

Research Article

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On the Diophantine Equation $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{1}{2}$

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Abstract

In this paper, we find all solutions of the Diophantine equation $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{1}{2}$ where x, y and z are positive integers.

Keywords: Diophantine equation

1. Introduction

A Diophantine equation is an equation in which only integer solutions are allowed. The researchers about Diophantine equations are ancient, various and no general method exists. The one well-known problem $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{n}$ has been studied by many researchers, see (1-3, 6).

In 2013, Rabago and Tagle (3) studied some elementary problems involving surface area and volume of a certain regular solid. This problem leads them to solving the specific case of Diophantine problem $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{n}$, i.e. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}$. Later, Sandor (5) offered an elementary approach to the solutions of the Diophantine equation $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}$.

In 2019, Pakapongpun (4) found positive integral solutions of the Diophantine equation $\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = \frac{1}{2}$.

In this paper, we find all solutions of the Diophantine equation $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{1}{2}$ where x, y and z are positive integers.

2. Solutions of the Diophantine equation

$$\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{1}{2}$$

From the Diophantine equation $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{1}{2}$, where x, y and z are positive integers, we get $\frac{2}{x} < \frac{1}{2}$, $\frac{3}{y} < \frac{1}{2}$ and $\frac{4}{z} < \frac{1}{2}$. This implies that $x \geq 5$, $y \geq 7$ and $z \geq 9$.

We consider the following 3 cases.

Case 1 : $x \leq y \leq z$ or $x \leq z \leq y$.

If $x \leq y \leq z$, then $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} \leq \frac{9}{x}$.

If $x \leq z \leq y$, then $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} \leq \frac{9}{x}$.

Therefore, $\frac{1}{2} = \frac{2}{x} + \frac{3}{y} + \frac{4}{z} \leq \frac{9}{x}$ which implies that $x \leq 18$. From $x \geq 5$ and $x \leq 18$, we get $5 \leq x \leq 18$. If $x = 5$, then $\frac{3}{y} + \frac{4}{z} = \frac{1}{10}$. We have

$$(y - 30)(z - 40) = 1200.$$

We use the function τ to ensure that the total number of divisors of 1200 is 30. Using factors of 1200, we have

$$y - 30 = 1, z - 40 = 1200;$$

$$y - 30 = 2, z - 40 = 600;$$

$$y - 30 = 3, z - 40 = 400;$$

$$y - 30 = 4, z - 40 = 300;$$

$$y - 30 = 5, z - 40 = 240;$$

$$\begin{aligned}
y - 30 &= 6, z - 40 = 200; \\
y - 30 &= 8, z - 40 = 150; \\
y - 30 &= 10, z - 40 = 120; \\
y - 30 &= 12, z - 40 = 100; \\
y - 30 &= 15, z - 40 = 80; \\
y - 30 &= 16, z - 40 = 75; \\
y - 30 &= 20, z - 40 = 60; \\
y - 30 &= 24, z - 40 = 50; \\
y - 30 &= 25, z - 40 = 48; \\
y - 30 &= 30, z - 40 = 40; \\
y - 30 &= 40, z - 40 = 30; \\
y - 30 &= 48, z - 40 = 25; \\
y - 30 &= 50, z - 40 = 24; \\
y - 30 &= 60, z - 40 = 20; \\
y - 30 &= 75, z - 40 = 16; \\
y - 30 &= 80, z - 40 = 15; \\
y - 30 &= 100, z - 40 = 12; \\
y - 30 &= 120, z - 40 = 10; \\
y - 30 &= 150, z - 40 = 8; \\
y - 30 &= 200, z - 40 = 6; \\
y - 30 &= 240, z - 40 = 5; \\
y - 30 &= 300, z - 40 = 4; \\
y - 30 &= 400, z - 40 = 3; \\
y - 30 &= 600, z - 40 = 2; \\
y - 30 &= 1200, z - 40 = 1.
\end{aligned}$$

The solutions (x, y, z) are (5,31,1240), (5,32,640), (5,33,440), (5,34,340), (5,35,280), (5,36,240), (5,38,190), (5,40,160), (5,42,140), (5,45,120), (5,46,115), (5,50,100), (5,54,90), (5,55,88), (5,60,80), (5,70,70), (5,78,65), (5,80,64), (5,90,60), (5,105,56), (5,110,55), (5,130,52), (5,150,50), (5,180,48), (5,230,46), (5,270,45), (5,330,44), (5,430,43), (5,630,42), (5,1230,41).

