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Iteration Methods for a General Variational Inequality System and Common Fixed Point Problems of Nonexpansive Mappings in q-Uniformly Smooth Banach Spaces

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Abstract

This research proposed the iteration method for finding a common fixed point of an infinite family of nonexpansive mappings and two inverse strongly accretive mappings in q-uniformly smooth Banach spaces. Furthermore, our method can also solve a new general variational inequality system and its strong convergence theorem is proved under some appropriate conditions. Our result improves and extends the previous outcomes in the literature.

Keywords: Banach space, Fixed point, Inverse-strongly accretive mapping, q-uniformly smooth, General variational inequality system

1. Introduction

Throughout this research, let E be a real Banach space. We recall that E is called:

- *uniformly convex* if for each $\epsilon \in (0,2]$ there exists $\delta > 0$ such that for any $x, y \in U$ where $U = \{x \in E : \| x \| = 1\}$ then $\| x - y \| \geq \epsilon$, $\| \frac{x+y}{2} \| \leq 1 - \delta$ holds.
- *smooth* if for each $x, y \in U$, $\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists.
- *uniformly smooth* if $\lim_{\tau \to 0} \frac{\rho(\tau)}{\tau} = 0$, where the *modulus of smoothness* of E is the mapping $\rho: [0,\infty) \to [0,\infty)$ defined by
$$\rho(\tau) = \sup\{\frac{1}{2}(\| x + y \| + \| x - y \|) - 1 : x, y \in E, \| x \| = 1, \| y \| = \tau\},$$
- *q-uniformly smooth* if for each $1 < q \leq 2$ there exists $c > 0$ such that modulus of smoothness $\rho(\tau) \leq c\tau^q, \forall \tau > 0$.

Let E^* be a dual space of E. The *generalized duality mapping* $J_q: E \to 2^{E^*}, q > 1$ is defined by
$$J_q(x) = \{f \in E^* : \langle x, f \rangle = \| x \|^q, \| f \| = \| x \|^{q-1}\},$$
for all $x \in E$.
If $q = 2$, the mapping $J_q = J_2 = J$ is said to be the *normalized duality*. For all $x \in E$, the properties of mapping J_q are shown as follow:
(i) $J_q(x) = \| x \|^{q-2} J_2(x), x \neq 0$;
(ii) $J_q(tx) = t^{q-1} J_q(x), t \geq 0$;
(iii) $J_q(-x) = -J_q(x)$.
We know that the mapping J_q is single-valued if E is smooth and can be written by j_q (1, 2).

Let $C \subset E$ be a nonempty closed convex subset. A fixed point problem is to find a set of fixed points of a mapping $T: C \to C$ denoted by $F(T)$ where $F(T) = \{x \in C | Tx = x\}$.

Let $\{T_n: C \to C\}_{n=1}^{\infty}$ be a sequence of an infinite family of mappings such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. $\{T_n\}$ is said to satisfy the $AKTT$-condition (3), if

$$\sum_{n=1}^{\infty} \sup_{\omega \in B} \| T_{n+1}\omega - T_n\omega \| < \infty,$$

for all bounded subset B of C.

A mapping $T: C \to C$ is called *nonexpansive* if for each $x, y \in C$ such that

$$\| Tx - Ty \| \leq \| x - y \|.$$

A mapping $A: C \to C$ is called *Lipschitzian* if for each $x, y \in C$, $L > 0$ such that

$$\| Ax - Ay \| \leq L \| x - y \|.$$

A mapping $A: C \to E$ is called β-*strongly accretive* if for each $x, y \in C$, $\beta > 0$ there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Ax - Ay, j_q(x-y)\rangle \geq \beta \| x - y \|^q.$$

A mapping $A: C \to E$ is called β-*inverse strongly accretive*, if for any $x, y \in C$, $\beta > 0$ there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\left\langle Ax - Ay, j_q(x-y)\right\rangle \geq \beta \| Ax - Ay \|^q.$$

A mapping $Q: C \to D$, $D \subset C$ is called *sunny* if

$$Q(Qx + t(x - Qx)) = Qx,$$

where $Qx + t(x - Qx), x \in C$ and $t \geq 0$. Q is called *retraction* if $Qx = x$, $\forall x \in D$. If Q is a retraction form C onto D then Q is called *sunny nonexpansive retraction* form C onto D (4-6).

Nowadays, variational inequality is one of the most attractive problems due to its widely use applications in many disciplines such as economics, engineering, medical sciences , operation research, structural analysis and many others. Undoubtedly, the algorithms for solving this problem have been studied and improved continuously not only in theoretical way but also in practical approach. Many authors endeavor to reach their goals in real world applications, see (7-16) and the related reference therein.

The famous classical variational inequality problem in the framework of 2-uniformly smooth Banach spaces was published in 2006 by Aoyama et al. (17) which is to find a point $x^* \in C$ such that

$$\langle Ax, j(x - x^*)\rangle \geq 0 \qquad (1.1)$$

for all $x \in C$. In 2010, Yao et al. (18) generated the system of variational inequalities in 2-uniformly smooth Banach spaces for finding $(x^*, y^*) \in C \times C$ satisfying the following conditions:

$$\begin{cases} \langle (I - \lambda A)y^* - x^*, j(x - x^*)\rangle \leq 0, \forall x \in C, \\ \langle (I - \sigma B)x^* - y^*, j(x - y^*)\rangle \leq 0, \forall x \in C. \end{cases} \qquad (1.2)$$

Later, in 2013, Song and Ceng (19) generalized the framework to a q-uniformly smooth Banach spaces and solve the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle (I - \lambda A)y^* - x^*, j_q(x - x^*)\rangle \leq 0, \forall x \in C, \\ \langle (I - \sigma B)x^* - y^*, j_q(x - y^*)\rangle \leq 0, \forall x \in C. \end{cases} \qquad (1.3)$$

Recently, in 2020, Wang and Pan (20) proposed the general variational inequality system in a 2-uniformly smooth Banach spaces:

$$\begin{cases} \langle (I - \lambda A)[tx^* + (1-t)y^*] - x^*, j(x - x^*)\rangle \leq 0, \\ \qquad \forall x \in C, \\ \langle (I - \sigma B)x^* - y^*, j(x - y^*)\rangle \leq 0, \forall x \in C. \end{cases} \qquad (1.4)$$

Due to its significance and the motivation for solving the system of variational inequalities (1.4), we extended their framework to q-uniformly smooth Banach spaces and therefore our mention variational system is stated as follow:

$$\begin{cases} \left\langle (I - \lambda A)[tx^* + (1-t)y^*] - x^*, j_q(x - x^*)\right\rangle \leq 0, \\ \qquad \forall x \in C, \\ \langle (I - \sigma B)x^* - y^*, j_q(x - y^*)\rangle \leq 0, \forall x \in C. \end{cases} \qquad (1.5)$$

2. Preliminaries

In this section, we recall some well known lemmas that will be used to support our proof in the next part.

