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Common Factors of Pell and Pell-Lucas numbers

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Abstract

In this paper, we present identities involving common factors of Pell and Pell-Lucas numbers and related identities consisting even and odd terms. We also present some generalized identities on the products of Pell and Pell-Lucas numbers. Binet's formula will employ to obtain the identities.

Keywords: Pell numbers, Pell-Lucas numbers, Binet's formula.

1. Introduction

The Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence and jacobsthal-Lucas sequence are most prominent examples of recursive sequences.

Many research papers are dedicated to Pell, Pell-Lucas and Modified Pell sequences, denoted by $P_n, Q_n and q_n$ where n a non negative integer, respectively. These sequences are particular cases of the sequences $W_n(a, b; p, q)$ defined by the general recurrence relation $W_n = pW_{n-1} - qW_{n-2}; n \geq 2$ with $W_0 = a, W_1 = b$ and a, b, p, q integers with $p > 0, q \neq 0$ by Horadam (5, 6).

In fact, in the Horadam notation we have $P_n = W_n(0,1; 2, -1); Q_n = W_n(2,2; 2, -1)$ and $q_n = W_n(1,1; 2, -1)$.

The Pell sequence$\{P_n\}$is defined by

$$P_{n+2} = 2P_{n+1} + P_n; n \geq 2 \quad (1.1)$$

with $P_0 = 0$ & $P_1 = 1$.

The Pell-Lucas sequence$\{Q_n\}$is defined by

$$Q_{n+2} = 2Q_{n+1} + Q_n; n \geq 2 \quad (1.2)$$

with $Q_0 = Q_1 = 2$.

The corresponding characteristic equation of (1.1) and (1.2) is

$$x^2 - 2x - 1 = 0 \quad (1.3)$$

and its roots are $\Re_1 = 1 + \sqrt{2}$ and $\Re_2 = 1 - \sqrt{2}$ and verify $\Re_1 + \Re_2 = 2; \Re_1 - \Re_2 = 2\sqrt{2}; \Re_1\Re_2 = -1$.

Their Binet's formulas are well known and given by

$$P_n = \frac{\Re_1^n - \Re_2^n}{\Re_1 - \Re_2} \quad (1.4)$$

$$Q_n = \Re_1^n + \Re_2^n \quad (1.5)$$

$$P_{-n} = (-1)^{1-n}\left(\frac{\Re_1^n - \Re_2^n}{\Re_1 - \Re_2}\right) \quad (1.6)$$

$$Q_{-n} = (-1)^{-n}(\Re_1^n + \Re_2^n) \quad (1.7)$$

2. Preliminaries

Thongmoon (13, 14), defined various identities of Fibonacci and Lucas numbers. Singh, Bhadouria and Sikhwal (11), present some generalized identities involving common factors of Fibonacci and Lucas numbers. Gupta and Panwar (3), present identities involving common factors of generalized Fibonacci, Jacobsthal and jacobsthal-Lucas numbers. Panwar, Singh and Gupta (9, 10), present Generalized Identities Involving Common factors of generalized Fibonacci, Jacobsthal and jacobsthal-Lucas numbers. Singh, Sisodiya and Ahmed (12), investigate some products of k-Fibonacci and k-Lucas numbers, also present some generalized identities on the products of k-Fibonacci and k-Lucas numbers to establish connection formulas between them with the help of Binet's formula. In this paper, we present identities involving common factors of Pell and Pell-Lucas numbers.

3. Identities of the Pell and Pell-Lucas numbers

In this section, we present identities involving common factors of Pell and Pell-Lucas numbers. We shall use Binet's formula for derivation.

Theorem 3.1: If P_n is the Pell numbers and Q_n is Pell-Lucas numbers, then

$$P_{2n-1}Q_{2n+1} = P_{4n} + 2, where\ n \geq 1 \qquad (3.1)$$

Proof:

$$P_{2n-1}Q_{2n+1} = \left(\frac{\Re_1^{2n-1} - \Re_2^{2n-1}}{\Re_1 - \Re_2}\right)\left(\Re_1^{2n+1} + \Re_2^{2n+1}\right)$$
$$= \frac{1}{\Re_1 - \Re_2}\left\{\left(\Re_1^{4n} - \Re_2^{4n}\right) + (\Re_1\Re_2)^{2n}\left(\frac{\Re_2}{\Re_1} - \frac{\Re_1}{\Re_2}\right)\right\}$$
$$= \left(\frac{\Re_1^{4n} - \Re_2^{4n}}{\Re_1 - \Re_2}\right) - (\Re_1\Re_2)^{2n-1}(\Re_1 + \Re_2)$$
$$= P_{4n} + 2$$

this completes the proof.

