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An Observation on the Natural Partial Order of Transformation Semigroups Restricted by an Equivalence Relation

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Abstract

Let $T(X)$ be the full transformation semigroup of a set X. We consider the subsemigroup of $T(X)$ defined by $E(X,\sigma)=\{\alpha\in T(X): \forall x,y\in X, (x,y)\in\sigma \text{ implies } x\alpha=y\alpha\}$ for an arbitrary equivalence relation σ on X. The natural partial order on the largest regular subsemigroup of $E(X,\sigma)$ is discussed in this paper, and we characterize when two regular elements of $E(X,\sigma)$ are related under this order. Also, their maximal, minimal and covering elements are described.

Keywords: Natural partial order, Maximal element, Minimal element, Covering element

1. Introduction

An element x of a semigroup S is said to be *regular* if there exists y in S such that $xyx=x$. The semigroup S is called a *regular semigroup* if all its elements are regular, and which is said to be an *inverse semigroup* if each of its idempotents commutes. Hence, every inverse semigroup is regular.

In 1952, Vagner (1) defined the natural partial order for any inverse semigroup S by defining $\leq$ on S as follows:

$$a\leq b \text{ if and only if } a=be \qquad (1.1)$$

for some idempotent $e\in S$.

Later, Nambooripad (2) extended this partial order $\leq$ on a regular semigroup S, which is defined by

$$a\leq b \text{ if and only if } a=eb=bf \qquad (1.2)$$

for some idempotents $e,f\in S$. For an inverse semigroup S, this relation is just the natural partial order (1.1).

In 1986, Mitsch (3) extended the partial order above to any semigroup S by defining $\leq$ on S as follows:

$$a\leq b \text{ if and only if } a=xb=by \text{ and } a=ay \qquad (1.3)$$

for some $x,y\in S^1$, where S^1 is the semigroup with identity obtained from S by adjoining an identity if necessary. This natural partial order coincides with the relation (1.2) if the semigroup S is regular.

Let X be a non-empty set. As usual, $T(X)$ denotes the semigroup (under composition) of all full transformations of X (that is, all mappings $\alpha: X\to X$). As early as 1955, Doss (4) proved that $T(X)$ is a regular semigroup and described its Green's relations. It is well-known that every semigroup is isomorphic to a subsemigroup of some full transformation semigroups (see(5)). Hence in order to study structure of semigroups, it suffices to consider in subsemigroups of $T(X)$.

Many papers describing the natural partial order on various subsemigroup of $T(X)$

can be found in (6-9). For examples, Kowol and Mitsch (8) studied various properties of this partial order on $T(X)$. For an equivalence σ on X, Sun, Pei and Cheng (7) characterized the partial order on the subsemigroup of $T(X)$ consisting of all $\alpha \in T(X)$ that preserve σ (that is, for all $x, y \in X$, if $(x, y) \in \sigma$, then $(x\alpha, y\alpha) \in \sigma$. Namnak and Laysirikul (9) discussed the natural partial order on the self-σ-preserving transformation semigroup (that is, for all $x \in X, (x, x\alpha) \in \sigma$).

Mendes-Goncalves and Sullivan (10) introduced a subsemigroup of $T(X)$ defined by $E(X,\sigma) = \{\alpha \in T(X): \forall x, y \in X, (x, y) \in \sigma$ implies $x\alpha = y\alpha\}$ where σ is an equivalence relation on a non-empty set X and call it the *semigroup of trans- formations restricted by an equivalence* σ. The authors characterized Green's relations on the largest regular subsemigroup of $E(X,\sigma)$. They also showed that if $|X| \geq 2$ and $\sigma \neq I_X = \{(x, x): x \in X\}$, then $E(X,\sigma)$ is not isomorphic to $T(Z)$ for any set Z. Sun and Wang (11) proved that $E(X,\sigma)$ is right abundant but not left abundant whenever the equivalence σ on the set $X(|X| \geq 3)$ is non-trivial, namely, $\sigma = I_X$ and $\sigma = X \times X$. Recently, Sawatraksa and Namnak investigated some conditions for isomorphism between two semigroups of type $E(X,\sigma)$ in (12). Han and Sun (6) described the semigroup $E(X,\sigma)$ with the natural partial order $\leq$ and provided a characterization for $\leq$. Moreover, they proved necessary and sufficient conditions for $\leq$ to be both left and right compatible with the multiplication and described the minimal and the maximal elements of $E(X,\sigma)$ with respect to this order. Sawatraksa, Namnak and Laysirikul (13) characterized the left regularity, the right regularity and the completely regularity for elements of $E(X,\sigma)$. Moreover, they presented a necessary and sufficient condition for the semigroup $E(X,\sigma)$ when it is left regular, right regular and completely regular.