Lemma 2.1 (21) *Let* $q > 1$. *Then the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q} b^{\frac{q}{q-1}}$$

for arbitrary positive real numbers a, b.

Lemma 2.2 (22) *Let C be a nonempty, closed and convex subset of a real q-uniformly smooth Banach space E, $L_2: C \to E$ be a κ-Lipschitzian and η-strongly accretive operator with constants $\kappa, \eta > 0$ and let* $0 < \mu < (\frac{q\eta}{c_q \kappa^q})^{\frac{1}{q-1}}$, $\tau = \mu(\eta - \frac{c_q \mu^{q-1}\kappa^q}{q})$, *then for* $t \in (0, min\{1, \frac{1}{\tau}\})$, *the* *mapping $S: C \to E$ defined by $S := (I - t\mu L_2)$ is a contraction with a constant $1 - t\tau$.*

Lemma 2.3 (22) *Let C be a nonempty, closed and convex subset of a real q-uniformly smooth Banach space E which admits weakly sequentially continuous generalized duality mapping j_q from E into E^*. Let $T: C \to C$ be a nonexpansive mapping. Then, for all $\{x_n\} \subset C$, if $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.4 (19) *Let C be a nonempty, closed and convex subset of a real reflexive and q-uniformly smooth Banach space E which admits a weakly sequentially continuous generalized duality mapping J_q from E into E^*. Let Q_C be a sunny nonexpansive retraction from E onto C, $V: C \to E$ a k-Lipschitzian and η-strongly accretive operator with constants k, $\eta > 0$. Suppose $f: C \to E$ is a L-Lipschitzian mapping with constant $L > 0$ and $T: C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $0 < \mu < (\frac{q\eta}{c_q \kappa^q})^{\frac{1}{q-1}}$ and $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - \frac{c_q \mu^{q-1} \kappa^q}{q})$. Then $\{x_t\}$ defined by $x_t = Q_C[t\gamma f x_t + (I - t\mu V)Tx_t]$ converges strongly to some point $x^* \in F(T)$ as $t \to 0$, which is the unique solution of the variational inequality:*

$$\langle \gamma f x^* - \mu V x^*, J_q(p - x^*) \rangle \leq 0, \forall p \in F(T).$$

Lemma 2.5 (23) *Let C be a closed convex subset of a strictly convex Banach space E. Let T_1 and T_2 be two nonexpansive mappings from C into itself with $F(T_1) \cap F(T_2) \neq \emptyset$. Define a mapping S by*

$$Sx = \lambda T_1 x + (1 - \lambda) T_2 x, \forall x \in C,$$

where λ is a constant in $(0,1)$. Then S is nonexpansive and $F(S) = F(T_1) \cap F(T_2)$.

Lemma 2.6 (3) *Suppose that $\{T_n\}$ satisfy the AKTT-condition such that*

(1) For each $x \in C, \{T_n x\}$ converge strongly to some point in C.

(2) Let the mapping $T: C \to C$ defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$. Then $\lim_{n\to\infty} \sup_{\omega \in B} \| T\omega - T_n \omega \| = 0$ for each bounded subset B of C.

Lemma 2.7 (24) *Let E be a real smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g: [0,2r] \to \mathbb{R}$ such that*

$$g(0) = 0 \text{ and}$$
$$g(\| x - y \|) \leq \| x \|^2 - 2\langle x, Jy \rangle + \| y \|^2,$$

for all $x, y \in B_r$ where $B_r = \{z \in E: \| z \| \leq r\}$.

Lemma 2.8 (25) *Let E be a real q-uniformly smooth Banach space, then there exists a constant $c_q > 0$ such that*

$$\| x + y \|^q \leq \| x \|^q + q\langle y, J_q(x) \rangle + c_q \| y \|^q, \quad \forall x, y \in E.$$

In particular, if E is real 2-uniformly smooth Banach space, then there exists a best smooth constant $K > 0$ such that

$$\| x + y \|^2 \leq \| x \|^2 + 2\langle y, J(x) \rangle + 2K \| y \|^2, \quad \forall x, y \in E.$$

Lemma 2.9 (26) *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the property:*

$$a_{n+1} \leq (1 - \alpha_n) a_n + b_n + \alpha_n c_n, n \in \mathbb{N},$$

where $\{\alpha\}, \{b_n\}, \{c_n\}$ satisfy the restrictions:

(1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $b_n \geq 0, \sum_{n=1}^{\infty} b_n < \infty$,

(3) $\limsup_{n\to\infty} c_n \leq 0$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10 (27) *Let C be a closed convex subset of a smooth Banach space E. Let $\tilde{C}$ be a nonempty subset of C. Let $Q: C \to \tilde{C}$ be a retraction and let j, j_q be the normalized duality mapping and generalized duality mapping on E, respectively. Then the following are equivalent:*

(1) Q is sunny and nonexpansive;

(2) $\| Qx - Qy \|^2 \leq \langle x - y, j(Qx - Qy) \rangle$, $\forall x, y \in E$;

(3) $\langle x - Qx, j(y - Qx) \rangle \leq 0, \forall x \in C, y \in \tilde{C}$;

(4) $\langle x - Qx, j_q(y - Qx) \rangle \leq 0, \forall x \in C, y \in \tilde{C}$.