Theorem 3.2:

$$P_{2n+1}Q_{2n} = P_{4n+1} + 1, where\ n \geq 0 \qquad (3.2)$$

Proof:

$$P_{2n+1}Q_{2n} = \left(\frac{\Re_1^{2n+1} - \Re_2^{2n+1}}{\Re_1 - \Re_2}\right)\left(\Re_1^{2n} + \Re_2^{2n}\right)$$
$$= \frac{1}{\Re_1 - \Re_2}\left\{\left(\Re_1^{4n+1} - \Re_2^{4n+1}\right) + (\Re_1\Re_2)^{2n}(\Re_1 - \Re_2)\right\}$$
$$= \left(\frac{\Re_1^{4n+1} - \Re_2^{4n+1}}{\Re_1 - \Re_2}\right) + (\Re_1\Re_2)^{2n}$$
$$= P_{4n+1} + 1$$

this completes the proof.

Theorem 3.3:

$$P_{2n+2}Q_{2n} = P_{4n+2} + 2, where\ n \geq 0 \qquad (3.3)$$

Proof:

$$P_{2n+2}Q_{2n} = \left(\frac{\Re_1^{2n+2} - \Re_2^{2n+2}}{\Re_1 - \Re_2}\right)\left(\Re_1^{2n} + \Re_2^{2n}\right)$$
$$= \frac{1}{\Re_1 - \Re_2}\{(\Re_1^{4n+2} - \Re_2^{4n+2}) + (\Re_1\Re_2)^{2n}(\Re_1^2 - \Re_2^2)\}$$
$$= \left(\frac{\Re_1^{4n+2} - \Re_2^{4n+2}}{\Re_1 - \Re_2}\right) + (\Re_1\Re_2)^{2n}(\Re_1 + \Re_2)$$
$$= P_{4n+2} + 2$$

this completes the proof.

*Following theorems can be solved with the help of **Binet's** formula.*

Theorem 3.4:

$$P_{2n+1}Q_{2n+2} = P_{4n+3} + 1, where\ n \geq 0 \qquad (3.4)$$

Theorem 3.5:

$$P_{2n}Q_{2n+1} = P_{4n+1} - 1, where\ n \geq 0 \qquad (3.5)$$

Theorem 3.6:

$$P_nQ_nQ_{2n+1} = P_{4n+1} - 1, where\ n \geq 0 \qquad (3.6)$$

Theorem 3.7:

$$P_{2n+2}Q_{2n+1} = P_{4n+3} - 1, where\ n \geq 0 \qquad (3.7)$$

Theorem 3.8:

$$P_{n+1}Q_{n+1}Q_{2n+1} = P_{4n+3} - 1, where\ n \geq 0 \qquad (3.8)$$

Theorem 3.9:

$$8P_{2n+2}P_{2n+1} = Q_{4n+3} + 2, where\ n \geq 0 \qquad (3.9)$$

4. Generalized Identities on the Products of Pell and Pell-Lucas Numbers

In this section we present generalized identities involving common factors of Pell and Pell-Lucas numbers. We shall use Binet's formula for derivation.

Theorem 4.1: If P_n & Q_n are the Pell and Pell-Lucas numbers, then

$$P_{2n+m}Q_{2n+1} = P_{4n+m+1} - P_{m-1} \qquad (4.1)$$

$where\ n \geq 0, m \geq 1$.