In the remainder, let σ be an equivalence relation on a set X and $\mathbb{E}$ be the set of all regular elements in $E(X,\sigma)$. The purpose of this paper is to investigate the natural partial order on $\mathbb{E}$ and characterize when two elements of $\mathbb{E}$ are related under this order. Also, their maximal, minimal and covering elements are described.

In this introductory section, we present a variety of notations and Theorems, the most of which will be indispensable for our research.

Theorem 1.1 (10) Let $\alpha \in E(X,\sigma)$. Then α is regular if and only if $|A \cap X\alpha| \leq 1$ for all $A \in X/\sigma$.

Theorem 1.2. The set $\mathbb{E}$ of all regular elements in $E(X,\sigma)$ forms a regular semigroup.

Proof. Let $\alpha, \beta \in \mathbb{E}$. Then $X\alpha\beta \subseteq X\beta$. For each $A \in X/\sigma$, we have that $A \cap X\alpha\beta \subseteq A \cap X\beta$. By Theorem 1.1, $|A \cap X\beta| \leq 1$ which implies that $|A \cap X\alpha\beta| \leq 1$. Hence by Theorem 1.1, we have $\alpha\beta$ is a regular element of $E(X,\sigma)$. Accordingly, $\mathbb{E}$ is a subsemigroup of $E(X,\sigma)$. Let $\alpha \in \mathbb{E}$. For each $x \in X\alpha$, we choose and fix an element $x' \in X$ such that $x = x'\alpha$. Let $A \in X/\sigma$ be such that $A \cap X\alpha \neq \emptyset$. By Theorem 1.1, there exists a unique element $x_A \in A \cap X\alpha$. Define $\mu_A: A \to A$ by $x\mu_A = x'_A$ for all $x \in A$. Let $\mu: X \to X$ be a mapping such that

$$\mu|_A = \begin{cases} \mu_A, & \text{if } A \cap X\alpha \neq \emptyset \\ c_A, & \text{otherwise} \end{cases}$$

for all $A \in X/\sigma$ where c_A is the constant mapping from A into $\{x'_B: B \in X/\sigma$ with $B \cap X\alpha \neq \emptyset$ and $x'_B\alpha = x_B\}$. Since X/σ is a partition of X, μ is well-defined. Obviously, $\mu \in E(X,\sigma)$. Since $X\mu = \{x'_A \in X : A \in X/\sigma$ and $A \cap X\alpha \neq \emptyset\}$ and α is regular, by Theorem 1.1, we have that $\mu \in \mathbb{E}$. For each $x \in X$, we get that $x\alpha \in A$ for some $A \in X/\sigma$. Thus $x\alpha\mu\alpha = x'_{A\alpha} = x_A$. Since $x\alpha, x_A \in A \cap X\alpha$ and by Theorem 1.1, it follows that $x\alpha = x_A$ and so α is a regular element of $\mathbb{E}$. Hence $\mathbb{E}$ is a regular semigroup.

2. Main results

In this section, we begin to present the natural partial order on $E(X,\sigma)$. For each $\alpha \in E(X,\sigma)$, let $\sigma(\alpha) = \{A\alpha^{-1}: A \in X/\sigma$ and $A\alpha^{-1} \neq \emptyset\}$. Then $\sigma(\alpha)$ is a partition of X. Let $Y \subseteq X$ and denote

$$\overline{Y} = \bigcup_{A \in X/\sigma, A \cap Y \neq \emptyset} A.$$

When $Y = \{x\}$ for some $x \in X$, we denote $\overline{Y}$ by $\overline{x}$. Clearly, $\overline{x}$ is the σ-class of x.