Lemma 2.11 (19) *Let C be a nonempty closed convex subset of a real q-uniformly smooth Banach space E. Let the mapping $A: C \to E$ be a $f\alpha$-inverse-strongly accretive operator. Then the following inequality holds:*

$$\| (I - \lambda A)x - (I - \lambda A)y \|^q \leq x - y \|^q - \lambda(q\alpha - c_q \lambda^{q-1}) \| Ax - Ay \|^q.$$

In particular, if $0 < \lambda \leq (\frac{q\alpha}{c_q})^{\frac{1}{q-1}}$, then $I - \lambda A$ is nonexpansive.

Lemma 2.12 (20) *Let E be a real Banach space. Let $\emptyset \neq C \subset E$ be a closed convex subset and $A, B: C \to E$ be two nonlinear mappings.*

Suppose that Q_C is a sunny nonexpansive retraction. For $\forall\lambda,\sigma > 0$ and $t \in [0,1]$, then the following assertions are equivalent:
(a) $(x^*, y^*) \in C \times C$ is a solution of problem (1.4);
(b) Let $\Psi: C \to C$ be a mapping defined by
$\Psi(x) = Q_C(I - \lambda A)[tx + (1-t)Q_C(I - \sigma B)x]$,
then let x^* be the fixed point of Ψ, that is $x^* = \Psi x^*$.
where $x^* = Q_C(I - \lambda A)[tx^* + (1-t)y^*]$, $y^* = Q_C(I - \sigma B)x^*$. Assume that $A, B: C \to E$ are α-inverse strongly accretive operator and β-inverse strongly operator, respectively. If $0 < \lambda < \frac{2\alpha}{c}$, $0 < \sigma < \frac{2\beta}{c}$, then Ψ is nonexpansive.

Lemma 2.13 *Let C be a nonempty closed convex subset of a real q-uniformly smooth Banach space E. Suppose Q_C is a sunny nonexpansive retraction from E onto C. Let the mapping $A, B: C \to E$ are α-inverse strongly accretive operator and β-inverse strongly operator, respectively. Assume $\Psi: C \to C$ is a mapping defined by*
$\Psi(x) = Q_C(I - \lambda A)[tx + (1-t)Q_C(I - \sigma B)x]$,
for all $\lambda, \sigma > 0$ and $t \in [0,1]$. If $0 < \lambda \leq (\frac{q\alpha}{c_q})^{\frac{1}{q-1}}$ and $0 < \sigma \leq (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, then Ψ is nonexpansive.
Proof. For all $x, y \in C$, by Lemma 2.11 and Lemma 2.12, we have that

$$\| \Psi(x) - \Psi(y) \| \leq \| x - y \|.$$

Therefore Ψ is nonexpansive.

Lemma 2.14 *Let C be a nonempty closed convex subset of a real q-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A, B: C \to E$ be two nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.5) if and only if*

$$x^* = Q_C(I - \lambda A)[tx^* + (1-t)y^*]$$

where $y^* = Q_C(I - \sigma B)x^*$, that is $x^* = \Psi(x^*)$, where Ψ is defined by Lemma 2.13.
Proof. From Lemma 2.10 and the definition of the sunny nonexpansive retraction, we deduce the problem (1.5) is equivalent to

$$\begin{cases} x^* = Q_C(I - \lambda A)[tx^* + (1-t)y^*] \\ y^* = Q_C(I - \sigma B)x^*, \end{cases}$$

which is solution of the problem (1.5).

Lemma 2.15 (28) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \qquad n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Theorem 3.1 *Let E be a q-uniformly smooth and uniformly convex Banach space and $\emptyset \neq C \subset E$ be a closed convex subset. Let $j_q: E \to E^*$ be a weakly sequentially continuous generalized duality mapping and Q_C be a sunny nonexpansive retraction from E onto C. Suppose that $A: C \to E$ is an α-inverse-strongly accretive, $B: C \to E$ is a β-inverse-strongly accretive, $\{T_i: C \to C\}_{i=1}^{\infty}$ is an infinite family of nonexpansive mappings and Ψ is defined by Lemma 2.13. Let $L_1: C \to E$ be a L-Lipschitzian, $L \geq 0$ and $L_2: C \to E$ be a κ-Lipschitzian and η-strongly accretive, $\kappa, \eta > 0$. Assume $t \in [0,1)$, $\{\alpha_n\}, \{\gamma_n\} \subset (0,1)$, $0 < \mu < (\frac{q\eta}{c_q \kappa^q})^{\frac{1}{q-1}}$, $0 < \lambda < (\frac{q\alpha}{c_q})^{\frac{1}{q-1}}$, c_q>0, $0 < \sigma < (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - \frac{c_q \mu^{q-1}\kappa^q}{q})$ and $F := \cap_{i=1}^{\infty} F(T_i) \cap F(\Psi) \neq \emptyset$.*
Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and

$$\begin{cases} z_n = Q_C(I - \sigma B)x_n \\ y_n = Q_C(I - \lambda A)(tx_n + (1-t)z_n), \\ x_{n+1} = Q_C[\alpha_n \gamma L_1 x_n + \gamma_n x_n \\ \quad +((1-\gamma_n)I - \alpha_n \mu L_2)T_n y_n], \end{cases} \tag{3.1}$$

which corresponds to the conditions:
(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$; and $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$;
(C2) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$,
$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.
Assume that $\{T_n\}_{n=1}^{\infty}$ and the $AKTT$-condition are satisfied. Let a mapping $T: C \to C$ be defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and suppose that $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves the following variational inequality:

$$\langle \gamma L_1 x^* - \mu L_2 x^*, j_q(z - x^*) \rangle \leq 0, \forall z \in F. \tag{3.2}$$

Proof. First of all, we shall prove that $\{x_n\}$ is bounded. Let $x^* \in F$, from Lemma 2.14, we have

$$x^* = Q_C(I - \lambda A)[tx^* + (1-t)y^*]$$

and $y^* = Q_C(I - \sigma B)x^*$.