Proof:

$$P_{2n+m}Q_{2n+1} = \left(\frac{\Re_1^{2n+m} - \Re_2^{2n+m}}{\Re_1 - \Re_2}\right)\left(\Re_1^{2n+1} + \Re_2^{2n+1}\right)$$
$$= \frac{1}{\Re_1 - \Re_2}\left\{\left(\Re_1^{4n+m+1} - \Re_2^{4n+m+1}\right) + \left(\Re_1\Re_2\right)^{2n}\left(\Re_1^m\Re_2 - \Re_2^m\Re_1\right)\right\}$$
$$= \left(\frac{\Re_1^{4n+m+1} - \Re_2^{4n+m+1}}{\Re_1 - \Re_2}\right) - (\Re_1\Re_2)^{2n}\left(\frac{\Re_1^{m-1} - \Re_2^{m-1}}{\Re_1 - \Re_2}\right)$$
$$= P_{4n+m+1} - P_{m-1}$$

this completes the proof.

Corollary 4.2: For different values of m, (4.1) can be expressed for even and odd numbers:

$(i)\ If\ m = 1\ then\ P_{2n+1}Q_{2n+1} = P_{4n+2}$

$(ii)\ If\ m = 2\ then\ P_{2n+2}Q_{2n+1} = P_{4n+3} - 1$

$(iii)\ If\ m = 3\ then\ P_{2n+3}Q_{2n+1} = P_{4n+4} - 2$

Following theorems can be solved with the help of Binet's formula.

Theorem 4.3: For $n \geq 0, m \geq 2$,

$$P_{2n+m}Q_{2n+2} = P_{4n+m+2} + P_{m-2} \qquad (4.2)$$

Corollary 4.4: For different values of m, (4.2) can be expressed for even and odd numbers:

$(i)\ If\ m = 2\ then\ P_{2n+2}Q_{2n+2} = P_{4n+4}$

$(ii)\ If\ m = 3\ then\ P_{2n+3}Q_{2n+2} = P_{4n+5} + 1$

$(iii)\ If\ m = 4\ then\ P_{2n+4}Q_{2n+2} = P_{4n+6} + 2$

Theorem 4.5: For $n \geq 0, m \geq 0$,

$$P_{2n+m}Q_{2n} = P_{4n+m} + P_m \qquad (4.3)$$

Corollary 4.6: For different values of m, (4.3) can be expressed for even and odd numbers:

$(i)\ If\ m = 0\ then\ P_{2n}Q_{2n} = P_{4n}$

$(ii)\ If\ m = 1\ then\ P_{2n+1}Q_{2n} = P_{4n+1} + 1$

$(iii)\ If\ m = 2\ then\ P_{2n+2}Q_{2n} = P_{4n+2} + 2$

Theorem 4.7: For $n \geq 0, m \geq 1$,
$$P_{2n-m}Q_{2n-1} = P_{4n-m-1} - P_{1-m} \quad (4.4)$$

Corollary 4.8: For different values of m, (4.4) can be expressed for even and odd numbers:
$(i)\ If\ m = 1\ then\ P_{2n-1}Q_{2n-1} = P_{4n-2}$
$(ii)\ If\ m = 2\ then\ P_{2n-2}Q_{2n-1} = P_{4n-3} - 1$
$(iii)\ If\ m = 3\ then\ P_{2n-3}Q_{2n-1} = P_{4n-4} + 2$

Theorem 4.9: For $n \geq 0, m \geq 0$,
$$P_{2n-m}Q_{2n} = P_{4n-m} + P_{-m} \quad (4.5)$$

Corollary 4.10: For different values of m, (4.5) can be expressed for even and odd numbers:
$(i)\ If\ m = 0\ then\ P_{2n}Q_{2n} = P_{4n}$
$(ii)\ If\ m = 1\ then\ P_{2n-1}Q_{2n} = P_{4n-1} + 1$
$(iii)\ If\ m = 2\ then\ P_{2n-2}Q_{2n} = P_{4n-2} - 2$

Theorem 4.11: For $n \geq 0, m \geq 0$,
$$P_nQ_{2n+m} = P_{3n+m} - (-1)^nP_{n+m} \quad (4.6)$$

Theorem 4.12: For $n \geq 0, m \geq 0$,
$$P_{2n+m}Q_n = P_{3n+m} + (-1)^nP_{n+m} \quad (4.7)$$

Theorem 4.13: For $n \geq 0, m \geq 0$,
$$P_{2n}Q_{2n+m} = P_{4n+m} - P_m \quad (4.8)$$

4. Conclusions

In this paper, we present many identities of common factors of Pell and Pell-Lucas numbers with the help of their Binet’s formula. The concept can be executed for generalized second order sequences as well as polynomials.

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