Han and Sun (6) described the semigroup $E(X,\sigma)$ with the natural partial order $\leq$ as follows.

Theorem 2.1(6) Let $\alpha, \beta \in E(X, \sigma)$. Then $\alpha \leq \beta$ on $E(X, \sigma)$ if and only if either $\alpha = \beta$ or the following statements hold.
(i) $|U\alpha| = 1|$ for all $U \in \sigma(\beta)$.
(ii) For each $x \in X$, if $x\beta \in X\alpha$, then $\bar{x}\alpha = \bar{x}\beta$.
(iii) For all $x \in X$, there exists $y \in X$ such that $\bar{x}\alpha = \bar{y}\beta$.

Next, we present the new conditions for check two elements of $E(X, \sigma)$ are related under natural partial order.

Theorem 2.2 Let $\alpha, \beta \in E(X, \sigma)$. Then $\alpha \leq \beta$ on $E(X, \sigma)$ if and only if either $\alpha = \beta$ or the following statements hold.
(i) For every $x \in X, x\beta \in X\alpha$ implies $x\beta = x\alpha$.
(ii) Whenever $A \in X/\sigma$, there exists $x \in X$ such that $A\alpha = \{x\beta\}$.
(iii) For all $x, y \in X, (x\beta, y\beta) \in \sigma$ implies $x\alpha = y\alpha$.
Proof. Assume that $\alpha \leq \beta$. Then there exist $\lambda, \mu \in E(X, \sigma)$ such that $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$. Let $x \in X$ be such that $x\beta \in X\alpha$. Then there is $x' \in X$ such that $x\beta = x'\alpha$. This implies that $x\beta = x'\alpha = x'\alpha\mu = x\beta\mu = x\alpha$. This proves that (i) holds. Let $A \in X/\sigma$. Then there exists $x \in X$ such that $A\lambda = \{x\}$. Therefore $A\alpha = A\lambda\beta = \{x\beta\}$. Hence (ii) holds. Let $x, y \in X$. If $(x\beta, y\beta) \in \sigma$, then since $\alpha = \beta\mu$ and $\mu \in E(X, \sigma)$, we deduce that $x\alpha = x\beta\mu = y\beta\mu = y\alpha$. Hence (iii) holds. Suppose that the converse conditions hold. Let $U \in \sigma(\beta)$ and $x, y \in U\alpha$. Then $x = x'\alpha$ and $y = y'\alpha$ for some $x', y' \in U$. By the definition of U, we have $(x'\beta, y'\beta) \in \sigma$. It follows from (iii) that $x = x'\alpha = y'\alpha = y$. This implies that $|U\alpha| = 1$ for all $U \in \sigma(\beta)$. From (i) and (ii), we see that the conditions (ii) and (iii) of Theorem 2.1 hold, respectively. Applying Theorem 2.1, we conclude that $\alpha \leq \beta$.

The following conclusion readily follows from Theorem 2.2.

Corollary 2.3. Let $\alpha, \beta \in E(X, \sigma)$ be such that $\alpha \leq \beta$ on $E(X, \sigma)$. Then the following statements hold.
(i) $X\alpha \subseteq X\beta$.
(ii) $\ker \beta \subseteq \ker \alpha$.
(iii) If $X\alpha = X\beta$, then $\alpha = \beta$.

A necessary and sufficient condition for two elements of $\mathbb{E}$ are related under natural partial order follows from Theorem 2.2 and 1.1.

Corollary 2.4. Let $\alpha, \beta \in \mathbb{E}$. Then $\alpha \leq \beta$ on $\mathbb{E}$ if and only if either $\alpha = \beta$ or the following statements hold.
(i) For all $x \in X, x\beta \in X\alpha$ implies $x\beta = x\alpha$.
(ii) For every $A \in X/\sigma$, there exists $x \in X$ such that $A\alpha = \{x\beta\}$.
(iii) $\ker \beta \subseteq \ker \alpha$.