It follows from (3.1) that

$$\| y_n - x^* \| = \| \Psi(x_n) - x^* \| \leq \| x_n - x^* \|. \quad (3.3)$$

From (3.1) and (3.3), we have

$$\begin{aligned}
&\| x_{n+1} - x^* \| \\
&= \| Q_C[\alpha_n \gamma L_1 x_n + \gamma_n x_n \\
&\quad + ((1-\gamma_n)I - \alpha_n \mu L_2)T_n y_n] - x^* \| \\
&\leq \| \alpha_n \gamma L_1 x_n + \gamma_n x_n \\
&\quad + ((1-\gamma_n)I - \alpha_n \mu L_2)T_n y_n - x^* \| \\
&= \| [(1-\gamma_n)I - \alpha_n \mu L_2](T_n y_n - x^*) \\
&\quad + \alpha_n(\gamma L_1 x_n - \mu L_2 x^*) + \gamma_n(x_n - x^*) \| \\
&\leq (1 - \gamma_n - \alpha_n \tau) \| T_n y_n - x^* \| \\
&\quad + \alpha_n \| \gamma L_1 x_n - \mu L_2 x^* \| + \gamma_n \| x_n - x^* \| \\
&\leq (1 - \gamma_n - \alpha_n \tau) \| x_n - x^* \| + \alpha_n \gamma L \| x_n - x^* \| \\
&\quad + \alpha_n \| \gamma L_1 x^* - \mu L_2 x^* \| + \gamma_n \| x_n - x^* \| \\
&= [1 - \alpha_n(\tau - \gamma L)] \| x_n - x^* \| \\
&\quad + \alpha_n(\tau - \gamma L)\frac{\| \gamma L_1 x^* - \mu L_2 x^* \|}{\tau - \gamma L}.
\end{aligned}$$

Therefore by the mathematical induction, we can conclude that for all $n \geq 1$,

$$\| x_n - x^* \| \leq \max\left\{ \| x_1 - x^* \|, \frac{\| \gamma L_1 x^* - \mu L_2 x^* \|}{\tau - \gamma L} \right\}.$$

Hence, $\{x_n\}$ is bounded and also $\{y_n\}$, $\{z_n\}$ are bounded. Next, we will show that $\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0$. By (3.1) and Lemma 2.13, we observe that

$$\begin{aligned}
&\| y_{n+1} - y_n \| \\
&= \| Q_C(I - \lambda A)(t x_{n+1} + (1-t) z_{n+1}) \\
&\quad - Q_C(I - \lambda A)(t x_n + (1-t) z_n) \| \\
&= \| Q_C(I - \lambda A)(t x_{n+1} + (1-t) Q_C(I - \sigma B) x_{n+1}) \\
&\quad - Q_C(I - \lambda A)(t x_n + (1-t) Q_C(I - \sigma B) x_n) \| \\
&= \| \Psi(x_{n+1}) - \Psi(x_n) \| \\
&\leq \| x_{n+1} - x_n \|,
\end{aligned}$$

and

$$\begin{aligned}
&\| T_{n+1} y_{n+1} - T_n y_n \| \\
&\leq \| T_{n+1} y_{n+1} - T_{n+1} y_n \| + \| T_{n+1} y_n - T_n y_n \| \\
&\leq \| y_{n+1} - y_n \| + \| T_{n+1} y_n - T_n y_n \| \\
&\leq \| x_{n+1} - x_n \| + \| T_{n+1} y_n - T_n y_n \|. \quad (3.4)
\end{aligned}$$

Again, it follows from (3.1) and Lemma 2.13, we have

$$\begin{aligned}
&\| x_{n+2} - x_{n+1} \| \\
&= \| Q_C[\alpha_{n+1} \gamma L_1 x_{n+1} + \gamma_{n+1} x_{n+1} \\
&\quad + ((1-\gamma_{n+1})I - \alpha_{n+1} \mu L_2) T_{n+1} y_{n+1}] \\
&\quad - Q_C[\alpha_n \gamma L_1 x_n + \gamma_n x_n \\
&\quad + ((1-\gamma_n)I - \alpha_n \mu L_2) T_n y_n] \| \\
&\leq \alpha_{n+1} \gamma \| L_1 x_{n+1} - L_1 x_n \| + \gamma_{n+1} \| x_{n+1} - x_n \| \\
&\quad + \| [(1-\gamma_{n+1})I - \alpha_{n+1} \mu L_2] \\
&\quad \times \left(T_{n+1} y_{n+1} - T_n y_n\right) \| + |\alpha_{n+1} - \alpha_n| \gamma \| L_1 x_n \| \\
&\quad + |\alpha_{n+1} - \alpha_n| \mu \| L_2 T_n y_n \| \\
&\quad + |\gamma_{n+1} - \gamma_n| \| T_n y_n - x_n \| \\
&\leq \alpha_{n+1} \gamma L \| x_{n+1} - x_n \| + \gamma_{n+1} \| x_{n+1} - x_n \| \\
&\quad + [(1-\gamma_{n+1})I - \alpha_{n+1} \tau][\| x_{n+1} - x_n \| \\
&\quad + \| T_{n+1} y_n - T_n y_n \|] \\
&\quad + |\alpha_{n+1} - \alpha_n| [\gamma \| L_1 x_n \| + \mu \| L_2 T_n y_n \|] \\
&\quad + |\gamma_{n+1} - \gamma_n| \| T_n y_n - x_n \| \\
&\leq [1 - \alpha_{n+1}(\tau - \gamma L)] \| x_{n+1} - x_n \| \\
&\quad + \| T_{n+1} y_n - T_n y_n \| \\
&\quad + (|\alpha_{n+1} - \alpha_n| + |\gamma_{n+1} - \gamma_n|) M_1, \quad (3.5)
\end{aligned}$$

where

$$M_1 = \sup_{n\geq 1}\{\gamma \| L_1 x_n \| + \mu \| L_2 T_n y_n \|, \| T_n y_n - x_n \|\} < \infty.$$

Since $\{T_n\}_{n=1}^{\infty}$ satisfies the $AKTT$-condition, we have

$$\sum_{n=1}^{\infty} \| T_{n+1} y_n - T_n y_n \| \leq \sum_{n=1}^{\infty} \sup_{y \in \{y_n\}} \| T_{n+1} y - T_n y \| < \infty. \quad (3.6)$$

Form the condition (C1), (C2), (3.5), (3.6) and Lemma 2.9, we can verify that

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \quad (3.7)$$

Later, we prove that $\lim_{n\to\infty} \| x_n - \Psi x_n \| = 0$ and $\lim_{n\to\infty} \| T x_n - x_n \| = 0$.

