{u_{n_j}\}$, we have from (3.25) and (3.26) that

$$\limsup_{j\to\infty x\in B}\|u_{n_j} - K_2^\infty K_1^\infty u_{n_j}\| = 0.$$

From Lemma 2.8, we have $w \in Fix(K_2^\infty K_1^\infty).$

Step 4.2 We shall show that $w \in \overline{\Gamma}.$
First, we will show $w \in EP(F_1).$
Since $u_n = T_{r_n}^{F_1} x_n$, we have

$$F_1\langle u_n, y\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \forall y \in C.$$

It follows from the monotonicity of F_1 that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F_1(y, u_n),$$

and hence replacing n by n_i, we get

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \ge F_1(y, u_{n_i}).$$

Since $\|u_n - x_n\| \to 0, \|K_2^n K_1^n u_n - u_n\| \to 0$ and $x_n \xrightarrow{w} w,$ as $n \to \infty,$ we get $u_{n_i} \xrightarrow{w} w$ and $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0.$ It follows by Lemma 2.1 (iv) that $0 \ge F_1(y, w), \forall w \in C.$ For t with $0 < t \le 1$ and $y \in C,$ let $y_t = ty + (1-t)w.$ Since $y \in C, w \in C,$ we have $y_t \in C,$ and hence, $F_1(y_t, w) \le 0.$ So, from Lemma 2.1 (i) and (iv), we have

$$\begin{aligned}0 &= F_1(y_t, y_t)\\ &\le tF_1(y_t, y) + (1-t)F_1(y_t, w) \le t(y_t, y_t).\end{aligned}$$

Therefore, $0 \le F_1(y_t, y).$ From Lemma 2.1 (iii), we have $0 \le F_1(w, y).$ This implies that $w \in EP(F_1).$

Next, we show that $Aw \in EP(F_2)$. Since $\|u_n - x_n\| \to 0, u_n \xrightarrow{w} w$ as $n \to \infty$, and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{w} w$, and since A is bounded linear operator, so $Ax_{n_k} \xrightarrow{w} Aw$.

Now, setting $m_{n_k} = Ax_{n_k} - T_{r_{n_k}}^{F_2} Ax_{n_k}$.

It follows from (3.21) that $\lim_{k\to\infty} m_{n_k} = 0$ and $Ax_{n_k} - m_{n_k} = T_{r_{n_k}}^{F_2} Ax_{n_k}$. Therefore, from Lemma 2.2, we have

$$F_2 \langle Ax_{n_k} - m_{n_k}, z\rangle + \frac{1}{r_{n_k}} \langle z - (Ax_{n_k} - m_{n_k}), (Ax_{n_k} - m_{n_k}) - Ax_{n_k}\rangle \geq 0, \forall z \in Q.$$

Since F_2 is upper semicontinuous, taking lim sup to above inequality as $k \to \infty$ and using condition (C3), we obtain

$$F_2 \langle Aw, z\rangle \geq 0, \forall z \in Q.$$

which means that $Aw \in EP(F_2)$ and hence $w \in \overline{\Gamma}$.

Therefore, we get $w \in \bigcap_{i=1}^{N} Fix(K_i) \cap \overline{\Gamma}$.

Step 5. Finally, we prove that the sequence $\{x_n\}$ converge strongly to

$$x^* = P_{\bigcap_{i=1}^{N} Fix(K_i) \cap \overline{\Gamma}} (I - \mu D + \gamma V) x^*$$

which is the unique solution of the variational inequality

$$\langle (\mu D - \gamma V) x^*, x - x^*\rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^{N} Fix(K_i) \cap \overline{\Gamma}. \tag{3.28}$$

Next, we claim that

$$\limsup_{n\to\infty} \langle (\mu D - \gamma V) x^*, x - x_n\rangle \leq 0,$$

where $x^* = P_{\bigcap_{i=1}^{N} Fix(K_i) \cap \overline{\Gamma}} (I - \mu D + \gamma V) x^*$.

Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ which converges weakly to *w*. From (3.25), we obtain

$$K_2^n K_1^n u_{n_j} \xrightarrow{w} w \text{ as } n \to \infty.$$

Since $x^* = P_{\bigcap_{i=1}^{N} Fix(K_i) \cap \overline{\Gamma}} (I - \mu D + \gamma V) x^*$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \langle (\mu D - \gamma V) x^*, x^* - x_n\rangle \\
&= \limsup_{n\to\infty} \langle (\mu D - \gamma V) x^*, x^* - u_n\rangle \\
&= \limsup_{j\to\infty} \langle (\mu D - \gamma V) x^*, x^* - u_{n_j}\rangle \\
&= \langle (\mu D - \gamma V) x^*, x^* - w\rangle \leq 0.
\end{aligned}$$

From (2.2), (3.1), (3.7) and Lemma 2.5, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n \gamma V(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n \mu D] K_2^n K_1^n u_n - x^*\|^2 \\
&= \|\alpha_n [\gamma V(x_n) - \mu D(x^*)] + \beta_n (x_n - x^*) \\
&\quad + [(1-\beta_n)I - \alpha_n \mu D] K_2^n K_1^n u_n \\
&\quad - [(1-\beta_n)I - \alpha_n \mu D] K_2^n K_1^n x^* \|^2 \\
&\leq \left\| \begin{matrix} \beta_n (x_n - x^*) + [(1-\beta_n)I - \alpha_n \mu D] K_2^n K_1^n u_n \\ -[(1-\beta_n)I - \alpha_n \mu D] K_2^n K_1^n x^* \end{matrix} \right\|^2 \\
&\quad + 2\alpha_n \langle \gamma V(x_n) - \mu D(x^*), x_{n+1} - x^*\rangle \\
&\leq \left\{ \beta_n \|x_n - x^*\| + \left\| \begin{matrix} [(1-\beta_n)I - \alpha_n \mu D] K_2^n K_1^n u_n \\ -[(1-\beta_n)I - \alpha_n \mu D] K_2^n K_1^n x^* \end{matrix} \right\| \right\}^2 \\
&\quad + 2\alpha_n \langle \gamma V(x_n) - \mu D(x^*), x_{n+1} - x^*\rangle \\
&= \left\{ \beta_n \|x_n - x^*\| + (1-\beta_n) \left\| \left(I - \frac{\alpha_n}{1-\beta_n} \mu D \right) K_2^n K_1^n u_n \right. \right. \\
&\quad \left. \left. - \left(I - \frac{\alpha_n}{1-\beta_n} \mu D \right) K_2^n K_1^n x^* \right\| \right\}^2 \\
&\quad + 2\alpha_n \langle \gamma V(x_n) - V(x^*), x_{n+1} - x^*\rangle \\
&\quad + 2\alpha_n \langle \gamma V(x^*) - \mu D(x^*), x_{n+1} - x^*\rangle \\
&\leq \{ \beta_n \|x_n - x^*\| + (1 - \beta_n - \alpha_n \tau) \|u_n - x^*\| \}^2 \\
&\quad + 2\alpha_n \gamma \rho \|x_n - x^*\| \|x_{n+1} - x^*\| \\
&\quad + 2\alpha_n \langle \gamma V(x^*) - \mu D(x^*), x_{n+1} - x^*\rangle \\
&\leq \left\{ \beta_n \|x_n - x^*\| + (1-\beta_n) \left(1 - \frac{\alpha_n \tau}{1-\beta_n} \right) \|u_n - x^*\| \right\}^2 \\
&\quad + \alpha_n \gamma \rho \left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \right) \\
&\quad + 2\alpha_n \langle \gamma V(x^*) - \mu D(x^*), x_{n+1} - x^*\rangle \\
&= (1 - \alpha_n \tau)^2 \|x_n - x^*\|^2 + \alpha_n \gamma \rho \|x_n - x^*\|^2 \\
&\quad + \alpha_n \gamma \rho \|x_{n+1} - x^*\|^2 + 2\alpha_n \langle \gamma V(x^*) - \mu D(x^*), x_{n+1} - x^*\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\|x_{n+1}-x_n\|^2\\
&\le \frac{(1-\alpha_n\tau)^2+\alpha_n\gamma\rho}{1-\alpha_n\gamma\rho}\|x_n-x^*\|^2\\
&+\frac{2\alpha_n}{1-\alpha_n\gamma\rho}\langle \gamma V(x^*)-\mu D(x^*), x_{n+1}-x^*\rangle\\
&=\left(1-\frac{2(\tau-\gamma\rho)\alpha_n}{1-\alpha_n\gamma\rho}\right)\|x_n-x^*\|^2+\frac{\alpha_n{}^2\tau^2}{1-\alpha_n\gamma\rho}\|x_n-x^*\|^2\\
&+\frac{2\alpha_n}{1-\alpha_n\gamma\rho}\langle \gamma V(x^*)-\mu D(x^*), x_{n+1}-x^*\rangle \qquad (3.29)\\
&\le\left(1-\frac{2(\tau-\gamma\rho)\alpha_n}{1-\alpha_n\gamma\rho}\right)\|x_n-x^*\|^2\\
&+\frac{2(\tau-\gamma\rho)\alpha_n}{1-\alpha_n\gamma\rho}\left\{\begin{array}{l}\dfrac{\alpha_n\tau^2M_1}{2(\tau-\gamma\rho)}\\ +\dfrac{1}{\tau-\gamma\rho}\langle \gamma V(x^*)-\mu D(x^*), x_{n+1}-x^*\rangle\end{array}\right\}\\
&=(1-\delta_n)\|x_n-x^*\|^2+\delta_n\sigma_n,
\end{aligned}$$