Let S be a semigroup. An element $a \in S$ is said to be *minimal* if $b \leq a$ implies that $b = a$ for all $b \in S$, and an element $a \in S$ is said to be *maximal* if $a \leq b$ implies that $a = b$ for all $b \in S$.

Lemma 2.5. Every non-regular element of $E(X, \sigma)$ is a maximal element of $E(X, \sigma)$.
Proof. Assume that α is not regular. Let $\beta \in E(X, \sigma)$ be such that $\alpha \leq \beta$. Suppose that $\alpha \neq \beta$. By assumption and Theorem 1.1, there exists $A \in X/\sigma$ such that $|A \cap X\alpha| \geq 2$. Let $a, b \in A \cap X\alpha$ be distinct. Then by Corollary 2.3(i), we have $a = a'\beta$ and $b = b'\beta$ for some $a', b' \in X$. It follows from Theorem 2.2(iii) that $a = a'\alpha = b'\alpha = b$. This is a contradiction. Hence $\alpha = \beta$. This means that α is a maximal element of $E(X, \sigma)$.

Recall that two elements a and a in a semigroup S is *incomparable* if $a \nleq b$ and $b \nleq a$.

Corollary 2.6. Every two non-regular element of $E(X, \sigma)$ is incomparable.

The next lemma is quote from (6).

Lemma 2.7. (6) Let $\alpha \in E(X, \sigma)$. Suppose that for each $A, B \in X/\sigma$, $A\alpha = B\alpha$ implies $A = B$. Then α is a maximal element of $E(X, \sigma)$.

Lemma 2.8. Let $\alpha \in E(X, \sigma)$. Suppose that $A \cap X\alpha \neq \emptyset$ for all $A \in X/\sigma$ and for each $A, B \in X/\sigma$, $A\alpha = B\alpha$ with $A \neq B$ implies $A\alpha \subseteq X\alpha$. Then α is a maximal element of $E(X, \sigma)$.
Proof. Let $\beta \in E(X, \sigma)$ be such that $\alpha \leq \beta$. Suppose that $\alpha \neq \beta$. Then $x\alpha \neq x\beta$ for some $x \in X$. By (i) of Theorem 2.2, $x\beta \notin X\alpha$. Let $A \in X/\sigma$ be such that $x\beta \in A$. Since $A \cap X\alpha \neq \emptyset$, we let $x' \in X$ be such that $x'\alpha \in A$. Then by (ii) of Theorem 2.2, we have $x'\alpha = x''\beta$ for some $x'' \in X$. From Theorem 2.2 (i), we obtain that $x''\beta = x''\alpha$. Since $(x\beta, x''\beta) = (x\beta, x'\alpha) \in \sigma$ and by (iii) of Theorem 2.2, we deduce that $x\alpha = x''\alpha$. Let $B, B' \in X/\sigma$ be such that $x \in B$ and $x'' \in B'$. If $B = B''$, then $x\beta = x''\beta = x''\alpha = x\alpha$ which leads to a contradiction. Hence $B \neq B'$. Since $x''\alpha = x'\alpha \in A$, $B'\alpha \subseteq A$. From $B\alpha = B'\alpha$ and by the hypothesis, it follows that $A \subseteq X\alpha$. This means that $x\beta \in X\alpha$. This is a contradiction. Hence $\alpha = \beta$. Consequently, α is maximal.

Theorem 2.9. Let $\alpha \in E(X,\sigma)$. Then α is a maximal element of $E(X,\sigma)$ if and only if one of the following statements holds:
(i) α is non-regular.
(ii) For every $A, B \in X/\sigma, A\alpha = B\alpha$ implies $A = B$.
(iii) $A \cap X\alpha \neq \emptyset$ for all $A \in X/\sigma$ and for every $A, B \in X/\sigma, A\alpha = B\alpha$ with $A \neq B$ implies $A\alpha \subseteq X\alpha$.