It follows from (3.1), Lemma 2.8 and Lemma 2.11, we have

$$\begin{aligned}
&\| z_n - y^* \|^q \\
&= \| Q_C(I - \sigma B) x_n - Q_C(I - \sigma B) x^* \|^q \\
&\leq \| (I - \sigma B) x_n - (I - \sigma B) x^* \|^q \\
&\leq \| x_n - x^* \|^q - \sigma\left(q\beta - c_q \sigma^{q-1}\right) \\
&\quad \| B x_n - B x^* \|^q. \quad (3.8)
\end{aligned}$$

Form equation (3.1) and (3.8), we get

$$\begin{aligned}
&\| y_n - x^* \|^q \\
&= \| Q_C(I - \lambda A)(t x_n + (1-t) z_n) \\
&\quad - Q_C(I - \lambda A)(t x^* + (1-t) y^*) \|^q \\
&\leq \| (I - \lambda A)(t x_n + (1-t) z_n) \\
&\quad - (I - \lambda A)(t x^* + (1-t) y^*) \|^q \\
&\leq \| (t x_n + (1-t) z_n) - (t x^* + (1-t) y^*) \|^q \\
&\quad - q\lambda\alpha \| A(t x_n + (1-t) z_n) \\
&\quad - A(t x^* + (1-t) y^*) \|^q \\
&\quad + c_q \lambda^q \| A(t x_n + (1-t) z_n) \\
&\quad - A(t x^* + (1-t) y^*) \|^q \\
&\leq t \| x_n - x^* \|^q + (1-t)[\| x_n - x^* \|^q \\
&\quad - \sigma(q\beta - c_q \sigma^{q-1}) \| B x_n - B x^* \|^q] \\
&\quad - \lambda(q\alpha - c_q \lambda^{q-1}) \| A(t x_n + (1-t) z_n) \\
&\quad - A(t x^* + (1-t) y^*) \|^q
\end{aligned}$$

$$=\| x_n - x^* \|^q - (1-t)\sigma\big(q\beta - c_q\sigma^{q-1}\big) \times \| Bx_n - Bx^* \|^q - \lambda(q\alpha - c_q\lambda^{q-1}) \| A(tx_n + (1-t)z_n) - A(tx^* + (1-t)y^*) \|^q.$$

Moreover, we know that

$$\begin{aligned}
&\| x_{n+1} - x^* \|^q \\
&= \| Q_C[\alpha_n\gamma L_1x_n + \gamma_nx_n + ((1-\gamma_n)I - \alpha_n\mu L_2)T_ny_n] - x^* \|^q \\
&\le \| \gamma_n(x_n - x^*) + (1-\gamma_n)(T_ny_n - x^*) + \alpha_n(\gamma L_1x_n - \mu L_2T_ny_n) \|^q \\
&\le \| \gamma_n(x_n - x^*) + (1-\gamma_n)(T_ny_n - x^*) \|^q + q\langle \alpha_n(\gamma L_1x_n - \mu L_2T_ny_n), j_q(\gamma_n(x_n - x^*) + (1-\gamma_n)(T_ny_n - x^*))\rangle + c_q \| \alpha_n(\gamma L_1x_n - \mu L_2T_ny_n) \|^q \\
&\le \gamma_n \| x_n - x^* \|^q + (1-\gamma_n) \| T_ny_n - x^* \|^q + q\alpha_n \| \gamma L_1x_n - \mu L_2T_ny_n \| \times \| \gamma_n(x_n - x^*) + (1-\gamma_n)(T_ny_n - x^*) \|^{q-1} + c_q\alpha_n^q \| \gamma L_1x_n - \mu L_2T_ny_n \|^q \\
&\le \gamma_n \| x_n - x^* \|^q + (1-\gamma_n) \| y_n - x^* \|^q + \alpha_nM_2 \\
&\le \| x_n - x^* \|^q - (1-\gamma_n)(1-t)\sigma\big(q\beta - c_q\sigma^{q-1}\big) \times \| Bx_n - Bx^* \|^q \\
&\quad - (1-\gamma_n)\lambda(q\alpha - c_q\lambda^{q-1}) \| A(tx_n + (1-t)z_n) - A(tx^* + (1-t)y^*) \|^q + \alpha_nM_2,
\end{aligned}$$

where

$$M_2 = \sup_{n\ge1}\{q \| \gamma L_1x_n - \mu L_2T_ny_n \| \times \| \gamma_n(x_n - x^*) + (1-\gamma_n)(T_ny_n - x^*) \|^{q-1} + c_q\alpha_n^{q-1} \| \gamma L_1x_n - \mu L_2T_ny_n \|^q\} < \infty.$$

By the fact that $a^r - b^r \le ra^{r-1}(a-b)$, $\forall r \ge 1$, we get

$$\begin{aligned}
&(1-\gamma_n)(1-t)\sigma\big(q\beta - c_q\sigma^{q-1}\big) \| Bx_n - Bx^* \|^q \\
&+ (1-\gamma_n)\lambda\big(q\alpha - c_q\lambda^{q-1}\big) \| A(tx_n + (1-t)z_n) - A(tx^* + (1-t)y^*) \|^q \\
&\le q \| x_n - x^* \|^{q-1} \| x_n - x_{n+1} \| + \alpha_nM_2.
\end{aligned}$$