where

$$M_1=\sup\left\{\|x_n-x^*\|^2 : n\ge 1\right\}, \delta_n=\frac{2(\tau-\gamma\rho)\alpha_n}{1-\alpha_n\gamma\rho}$$

and

$$\sigma_n=\frac{\alpha_n\tau^2M_1}{2(\tau-\gamma\rho)}+\frac{1}{\tau-\gamma\rho}\langle \gamma V(x^*)-\mu D(x^*), x_{n+1}-x^*\rangle.$$

It can see that $\delta_n\to 0, \sum_{n=1}^{\infty}\delta_n=\infty$ and $\limsup_{n\to\infty}\sigma_n\le 0.$ Hence, by Lemma 2.9, the sequence $\{x_n\}$ converges strongly to x^*.
This is complete the proof.

Corollary 3.1. *Let H_1 and H_2 be two real Hilbert spaces and let $C\subset H_1$ and $Q\subset H_1$ be nonempty closed convex subsets. Let $A: H_1\to H_2$ be a bounded linear operator. Let $F_1, h_1 : C\times C\to\mathbb{R}$ and $F_2, h_2 : Q\times Q\to\mathbb{R}$ are bifunctions satisfying Lemma 2.1 and h_1, h_2 are monotone and F_2 is upper semicontinuous. Let $D: H_1\to H_1$ be a k – Lipschitzian continuous and η - strongly monotone mapping with $k>0$ and $\eta>0$, $V: H_1\to H_1$ be a ρ - Lipschitzian continuous mapping with $\rho>0$. Let $0<\mu<\frac{2\eta}{k^2}$ and $0<\gamma\rho<\tau$, where $\tau=1-\sqrt{1-\mu(2\eta-\mu k^2)}$. For a given $x_0\in C$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by iterative algorithm:*

$$\begin{cases}u_n=T_{r_n}^{F_1}(x_n+\xi A^*(T_{r_n}^{F_2}-I)Ax_n),\\ x_{n+1}=\alpha_n\gamma V(x_n)+\beta_n x_n\\ +[(1-\beta_n)I-\alpha_n\mu D]u_n, \forall n\ge 1,\end{cases} \qquad (3.30)$$

where $\{r_n\}\subset(0,\infty)$, $\xi\in\left(0,\frac{1}{L}\right)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Assume that the following conditions are satisfied:

(C1) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty;$

(C2) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1;$

(C3) $\liminf_{n\to\infty}r_n>0$ *and* $\lim_{n\to\infty}|r_{n+1}-r_n|=0;$

Then, the sequence $\{x_n\}$ generated by (3.30) converges strongly to $x^*\in\cap\overline{\Gamma}$, which solves the variational inequality

$$\langle(\mu D-\gamma V)x^*, x-x^*\rangle\ge 0, \forall x\in\overline{\Gamma}.$$

Proof Put $K_ix=x$ for all $i=1,2,\dots,N$ and $x\in H_1$, and take the finite family of sequences $\{\sigma_n^i\}_{i=1}^N$ in (ξ_1,ξ_2) for some $\xi_1,\xi_2\in(0,1)$ such that $\lim_{n\to\infty}|\sigma_{n+1}^i-\sigma_n^i|=0$ for all $i=1,2,\dots,N$. In this case, $K_N^nK_{N-1}^n,\dots,K_1^n$ is the identity mapping I of H_1. It can see that the all conditions of Theorem 3.1 are satisfied. Then conclusion of Corollary 3.1 is obtained.

4. Conclusions

In this paper, we first propose a modified iterative scheme (3.1) in Theorem 3.1 and then we prove some strong convergence of the sequence $\{x_n\}$ generated by (3.1) to a common solution of the set of fixed points of a finite family of nonexpansive mappings and the split equilibrium problem in Hilbert spaces. We divide the proof into 5 steps and our theorem is extend and improve the corresponding results of Kazmi and Rizvi (22).

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