Proof. Assume that α is a maximal and a regular element of $E(X,\sigma)$ and $A\alpha = B\alpha$ where $A, B \in X/\sigma$ with $A \neq B$. Suppose that $C \cap X\alpha = \emptyset$ for some $C \in X/\sigma$. Let $c \in C$. Then $c \notin X\alpha$. Define $\beta: X \to X$ by

$$x\beta = \begin{cases} c, & \text{if } x \in A, \\ x\alpha, & \text{otherwise.} \end{cases}$$

Since $\alpha \in E(X,\sigma)$, we deduce that $\beta \in E(X,\sigma)$. Obviously, $\alpha \neq \beta$. Next, we will show that $\alpha < \beta$ by using Theorem 2.2. For each $x \in X$, if $x\beta \in X\alpha$, then $x\beta \neq c$. From the definition of β, we have $x \notin A$ and so $x\beta = x\alpha$. Hence (i) holds. Let $D \in X/\sigma$. If $D \neq A$, then $x\beta = x\alpha$ for all $x \in D$ and thus $D\alpha = \{d\beta\}$ for some $d \in D$. If $D = A$, then $D\alpha = A\alpha = B\alpha$. We choose $b \in B$. From $B \neq A$, we have that $b\beta = b\alpha$. This implies that $D\alpha = B\alpha = \{b\alpha\} = \{b\beta\}$. Hence (ii) holds. Let $a, b \in X$ be such that $(a\beta, b\beta) \in \sigma$. We distinguish three cases as follows.
Case 1. $a, b \in A$. Then $a\alpha = b\alpha$.
Case 2. $a \in A$ and $b \notin A$. Then $a\beta = c$ and $b\beta = b\alpha$. Since $(c, b\alpha) = (a\beta, b\beta) \in \sigma$, $b\alpha \in C$ which is a contradiction with $C \cap X\alpha = \emptyset$. Hence this case is impossible.
Case 3. $a \notin A$ and $b \notin A$. Then $a\beta = a\alpha$ and $b\beta = b\alpha$. Since $(a\alpha, b\alpha) = (a\beta, b\beta) \in \sigma$ and by Theorem 1.1, it follows that $a\alpha = b\alpha$.
We conclude that $\alpha < \beta$ by Theorem 2.2, which is a contradiction with maximality of α. Hence $C \cap X\alpha \neq \emptyset$ for all $C \in X/\sigma$. Let $A_1, A_2 \in X/\sigma$ be such that $A_1\alpha = A_2\alpha$ with $A_1 \neq A_2$. Suppose that $\overline{A_1\alpha} \nsubseteq X\alpha$. Then there exists $y \in \overline{A_1\alpha}/X\alpha$. Define $\gamma: X \to X$ by

$$x\gamma = \begin{cases} y, & \text{if } x \in A_1, \\ x\alpha, & \text{otherwise.} \end{cases}$$

Then $\gamma \in E(X,\sigma)$ and $\alpha \neq \gamma$. Similar to above proof, α and γ satisfy the conditions (i) and (ii) in Theorem 2.2. Next, we will show that α and γ satisfy the condition (iii) in Theorem 2.2. Let $a, b \in X$ be such that $(a\gamma, b\gamma) \in \sigma$. We distinguish three cases as follows.
Case 1. $a, b \in A_1$. Then $a\alpha = b\alpha$.
Case 2. $a \in A_1$ and $b \notin A_1$. Then $a\gamma = y$ and $b\gamma = b\alpha$. Since $(y, b\alpha) = (a\gamma, b\gamma) \in \sigma, b\alpha \in A_1\alpha$. Therefore $(a\alpha, b\alpha) \in \sigma$. It follows from Theorem 1.1 that $a\alpha = b\alpha$.
Case 3. $a \notin A_1$ and $b \notin A_1$. Then $a\beta = a\alpha$ and $b\beta = b\alpha$. From $(a\alpha, b\alpha) = (a\beta, b\beta) \in \sigma$ and by Theorem 1.1, we have $a\alpha = b\alpha$.
This implies that $\alpha < \gamma$ Theorem 2.2. This is a contradiction with maximality of α. Hence $A_1\alpha \subseteq X\alpha$.