Since $0 < \lambda < (\frac{q\alpha}{c_q})^{\frac{1}{q-1}}$, $0 < \sigma < (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, (3.7) and by the conditions $(C1)$ and $(C2)$, we conclude that

$$\lim_{n\to\infty} \| Bx_n - Bx^* \| = 0 \tag{3.9}$$

and

$$\lim_{n\to\infty} \| A(tx_n + (1-t)z_n) - A(tx^* + (1-t)y^*) \| = 0. \tag{3.10}$$

Setting $r_1 = \sup_{n\ge1}\{\| x_n - x^* \|, \| z_n - y^* \|\}$, it follows from Lemma 2.7 and Lemma 2.10, we have

$$\begin{aligned}
&\| z_n - y^* \|^2 \\
&= \| Q_C(x_n - \sigma Bx_n) - Q_C(x^* - \sigma Bx^*) \|^2 \\
&\le \langle (x_n - \sigma Bx_n) - (x^* - \sigma Bx^*), j(z_n - y^*)\rangle \\
&= \langle (x_n - x^*) + \sigma(Bx^* - Bx_n), j(z_n - y^*)\rangle \\
&= \langle x_n - x^*, j(z_n - y^*)\rangle + \sigma\langle Bx^* - Bx_n, j(z_n - y^*)\rangle \\
&\le \frac{1}{2}[\| x_n - x^* \|^2 + \| z_n - y^* \|^2 - g_1(\| (x_n - x^*) - (z_n - y^*) \|)] + \sigma \| Bx^* - Bx_n \| \| z_n - y^* \|.
\end{aligned}$$

Then,

$$\| z_n - y^* \|^2 \le \| x_n - x^* \|^2 - g_1(\| (x_n - x^*) - (z_n - y^*) \|) + 2\sigma \| Bx^* - Bx_n \| \| z_n - y^* \|, \tag{3.11}$$

Furthermore, setting

$$r_2 = \sup_{n\ge1}\{\| x_n - x^* \|, \| y_n - x^* \|\}$$

and

$$r_3 = \sup_{n\ge1}\{\| y_n - x^* \|, \| z_n - y^* \|\},$$

we compute

$$\begin{aligned}
&\| y_n - x^* \|^2 \\
&= \| Q_C(I - \lambda A)(tx_n + (1-t)z_n) - Q_C(I - \lambda A)(tx^* + (1-t)y^*) \|^2 \\
&\le \langle (I - \lambda A)(tx_n + (1-t)z_n) - (I - \lambda A)(tx^* + (1-t)y^*), j(y_n - x^*)\rangle \\
&\le \frac{t}{2}[\| x_n - x^* \|^2 + \| y_n - x^* \|^2 - g_2(\| x_n - y_n \|)] \\
&\quad + \tfrac{1-t}{2}[\| z_n - y^* \|^2 + \| y_n - x^* \|^2 - g_3(\| (z_n - y^*) - (y_n - x^*) \|)] \\
&\quad + \lambda \| A(tx^* + (1-t)y^*) - A(tx_n + (1-t)z_n) \| \| y_n - x^* \|,
\end{aligned}$$

where $g_2: [0, 2r_2) \to [0,\infty)$ and $g_3: [0, 2r_3) \to [0,\infty)$ are continuous, strictly increasing and convex functions. Since $0 \le t < 1$ and from (3.11), so

$$\begin{aligned}
&\| y_n - x^* \|^2 \\
&\le t \| x_n - x^* \|^2 + (1-t) \| z_n - y^* \|^2 - tg_2(\| x_n - y_n \|) - (1-t)g_3(\| (z_n - y^*) - (y_n - x^*) \|) \\
&\quad + 2\lambda \| A(tx^* + (1-t)y^*) - A(tx_n + (1-t)z_n) \| \| y_n - x^* \| \\
&\le \| x_n - x^* \|^2 - tg_2(\| x_n - y_n \|) - (1-t)g_3(\| (z_n - y^*) - (y_n - x^*) \|) \\
&\quad - (1-t)g_1(\| (x_n - x^*) - (z_n - y^*) \|) + 2\sigma(1-t) \| Bx^* - Bx_n \| \| z_n - y^* \| \\
&\quad + 2\lambda \| A(tx^* + (1-t)y^*) - A(tx_n + (1-t)z_n) \| \| y_n - x^* \|.
\end{aligned}$$

Consider,

$$\begin{aligned}
&\| x_{n+1} - x^* \|^2 \\
&= \| Q_C[\alpha_n\gamma L_1x_n + \gamma_nx_n + ((1-\gamma_n)I - \alpha_n\mu L_2)T_ny_n] - x^* \|^2 \\
&\le \| \gamma_n(x_n - x^*) + (1-\gamma_n)(T_ny_n - x^*) + \alpha_n(\gamma L_1x_n - \mu L_2T_ny_n) \|^2
\end{aligned}$$

$$\begin{aligned}
&\le \| \gamma_n(x_n - x^*) + (1-\gamma_n)(T_n y_n - x^*) \|^2 \\
&\quad + 2\alpha_n \langle \gamma L_1 x_n - \mu L_2 T_n y_n, j(\gamma_n(x_n - x^*) \\
&\quad + (1-\gamma_n)(T_n y_n - x^*) \\
&\quad + \alpha_n(\gamma L_1 x_n - \mu L_2 T_n y_n)) \rangle \\
&\le \gamma_n \| x_n - x^* \|^2 + (1-\gamma_n) \| y_n - x^* \|^2 + \alpha_n M_3 \\
&\le \| x_n - x^* \|^2 \\
&\quad - (1-\gamma_n) t g_2(\| x_n - y_n \|) \\
&\quad - (1-\gamma_n)(1-t) g_3(\| (z_n - y^*) - (y_n - x^*) \|) \\
&\quad - (1-\gamma_n)(1-t) g_1(\| (x_n - x^*) - (z_n - y^*) \|) \\
&\quad + 2\sigma(1-\gamma_n)(1-t) \| Bx^* - Bx_n \| \| z_n - y^* \| \\
&\quad + 2\lambda(1-\gamma_n) \| A(tx^* + (1-t)y^*) \\
&\quad - A(tx_n + (1-t)z_n) \| \| y_n - x^* \|] + \alpha_n M_3,
\end{aligned}$$

where

$$M_3 = \sup_{n \ge 0} \{ 2 \langle \gamma L_1 x_n - \mu L_2 T_n y_n, J(\gamma_n(x_n - p) + (1-\gamma_n)(T_n y_n - p) + \alpha_n(\gamma L_1 x_n - \mu L_2 T_n y_n)) \rangle \} < \infty.$$