In the same manner,

if $x = 6$, then $\frac{3}{y} + \frac{4}{z} = \frac{1}{6}$. We have

$$(y - 18)(z - 24) = 432.$$

The solutions (x, y, z) are (6,19,456), (6,20,240), (6,21,168), (6,22,132), (6,24,96), (6,26,78), (6,27,72), (6,30,60), (6,34,51), (6,36,48), (6,42,42), (6,45,40), (6,54,36), (6,66,33), (6,72,32), (6,90,30), (6,126,28), (6,162,27), (6,234,26), (6,450,25).

If $x = 7$, then $\frac{3}{y} + \frac{4}{z} = \frac{3}{14}$. We have

$$(y - 14)(3z - 56) = 784.$$

The solutions (x, y, z) are (7,15,280), (7,18,84), (7,21,56), (7,30,35), (7,42,28), (7,63,24), (7,126,21), (7,210,20), (7,798,19).

If $x = 8$, then $\frac{3}{y} + \frac{4}{z} = \frac{1}{4}$. We have

$$(y - 12)(z - 16) = 192.$$

The solutions (x, y, z) are (8,13,208), (8,14,112), (8,15,80), (8,16,64), (8,18,48), (8,20,40), (8,24,32), (8,28,28), (8,36,24), (8,44,22), (8,60,20), (8,76,19), (8,108,18), (8,204,17).

If $x = 10$, then $\frac{3}{y} + \frac{4}{z} = \frac{3}{10}$. We have

$$(y - 10)(3z - 40) = 400.$$

The solutions (x, y, z) are (10,12,80), (10,15,40), (10,18,30), (10,30,20), (10,60,16), (10,90,15), (10,210,14).

If $x = 12$, then $\frac{3}{y} + \frac{4}{z} = \frac{1}{3}$. We have

$$(y - 9)(z - 12) = 108.$$

Since $x \leq y \leq z$ or $x \leq z \leq y$, the solutions (x, y, z) are (12,12,48), (12,13,39), (12,15,30), (12,18,24), (12,21,21), (12,27,18), (12,36,16), (12,45,15), (12,63,14), (12,117,13).

If $x = 16$, then $\frac{3}{y} + \frac{4}{z} = \frac{3}{8}$. We have

$$(y - 8)(3z - 32) = 256.$$

Since $x \leq y \leq z$ or $x \leq z \leq y$, the solution (x, y, z) is (16,24,16).

If $x = 9$, then $\frac{3}{y} + \frac{4}{z} = \frac{5}{18}$.

Case $z \leq y$. We get $\frac{5}{18} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{z}$, so $9 \leq z \leq 25$.

Since y is a positive integer, the solutions (x, y, z) are (9,270,15), (9,108,16), (9,54,18), (9,27,24).

Case $y \leq z$. We get $\frac{5}{18} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{y}$, so $9 \leq y \leq 25$.

Since z is a positive integer, the solutions (x, y, z) are (9,11,792), (9,12,144), (9,14,63), (9,18,36).

If $x = 11$, then $\frac{3}{y} + \frac{4}{z} = \frac{7}{22}$.

Case $z \leq y$. We get $\frac{7}{22} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{z}$, so $11 \leq z \leq 22$.

Since y is a positive integer, the solutions (x, y, z) are (11,286,13), (11,44,16), (11,22,22).

Case $y \leq z$. We get $\frac{7}{22} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{y}$, that is $11 \leq y \leq 22$.

Since z is a positive integer, the solutions (x, y, z) are (11,11,88), (11,22,22).

If $x = 13$, then $\frac{3}{y} + \frac{4}{z} = \frac{9}{26}$.

Case $z \leq y$. We get $\frac{9}{26} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{z}$, so $13 \leq z \leq 20$.

Since y is a positive integer, the solution (x, y, z) is $(13, 78, 13)$.

Case $y \leq z$. We get $\frac{9}{26} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{y}$, so $13 \leq y \leq 20$.

Since z is a positive integer, there is no solution.

If $x = 14$, then $\frac{3}{y} + \frac{4}{z} = \frac{5}{14}$.