The converse of Theorem follows from Lemmas 2.5, 2.7 and 2.8.

Corollary 2.10. Let $\alpha \in \mathbb{E}$. Then α is a maximal element of $\mathbb{E}$ if and only if one of the following statements holds:
(i) For every $A, B \in X/\sigma, A\alpha = B\alpha$ implies $A = B$.
(ii) $A \cap X\alpha \neq \emptyset$ for all $A \in X/\sigma$ and for every $A, B \in X/\sigma, A\alpha = B\alpha$ with $A \neq B$ implies $A\alpha \subseteq X\alpha$.

Proof. Assume that α is a maximal and $A\alpha = B\alpha$ where $A, B \in X/\sigma$ with $A \neq B$. Suppose that $C \cap X\alpha = \emptyset$ for some $C \in X/\sigma$. Let β be defined as in the proof of Theorem 2.9. Then $\beta \in E(X,\sigma)$ and $\alpha < \beta$. By the definition of β, we have $C \cap X\beta = \{c\}$ and $D \cap X\beta = D \cap X\alpha$ for all $D \in X/\sigma \backslash \{C\}$. By the regularity of $\alpha, |D \cap X\beta| = |D \cap X\alpha| \leq 1$ for all $D \in X/\sigma \backslash \{C\}$. This implies that $\beta \in \mathbb{E}$ by Theorem 1.1. This is a contradiction with the maximality of α. Hence $C \cap X\alpha \neq \emptyset$ for all $C \in X/\sigma$. Let $A_1, A_2 \in X/\sigma$ be such that $A_1\alpha = A_2\alpha$ with $A_1 \neq A_2$. Suppose that $\overline{A_1\alpha} \subseteq X\alpha$. Let γ be defined as in the proof of Theorem 2.9. Then $\gamma \in E(X,\sigma)$ and $\alpha < \gamma$. Similar to above proof, we get that $\gamma \in \mathbb{E}$. This is a contradiction with maximality of α. Hence $A_1\alpha \subseteq X\alpha$.

The converse of theorem follows form Theorem 2.9.

Next, we describe the minimal elements in the semigroup $\mathbb{E}$.

Theorem 2.11.(6) Let $\alpha \in E(X,\sigma)$. Then α is a maximal element of $E(X,\sigma)$ if and only if α is constant.

Corollary 2.12. Let $\alpha \in \mathbb{E}$. Then α is a minimal element of $\mathbb{E}$ if and only if α is constant.

Proof. Assume that α is minimal. Let $y_1, y_2 \in X\alpha$. Define $\beta_1, \beta_2 \in \mathbb{E}$ by $x\beta_1 = y_1$ for all $x \in X$

and $x\beta_2 = y_2$ for all $x \in X$. By Theorem 2.11, it implies that $\beta_1 \leq \alpha$ and $\beta_2 \leq \alpha$. From assumption, we get that $\beta_1 = \alpha = \beta_2$. Therefore $y_1 = y_2$. Hence α is constant. The converse of corollary follows from Theorem 2.11.

Let ρ be a partial order on a semigroup S. An element $b \in S$ is called an *upper cover* for $a \in S$ if $(a, b) \in \rho$ and there exists no $c \in S$ such that $(a, c) \in \rho$ and $(c, b) \in \rho$. A *lower cover* is defined dually.

Finally, the following results are concerned with the existence of an upper cover and a lower cover for elements of $E(X, \sigma)$ and $\mathbb{E}$.

Theorem 2.13. Let $\alpha \in E(X, \sigma)$. If α is not a maximal element of $E(X, \sigma)$, then α has an upper cover.

Proof. Suppose that α is not maximal. Then by Theorem 2.9, we have α is regular and $A\alpha = B\alpha$ for some $A, B \in X/\sigma$ with $A \neq B$. We consider two cases as follows.