It follows that

$$\begin{aligned}
&(1-\gamma_n)(1-t) g_1(\| (x_n - x^*) - (z_n - y^*) \|) \\
&+ (1-\gamma_n) t g_2(\| x_n - y_n \|) \\
&+ (1-\gamma_n)(1-t) g_3(\| (z_n - y^*) - (y_n - x^*) \|) \\
&\le \| x_n - x^* \|^2 - \| x_{n+1} - x^* \|^2 \\
&\quad + 2\sigma(1-\gamma_n)(1-t) \| Bx^* - Bx_n \| \| z_n - y^* \| \\
&\quad + 2\lambda(1-\gamma_n) \| A(tx^* + (1-t)y^*) \\
&\quad - A(tx_n + (1-t)z_n) \| \| y_n - x^* \|] + \alpha_n M_3 \\
&\le \| x_n - x_{n+1} \| (\| x_n - x^* \| + \| x_{n+1} - x^* \|) \\
&\quad + 2\sigma(1-\gamma_n)(1-t) \| Bx^* - Bx_n \| \| z_n - y^* \| \\
&\quad + 2\lambda(1-\gamma_n) \| A(tx^* + (1-t)y^*) \\
&\quad - A(tx_n + (1-t)z_n) \| \| y_n - x^* \|] + \alpha_n M_3.
\end{aligned}$$

It follows from (3.7), (3.9), (3.10), condition $(C1)$, $(C2)$ and the properties of g, we conclude that

$$\lim_{n\to\infty} \| (x_n - x^*) - (z_n - y^*) \| = 0, \qquad (3.12)$$

$$\lim_{n\to\infty} \| x_n - y_n \| = 0 \qquad (3.13)$$

and

$$\lim_{n\to\infty} \| (z_n - y^*) - (y_n - x^*) \| = 0. \qquad (3.14)$$

So,

$$\| x_n - \Psi x_n \| = \| x_n - y_n \| \to 0 \quad as \quad n \to \infty. \qquad (3.15)$$

Observe that

$$\begin{aligned}
&\| T_n y_n - x_n \| \\
&\le \| x_{n+1} - x_n \| + \| x_{n+1} - T_n y_n \| \\
&= \| x_{n+1} - x_n \| + \| Q_C[\alpha_n \gamma L_1 x_n + \gamma_n x_n \\
&\quad + ((1-\gamma_n)I - \alpha_n \mu L_2) T_n y_n] - T_n y_n \| \\
&\le \| x_{n+1} - x_n \| + \| [\alpha_n \gamma L_1 x_n + \gamma_n x_n \\
&\quad + ((1-\gamma_n)I - \alpha_n \mu L_2) T_n y_n] - T_n y_n \| \\
&= \| x_{n+1} - x_n \| \\
&\quad + \| \alpha_n(\gamma L_1 x_n - \mu L_2 T_n y_n) + \gamma_n(x_n - T_n y_n) \| \\
&\le \| x_{n+1} - x_n \| + \alpha_n \| \gamma L_1 x_n - \mu L_2 T_n y_n \| \\
&\quad + \gamma_n \| x_n - T_n y_n \|,
\end{aligned}$$

which implies that

$$\| T_n y_n - x_n \| \le \frac{1}{1-\gamma_n} (\| x_{n+1} - x_n \| + \alpha_n \| \gamma L_1 x_n - \mu L_2 T_n y_n \|). \qquad (3.16)$$

Applied condition $(C1)$, $(C2)$ and (3.7) in (3.16), we get

$$\lim_{n\to\infty} \| T_n y_n - x_n \| = 0. \qquad (3.17)$$

Thus, we have

$$\begin{aligned}
&\| T_n y_n - y_n \| \\
&= \| T_n y_n - x_n \| + \| x_n - y_n \| \to 0 \\
&\quad as \quad n \to \infty, \qquad (3.18)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\| T_n x_n - x_n \| \\
&\le \| T_n x_n - T_n y_n \| + \| T_n y_n - y_n \| + \| y_n - x_n \| \\
&\le 2 \| y_n - x_n \| + \| T_n y_n - y_n \| \to 0 \\
&\quad as \quad n \to \infty. \qquad (3.19)
\end{aligned}$$

By Lemma 2.6, we have

$$\limsup_{n\to\infty} \| Tx_n - T_n x_n \| \le \lim_{n\to\infty} \sup_{x \in \{x_n\}} \| Tx - T_n x \| = 0. \qquad (3.20)$$

Thus,

$$\| Tx_n - x_n \| \le \| Tx_n - T_n x_n \| + \| T_n x_n - x_n \| \to 0 \quad as \quad n \to \infty. \qquad (3.21)$$

Now, we show that $x^* \in F := \cap_{i=1}^{\infty} F(T_i) \cap F(\Psi)$. Let $\delta \in (0,1)$ be a constant and $U: C \to C$ be defined by $Ux = \delta Tx + (1-\delta)\Psi x$, where Ψ is defined by Lemma 2.13. By Lemma 2.5 and Lemma 2.13, we conclude that U is a nonexpansive and

$$F(U) = F(T) \cap F(\Psi) = \cap_{i=0}^{\infty} F(T_i) \cap F(\Psi).$$

Setting $x_t = Q_C[t\gamma L_1 x_t + (I - t\mu L_2)Ux_t]$, by Lemma 2.4, we get $\{x_t\}$ converges strongly to the unique solution of the variational inequality (3.2), that is $x^* \in F(U)$. Consider,