Case $z \leq y$. We get $\frac{5}{14} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{z}$, so $14 \leq z \leq 19$.

Since y is a positive integer, the solutions (x, y, z) are $(14, 42, 14)$, $(14, 28, 16)$.

Case $y \leq z$. We get $\frac{5}{14} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{y}$, so $14 \leq y \leq 19$.

Since z is a positive integer, the solutions (x, y, z) are $(14, 14, 28)$, $(14, 18, 21)$.

If $x = 15$, then $\frac{3}{y} + \frac{4}{z} = \frac{11}{30}$.

Case $z \leq y$. We get $\frac{11}{30} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{z}$, so $15 \leq z \leq 19$.

Since y is a positive integer, the solution (x, y, z) is $(15, 30, 15)$.

Case $y \leq z$. We get $\frac{11}{30} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{y}$, so $15 \leq y \leq 19$.

Since z is a positive integer, the solutions (x, y, z) are $(15, 15, 24)$, $(15, 18, 20)$.

If $x = 17$, then $\frac{3}{y} + \frac{4}{z} = \frac{13}{34}$.

Case $z \leq y$. We get $\frac{13}{34} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{z}$, so $17 \leq z \leq 18$.

Since y is a positive integer, there is no solution.

Case $y \leq z$. We get $\frac{13}{34} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{y}$, so $17 \leq y \leq 18$.

Since z is a positive integer, there is no solution.

If $x = 18$, then $\frac{3}{y} + \frac{4}{z} = \frac{7}{18}$.

Case $z \leq y$. We get $\frac{7}{18} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{z}$, so $z = 18$.

The solution (x, y, z) is $(18, 18, 18)$.

Case $y \leq z$. We get $\frac{7}{18} = \frac{3}{y} + \frac{4}{z} \leq \frac{7}{y}$, so $y = 18$.

The solution (x, y, z) is $(18, 18, 18)$.

Case 2 : $y < x \leq z$ or $y \leq z < x$.

If $y < x \leq z$, then $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} < \frac{9}{y}$.

If $y \leq z < x$, then $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} < \frac{9}{y}$.

Therefore, $\frac{1}{2} = \frac{2}{x} + \frac{3}{y} + \frac{4}{z} < \frac{9}{y}$ which implies that $y < 18$. From y is a positive integer, $y \geq 7$ and $y < 18$, we get $7 \leq y \leq 17$.

If $y = 7$, then $\frac{2}{x} + \frac{4}{z} = \frac{1}{14}$. We have

$$(x - 28)(z - 56) = 1568.$$

We use the function τ to ensure that the total number of divisors of 1568 is 18. Using factors of 1568, we have

$$x - 28 = 1, z - 56 = 1568;$$

$$x - 28 = 2, z - 56 = 784;$$

$$x - 28 = 4, z - 56 = 392;$$

$$x - 28 = 7, z - 56 = 224;$$

$$x - 28 = 8, z - 56 = 196;$$

$$x - 28 = 14, z - 56 = 112;$$

$$x - 28 = 16, z - 56 = 98;$$

$$x - 28 = 28, z - 56 = 56;$$

$$x - 28 = 32, z - 56 = 49;$$

$$x - 28 = 49, z - 56 = 32;$$

$$x - 28 = 56, z - 56 = 28;$$

$$x - 28 = 98, z - 56 = 16;$$

$$x - 28 = 112, z - 56 = 14;$$

$$x - 28 = 196, z - 56 = 8;$$

$$x - 28 = 224, z - 56 = 7;$$

$$x - 28 = 392, z - 56 = 4;$$

$$x - 28 = 784, z - 56 = 2;$$

$$x - 28 = 1568, z - 56 = 1.$$

The solutions (x, y, z) are $(29, 7, 1624)$, $(30, 7, 840)$, $(32, 7, 448)$, $(35, 7, 280)$, $(36, 7, 252)$, $(42, 7, 168)$, $(44, 7, 154)$, $(56, 7, 112)$, $(60, 7, 105)$, $(77, 7, 88)$, $(84, 7, 84)$, $(126, 7, 72)$, $(140, 7, 70)$, $(224, 7, 64)$, $(252, 7, 63)$, $(420, 7, 60)$, $(812, 7, 58)$, $(1596, 7, 57)$.