Case 1. $C \cap X\alpha = \emptyset$ for some $C \in X/\sigma$. Let β be defined as in the proof of Theorem 2.9. Then $\alpha < \beta$. Suppose that $\alpha < \lambda \leq \beta$ for some $\lambda \in E(X, \sigma)$. By Corollary 2.3 (i), we have $X\alpha \subset X\lambda \subseteq X\beta = X\alpha \cup \{c\}$. This implies that $X\lambda = X\beta$ and so $\lambda = \beta$ from (iii) in Corollary 2.3. Hence β is an upper cover of α.

Case 2. $A_1\alpha = A_2\alpha$ where $A_1, A_2 \in X/\sigma$ with $A_1 \neq A_2$ and $\overline{A_1\alpha} \nsubseteq X\alpha$. Let γ be defined as in the proof of Theorem 2.9. Similar to the Case 1., we obtain that γ is an upper cover of α.

The proof of the next result is similar to Theorem 2.13.

Theorem 2.14. Let $\alpha \in \mathbb{E}$. If α is not a maximal element of $\mathbb{E}$, then α has an upper cover.

Theorem 2.15. Every non-regular element of $E(X, \sigma)$ has a lower cover.

Proof. Let α be a non-regular element of $E(X, \sigma)$. For each $A \in X/\sigma$ with $|A \cap X\alpha| \geq 2$, we choose and fix an element $x_A \in A \cap X\alpha$. Define $\beta_A : A\alpha^{-1} \to X$ by $x\beta_A = x_A$ for all $x \in A\alpha^{-1}$. Let $\beta : X \to X$ be a mapping defined by

$$\beta|_{A^{-1}} = \begin{cases} \beta_A, & \text{if } |A \cap X\alpha| \geq 2, \\ \alpha|_{A\alpha^{-1}}, & \text{if } |A \cap X\alpha| = 1, \end{cases}$$

for all $A \in X/\sigma$ with $A \cap X\alpha \neq \emptyset$. Then $\beta \in E(X, \sigma)$ and $|A \cap X\beta| \leq 1$ for all $A \in X/\sigma$. By Theorem 1.1, β is regular. Obviously, $X\beta \subseteq X\alpha$. Since α is non-regular and by Theorem 1.1, $|A \cap X\alpha| \geq 2$ for some $A \in X/\sigma$. Thus $X\beta \neq X\alpha$. This implies that $\alpha \neq \beta$.

Next, we will prove that $\beta < \alpha$ by applying Theorem 2.2. Let $x \in X$ be such that $x\alpha \in X\beta$. Thus $x\alpha \in B$ for some $B \in X/\sigma$. If $|B \cap X\alpha| = 1$, then $x\beta = x\alpha$. If $|B \cap X\alpha| \geq 2$, then $x\beta = x_B$ with $x_B \in B$. Since $x_B, x\alpha \in B \cap X\beta$ and by Theorem 1.1, we deduce that $x\alpha = x_B = x\beta$. Hence (i) holds. Let $C \in X/\sigma$. Then $C\alpha \subseteq C'$ for some $C' \in X/\sigma$. If $|C' \cap X\alpha| = 1$, then $x\beta = x\alpha$ for all $x \in C$. Therefore $C\beta = \{x\alpha\}$ for some $x \in X$. If $|C' \cap X\alpha| \geq 2$, then $C\beta = \{xC'\}$ where $xC' \in C' \cap X\alpha$. Since $xC' \in X\alpha$, there exists $y \in X$ such that $C\beta = \{xC'\} = \{xC'\} = \{y\alpha\}$. This implies that (ii) holds. Let $a, b \in X$ be such that $(a\alpha, b\alpha) \in \sigma$. Then $a\alpha, b\alpha \in D$ for some $D \in X/\sigma$. If $|D \cap X\alpha| = 1$, then $a\beta = a\alpha = b\alpha = b\beta$. If $|D \cap X\alpha| \geq 2$, then $a\beta = x_D = b\beta$ with $x_D \in B$. Hence (iii) holds. We conclude that $\beta < \alpha$.