$\| x_n - Ux_n \|$
$\leq \| \delta(x_n - Tx_n) + (1-\delta)(x_n - \Psi x_n) \|$
$\leq \delta \| x_n - Tx_n \| + (1-\delta) \| x_n - \Psi x_n \|$
$= \delta \| x_n - Tx_n \| + (1-\delta) \| x_n - y_n \|.$
From (3.13) and (3.21), we have

$$\| x_n - Ux_n \| \to 0 \quad as \quad n \to \infty. \qquad (3.22)$$

Next, we need to show $\limsup_{n\to\infty} \langle \gamma L_1 x^* - \mu L_2 x^*, j_q(y_n - x^*) \rangle \leq 0$, where x^* is the solution of the variational inequality (3.2).
It follows from (3.22) and Lemma 2.5 that $z \in F(U)$. Since the Banach space E has a weakly sequentially continuous generalized duality mapping $j_q: E \to E^*$ and $x_{n_k} \rightharpoonup z$, we obtain that

$$\begin{aligned} &\limsup_{n\to\infty} \langle \gamma L_1 x^* - \mu L_2 x^*, j_q(x_n - x^*) \rangle \\ &= \lim_{k\to\infty} \langle \gamma L_1 x^* - \mu L_2 x^*, j_q(x_{n_k} - x^*) \rangle \\ &= \langle \gamma L_1 x^* - \mu L_2 x^*, j_q(z - x^*) \rangle \leq 0. \end{aligned} \qquad (3.23)$$

Finally, we prove that
$$\| x_n - x^* \| \to 0 \quad as \quad n \to \infty.$$
Setting $u_n = \alpha_n \gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu L_2) T_n x_{n+1}, \forall n \geq 0$, it follows from Lemma 2.1, Lemma 2.2 and Lemma 2.10 that

$$\begin{aligned} &\| x_{n+1} - x^* \|^q \\ &\leq \langle u_n - x^*, j_q(x_{n+1} - x^*) \rangle \\ &\leq (1 - \gamma_n - \alpha_n \tau) \| T_n y_n - x^* \| \| x_{n+1} - x^* \|^{q-1} \\ &\quad + \gamma_n \| x_n - x^* \| \| x_{n+1} - x^* \|^{q-1} \\ &\quad + \alpha_n \langle \gamma L_1 x_n - \gamma L_1 x^*, j_q(x_{n+1} - x^*) \rangle \\ &\quad + \alpha_n \langle \gamma L_1 x^* - \mu L_2 x^*, j_q(x_{n+1} - x^*) \rangle \\ &\leq [1 - \alpha_n(\tau - \gamma L)] \| x_n - x^* \| \| x_{n+1} - x^* \|^{q-1} \\ &\quad + \alpha_n \langle \gamma L_1 x^* - \mu L_2 x^*, j_q(x_{n+1} - x^*) \rangle \\ &\leq [1 - \alpha_n(\tau - \gamma L)] [\frac{1}{q} \| x_n - x^* \|^q \\ &\quad + \frac{q-1}{q} \| x_{n+1} - x^* \|^q] \\ &\quad + \alpha_n \langle \gamma L_1 x^* - \mu L_2 x^*, j_q(x_{n+1} - x^*) \rangle, \end{aligned}$$
therefore

$$\begin{aligned} &\| x_{n+1} - x^* \|^q \\ &\leq \frac{1 - \alpha_n(\tau - \gamma L)}{1 + (q-1)\alpha_n(\tau - \gamma L)} \| x_n - x^* \|^q \\ &\quad + \frac{q\alpha_n}{1+(q-1)\alpha_n(\tau-\gamma L)} \langle \gamma L_1 x^* - \mu L_2 x^*, \\ &\qquad j_q(x_{n+1} - x^*) \rangle \\ &\leq [1 - \alpha_n(\tau - \gamma L)] \| x_n - x^* \|^q \\ &\quad + \frac{q\alpha_n}{1+(q-1)\alpha_n(\tau-\gamma L)} \langle \gamma L_1 x^* - \mu L_2 x^*, \\ &\qquad j_q(x_{n+1} - x^*) \rangle. \end{aligned} \qquad (3.24)$$

Now, from (C1), (3.23) and applying Lemma 2.15 to (3.24), we can verify that $\| x_n - x^* \| \to 0$, that is $x_n \to x^*$ as $n \to \infty$. This completes the proof.

Corollary 3.2 *Let E be a 2-uniformly smooth and uniformly convex Banach space and $\emptyset \neq C \subset E$ be a closed convex subset. Let $j: E \to E^*$ be a weakly sequentially continuous generalized duality mapping with the best smooth constant K and Q_C be a sunny nonexpansive retraction from E onto C. Suppose that $A: C \to E$ is an α-inverse-strongly accretive, $B: C \to E$ is a β-inverse-strongly accretive, $T: C \to C$ is a nonexpansive mappings and Ψ is defined by Lemma 2.13. Let $L_1: C \to E$ be a L-Lipschitzian, $L \geq 0$ and $L_2: C \to E$ be a κ-Lipschitzian and η-strongly accretive, $\kappa, \eta > 0$. Assume $\{\alpha_n\}, \{\gamma_n\} \subset (0,1)$,*
$0 < \mu < \frac{\eta}{K^2\kappa^2}, 0 < \lambda < \frac{\alpha}{K^2},$
$0 < \sigma < \frac{\beta}{K^2}, \quad 0 \leq \gamma L < \tau$ *where* $\tau = \mu(\eta - K^2\mu\kappa^2)$ *and* $F := F(T) \cap F(\Psi) \neq \emptyset$. *Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and*

$$\begin{cases} z_n = Q_C(I - \sigma B)x_n \\ y_n = Q_C(I - \lambda A)(tx_n + (1-t)z_n), \\ x_{n+1} = Q_C[\alpha_n \gamma L_1 x_n + \gamma_n x_n \\ +((1-\gamma_n)I - \alpha_n \mu L_2) T y_n], \end{cases} \qquad (3.25)$$

which corresponds to the conditions:
(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$; and $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$;
(C2) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$,
$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.
Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves the following variational inequality:
$$\langle \gamma L_1 x^* - \mu L_2 x^*, J(z - x^*) \rangle \leq 0, \forall z \in F.$$

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