If $y = 8$, then $\frac{2}{x} + \frac{4}{z} = \frac{1}{8}$. We have

$$(x - 16)(z - 32) = 512.$$

The solutions (x, y, z) are $(17, 8, 544)$, $(18, 8, 288)$, $(20, 8, 160)$, $(24, 8, 96)$, $(32, 8, 64)$, $(48, 8, 48)$, $(80, 8, 40)$, $(144, 8, 36)$, $(272, 8, 34)$, $(528, 8, 33)$.

If $y = 9$, then $\frac{2}{x} + \frac{4}{z} = \frac{1}{6}$. We have

$$(x - 12)(z - 24) = 288.$$

The solutions (x, y, z) are $(13, 9, 312)$, $(14, 9, 168)$, $(15, 9, 120)$, $(16, 9, 96)$, $(18, 9, 72)$, $(20, 9, 60)$, $(21, 9, 56)$, $(24, 9, 48)$, $(28, 9, 42)$, $(30, 9, 40)$, $(36, 9, 36)$, $(44, 9, 33)$, $(48, 9, 32)$, $(60, 9, 30)$, $(84, 9, 28)$, $(108, 9, 27)$, $(156, 9, 26)$, $(300, 9, 25)$.

If $y = 10$, then $\frac{2}{x} + \frac{4}{z} = \frac{1}{5}$. We have

$$(x - 10)(z - 20) = 200.$$

The solutions (x, y, z) are $(11, 10, 220)$, $(12, 10, 120)$, $(14, 10, 70)$, $(15, 10, 60)$, $(18, 10, 45)$, $(20, 10, 40)$, $(30, 10, 30)$, $(35, 10, 28)$, $(50, 10, 25)$, $(60, 10, 24)$, $(110, 10, 22)$, $(210, 10, 21)$.

If $y = 12$, then $\frac{2}{x} + \frac{4}{z} = \frac{1}{4}$. We have

$$(x - 8)(z - 16) = 128.$$

Since $y < x \leq z$ or $y \leq z < x$, the solutions (x, y, z) are $(16, 12, 32)$, $(24, 12, 24)$, $(40, 12, 20)$, $(72, 12, 18)$, $(136, 12, 17)$.

If $y = 14$, then $\frac{2}{x} + \frac{4}{z} = \frac{2}{7}$. We have

$$(x - 7)(2z - 28) = 196.$$

Since $y < x \leq z$ or $y \leq z < x$, and z is a positive integer, the solutions (x, y, z) are $(21, 14, 21)$, $(56, 14, 16)$, $(105, 14, 15)$.

If $y = 11$, then $\frac{2}{x} + \frac{4}{z} = \frac{5}{22}$.

Case $x \leq z$. We get $\frac{5}{22} = \frac{2}{x} + \frac{4}{z} \leq \frac{6}{x}$, so $12 \leq x \leq 26$. Since z is a positive integer, the solution (x, y, z) is $(12, 11, 66)$.

Case $z < x$. We get $\frac{5}{22} = \frac{2}{x} + \frac{4}{z} < \frac{6}{z}$, so $11 \leq z \leq 26$. Since x is a positive integer, the solutions (x, y, z) are $(396, 11, 18)$, $(44, 11, 22)$, $(33, 11, 24)$.

If $y = 13$, then $\frac{2}{x} + \frac{4}{z} = \frac{7}{26}$.

Case $x \leq z$. We get $\frac{7}{26} = \frac{2}{x} + \frac{4}{z} \leq \frac{6}{x}$, so $14 \leq x \leq 22$. Since z is a positive integer, there is no solution.

Case $z < x$. We get $\frac{7}{26} = \frac{2}{x} + \frac{4}{z} < \frac{6}{z}$, so $13 \leq z \leq 22$. Since x is a positive integer, the solutions (x, y, z) are $(780, 13, 15)$, $(104, 13, 16)$.

If $y = 15$, then $\frac{2}{x} + \frac{4}{z} = \frac{3}{10}$.

Case $x \leq z$. We get $\frac{3}{10} = \frac{2}{x} + \frac{4}{z} \leq \frac{6}{x}$, so $16 \leq x \leq 20$. Since z is a positive integer, the solution (x, y, z) is $(20, 15, 20)$.