Finally, we will prove that β is a lower cover of α. Let $\gamma \in E(X, \sigma)$ be such that $\beta \leq \gamma < \alpha$. By Corollary 2.3 (i), $X\beta \subseteq X\gamma \subseteq X\alpha$. Let $x \in X$. Then $x\alpha \in A$ for some $A \in X/\sigma$. If $|A \cap X\alpha| = 1$, then $x\beta = x\alpha$, so that $x\alpha \in X\gamma$ and by Theorem 2.2(i), we have that $x\gamma = x\alpha = x\beta$. If $|A \cap X\alpha| \geq 2$, then $x\beta = x_A$. Since $x_A \in X\alpha$, $x_A = x'\alpha$ for some $x' \in X$. From $(x\alpha, x'\alpha) = (x\alpha, x_A) \in \sigma$ and by (iii) of Theorem 2.2, we have that $x\gamma = x'\gamma$. Since $x'\alpha = x'\alpha = x_A \in X\beta \subseteq X\gamma$ and by (i) of Theorem 2.2, we deduce that $x'\gamma = x'\alpha$. This implies that $x\beta = x\gamma$ and hence $\beta = \gamma$. We conclude that β is a lower cover of α, as required.

Theorem 2.16. Let α be a regular element in $E(X, \sigma)$. If α is not a minimal element of $E(X, \sigma)$, then α has a lower cover.

Proof. If α is not minimal, then α is non-constant. Let $y_1, y_2 \in X\alpha$ be distinct. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} y_1, & \text{if } x\alpha = y_2, \\ x\alpha, & \text{otherwise.} \end{cases}$$

Then $\beta \in E(X, \sigma)$ and $\alpha \neq \beta$. Now we will show that $\beta < \alpha$ by using Theorem2.2. For each $x \in X$, if $x\alpha \in X\beta = X\alpha \setminus \{y_2\}$, then $x\alpha \neq y_2$. Thus $x\beta = x\alpha$. Hence (i) holds. Let $A \in X/\sigma$. If $A\alpha = \{y_2\}$, then $A\beta = \{y_1\}$. Since $y_1 \in X\alpha$, there exists $x \in X$ such that $A\beta = \{y_1\} = \{x\alpha\}$. If $A\alpha \neq \{y_2\}$, then $A\beta = A\alpha$ and so $A\beta = \{x\alpha\}$ for some $x \in A$. Therefore (ii) holds. Let $x, y \in$

X be such that $(x\alpha, y\alpha) \in \sigma$. Then by Theorem 1.1, we obtain that $x\alpha = y\alpha$. If $x\alpha = y\alpha = y_2$, then $x\beta = y_1 = y\beta$. If $x\alpha = y\alpha \neq y_2$, then $x\beta = x\alpha = y\alpha = y\beta$. We conclude that (iii) holds. Hence $\beta < \alpha$.

Finally, to show that β is a lower cover of α, let $\gamma \in E(X,\sigma)$ be such that $\beta \leq \gamma < \alpha$. Then by Corollary 2.3(i), $X\alpha\setminus\{y_2\} = X\beta \subseteq X\gamma \subset X\alpha$, which implies that $X\beta = X\gamma$. It follows from (iii) in Corollary 2.3 that $\beta = \gamma$. Consequently, α has a lower cover.

Theorem 2.17. Let $\alpha \in \mathbb{E}$. If α is not a minimal element of $\mathbb{E}$, then α has a lower cover.

Proof. If α is not minimal, then α is non-constant. Let $y_1, y_2 \in X\alpha$ be distinct. Let β be defined as in the proof of Theorem 2.16. Then $\beta \in E(X,\sigma)$ is a lower cover of α. It remains to show that $\beta \in \mathbb{E}$. For each $A \in X/\sigma$, we have $A \cap X\beta = A \cap (X\alpha\setminus\{y_2\}) \subseteq A \cap X\alpha$. By Theorem 1.1, we deduce that $|A \cap X\beta| = |A \cap (X\alpha\setminus\{y_2\})| \leq |A \cap X\alpha| = 1$. Hence $\beta \in \mathbb{E}$.

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