Case $z < x$. We get $\frac{3}{10} = \frac{2}{x} + \frac{4}{z} < \frac{6}{z}$, so $15 \leq z \leq 19$. Since x is a positive integer, the solutions (x, y, z) are $(60, 15, 15)$, $(40, 15, 16)$.

If $y = 16$, then $\frac{2}{x} + \frac{4}{z} = \frac{5}{16}$.

Case $x \leq z$. We get $\frac{5}{16} = \frac{2}{x} + \frac{4}{z} \leq \frac{6}{x}$, so $17 \leq x \leq 19$. Since z is a positive integer, there is no solution.

Case $z < x$. We get $\frac{5}{16} = \frac{2}{x} + \frac{4}{z} < \frac{6}{z}$, so $16 \leq z \leq 19$. Since x is a positive integer, the solution (x, y, z) is $(32, 16, 16)$.

If $y = 17$, then $\frac{2}{x} + \frac{4}{z} = \frac{11}{34}$.

Case $x \leq z$. We get $\frac{11}{34} = \frac{2}{x} + \frac{4}{z} \leq \frac{6}{x}$, so $x = 18$. Since z is a positive integer, there is no solution.

Case $z < x$. We get $\frac{11}{34} = \frac{2}{x} + \frac{4}{z} < \frac{6}{z}$, so $17 \leq z \leq 18$. Since x is a positive integer, there is no solution.

Case 3: $z < x \leq y$ or $z < y < x$.

If $z < x \leq y$, then $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} < \frac{9}{z}$.

If $z < y < x$, then $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} < \frac{9}{z}$.

Therefore, $\frac{1}{2} = \frac{2}{x} + \frac{3}{y} + \frac{4}{z} < \frac{9}{z}$ which implies that $z < 18$. From z is a positive integer, $z \geq 9$ and $z < 18$, we get $9 \leq z \leq 17$.

If $z = 9$, then $\frac{2}{x} + \frac{3}{y} = \frac{1}{18}$. We have

$$(x - 36)(y - 54) = 1944.$$

We use the function τ to ensure that the total number of divisors of 1944 is 24. Using factors of 1944, we have

$$\begin{aligned} x - 36 &= 1, y - 54 = 1944; \\ x - 36 &= 54, y - 54 = 36; \\ x - 36 &= 2, y - 54 = 972; \\ x - 36 &= 72, y - 54 = 27; \\ x - 36 &= 3, y - 54 = 648; \\ x - 36 &= 81, y - 54 = 24; \\ x - 36 &= 4, y - 54 = 486; \\ x - 36 &= 108, y - 54 = 18; \\ x - 36 &= 6, y - 54 = 324; \\ x - 36 &= 162, y - 54 = 12; \\ x - 36 &= 8, y - 54 = 243; \\ x - 36 &= 216, y - 54 = 9; \\ x - 36 &= 9, y - 54 = 216; \\ x - 36 &= 243, y - 54 = 8; \\ x - 36 &= 12, y - 54 = 162; \\ x - 36 &= 324, y - 54 = 6; \\ x - 36 &= 18, y - 54 = 108; \\ x - 36 &= 486, y - 54 = 4; \\ x - 36 &= 24, y - 54 = 81; \\ x - 36 &= 648, y - 54 = 3; \\ x - 36 &= 27, y - 54 = 72; \\ x - 36 &= 972, y - 54 = 2; \\ x - 36 &= 36, y - 54 = 54; \\ x - 36 &= 1944, y - 54 = 1. \end{aligned}$$

The solutions (x, y, z) are (37,1998,9), (38,1026,9), (39,702,9), (40,540,9), (42,378,9), (44,297,9), (45,270,9), (48,216,9), (54,162,9), (60,135,9), (63,126,9), (72,108,9), (90,90,9), (108,81,9), (117,78,9), (144,72,9), (198,66,9), (252,63,9), (279,62,9), (360,60,9), (522,58,9), (684,57,9), (1008,56,9), (1980,55,9).

In the same manner,

if $z = 10$, then $\frac{2}{x} + \frac{3}{y} = \frac{1}{10}$. We have

$$(x - 20)(y - 30) = 600.$$

The solutions (x, y, z) are (21,630,10), (22,330,10), (23,230,10), (24,180,10), (25,150,10), (26,130,10), (28,105,10), (30,90,10), (32,80,10), (35,70,10), (40,60,10), (44,55,10), (45,54,10), (50,50,10), (60,45,10), (70,42,10), (80,40,10), (95,38,10), (120,36,10), (140,35,10), (170,34,10), (220,33,10), (320,32,10), (620,31,10).

If $z = 11$, then $\frac{2}{x} + \frac{3}{y} = \frac{3}{22}$. We have

$$(3x - 44)(y - 22) = 968.$$

Since x is a positive integer, the solutions (x, y, z) are (15,990,11), (16,264,11), (22,66,11), (44,33,11), (55,30,11), (176,24,11).

If $z = 12$, then $\frac{2}{x} + \frac{3}{y} = \frac{1}{6}$. We have

$$(x - 12)(y - 18) = 216.$$

The solutions (x, y, z) are (13,234,12), (14,126,12), (15,90,12), (16,72,12), (18,54,12), (20,45,12), (21,42,12), (24,36,12), (30,30,12), (36,27,12), (39,26,12), (48,24,12), (66,22,12), (84,21,12), (120,20,12), (228,19,12).

If $z = 14$, then $\frac{2}{x} + \frac{3}{y} = \frac{3}{14}$. We have

$$(3x - 28)(y - 14) = 392.$$

Since $z < x \leq y$ or $z < y < x$, and y is a positive integer, the solutions (x, y, z) are (28,21,14), (42,18,14), (140,15,14).

If $z = 16$, then $\frac{2}{x} + \frac{3}{y} = \frac{1}{4}$. We have

$$(x - 8)(y - 12) = 96.$$

Since $z < x \leq y$ or $z < y < x$, the solutions (x, y, z) are (20,20,16), (24,18,16).

If $z = 13$, then $\frac{2}{x} + \frac{3}{y} = \frac{5}{26}$.

Case $x \leq y$. We get $\frac{5}{26} = \frac{2}{x} + \frac{3}{y} \leq \frac{5}{x}$, so $14 \leq x \leq 26$.

Since y is a positive integer, the solution (x, y, z) is (26,26,13).

Case $y < x$. We get $\frac{5}{26} = \frac{2}{x} + \frac{3}{y} < \frac{5}{y}$, so $14 \leq y \leq 25$.

Since x is a positive integer, the solutions (x, y, z) are (416,16,13), (78,18,13).

If $z = 15$, then $\frac{2}{x} + \frac{3}{y} = \frac{7}{30}$.

Case $x \leq y$. We get $\frac{7}{30} = \frac{2}{x} + \frac{3}{y} \leq \frac{5}{x}$, so $16 \leq x \leq 21$.

Since y is a positive integer, there is no solution.

Case $y < x$. We get $\frac{7}{30} = \frac{2}{x} + \frac{3}{y} < \frac{5}{y}$, so $16 \leq y \leq 21$.

Since x is a positive integer, the solutions (x, y, z) are (30,18,15), (24,20,15).

If $z = 17$, then $\frac{2}{x} + \frac{3}{y} = \frac{9}{34}$.

Case $x \leq y$. We get $\frac{9}{34} = \frac{2}{x} + \frac{3}{y} \leq \frac{5}{x}$, so $x = 18$.

Since y is a positive integer, there is no solution.

Case $y < x$. We get $\frac{9}{34} = \frac{2}{x} + \frac{3}{y} < \frac{5}{y}$, so $y = 18$.

Since x is a positive integer, there is no solution.

On the other hand, we can compute directly that all above (x, y, z) satisfy the equation $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{1}{2}$. Hence, we obtain all solutions of the Diophantine equation $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{1}{2}$ where x, y and z are positive integers. $\square$

Moreover, If (x_0, y_0, z_0) is a positive integral solution of the Diophantine equation $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{1}{2}$ such that $4|z_0$, then $(\frac{z_0}{4}, x_0, y_0)$ is a positive integral solution of the equation $\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = \frac{1}{2}$, in (4). On the other hand, if (x_0, y_0, z_0) is a positive integral solution of the Diophantine equation $\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = \frac{1}{2}$, then $(y_0, z_0, 4x_0)$ is also a positive integral solution of the Diophantine equation $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{1}{2}$.

3. Conclusions

In this paper, we get all solutions of the Diophantine equation $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{1}{2}$ where x, y and z are positive integers.

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