PROGRESS IN APPLIED SCIENCE AND TECHNOLOGY
Faculty of Science and Technology
Rajamangala University of Technology Thanyaburi
Prog Appl Sci Tech. 2022; 12(3):1-6
ISSN (Print): 2730-3012
ISSN (Online): 2730-3020
https://ph02.tci-thaijo.org/index.php/past

Research Article

Received: March 27, 2022
Revised: August 27, 2022
Accepted: September 14, 2022

DOI: 10.14456/past.2022.7

Multivalued Nonlinear Weakly Picard Operators in Metric Spaces

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Abstract

In this paper, the concept of (F_s, L)-contraction was presented and a new fixed-point theorem for such contractions would be established. We provide applications to prove that there is a fixed point for cyclic mappings. We also received fixed-point results for the weak contraction type mappings.

Keywords: Multivalued Nonlinear, Weakly Picard Operators, Metric Spaces

1. Introduction

Let (Ω, d) be a metric space and $\Gamma(\Omega)$ be the class of nonempty subsets of Ω. Denote by $CB(\Omega)$ (resp. $K(\Omega)$) the class of nonempty bounded and closed (resp. all nonempty compact) subsets of Ω. For $U, V \in CB(\Omega)$, consider the Pompeiu-Hausdorff functional

$$\mathcal{H}(U,V) \coloneqq \max\left\{\sup_{u\in U}\inf_{v\in V} d(u,v), \sup_{v\in V}\inf_{u\in U} d(u,v)\right\}. \tag{1.1}$$

For $\zeta \in \Omega$, define $D(\zeta, V) = \inf_{\theta\in V} d(\zeta, \theta)$.

Nadler (1) used the notion of the Pompeiu-Hausdorff metric to ensure the existence of fixed points for multivalued contractive mappings. Berinde (2) introduced the following notion which was later named from weak contraction to almost contraction by Berinde (3).

Let (Ω, d) be a metric space and a mapping $Z : \Omega \to \Omega$ is said to be an almost contraction or an (δ, L)-contraction if there are $\delta \in (0,1)$, $L \geq 0$ and $\eta, \mu \in \Omega$ such that

$$d(Z\eta, Z\mu) \leq \delta d(\eta, \mu) + Ld(\mu, Z\eta). \tag{1.2}$$

Indeed, the notion of multivalued almost contractions as follows: A mapping $Z: \Omega \to CB(\Omega)$ is an almost contraction if there are $\delta \in (0,1)$ and $L \geq 0$, for $\eta, \mu \in \Omega$, the following inequality holds:

$$\mathcal{H}(Z\eta, Z\mu) \leq \delta d(\eta, \mu) + LD(\mu, Z\eta). \tag{1.3}$$

Berinde (4) established the Nadler fixed point theorem in (1). Popescu (5) introduced the concept of (s, r)-contraction multivalued operators and obtained some (strict) fixed point results. Let $Z: \Omega \to CB(\Omega)$ be a multivalued operator on a complete metric space (Ω, d) with Z is an (s, r)-contractive if $r \in [0,1), s \geq r$ and $\eta, \mu \in \Omega$

$$D(\mu, Z\eta) \leq sd(\mu, \eta) \quad \textit{implies} \quad \mathcal{H}(Z\eta, Z\mu) \leq rN(\eta, \mu), \tag{1.4}$$

where

$$N(\eta, \mu) = \max\left\{d(\eta, \mu), D(\eta, Z\eta), D(\mu, Z\mu), \frac{D(\eta, Z\mu) + D(\mu, Z\eta)}{2}\right\}.$$

Then Z has a fixed point. Moreover, if $s \geq 1$, such a fixed point is unique.

Latter, Kamran et al. (6) improved the results of Popescu (5) to weakly (s,r)-contractive multivalued operators. Let $Z: \Omega \to CB(\Omega)$ be a multivalued operator on a metric space with Z is a weakly (s,r)-contraction if there are $r \in [0,1), s \geq r$ and $L \geq 0$ such that

$$D(\mu, Z\eta) \leq sd(\mu,\eta) \quad implies \quad \mathcal{H}(Z\eta, Z\mu) \leq rM(\eta,\mu), \tag{1.5}$$

where

$$M(\eta,\mu) = max\left\{d(\eta,\mu), D(\eta,Z\eta), D(\mu,Z\mu), \frac{D(\eta,Z\mu)+D(\mu,Z\eta)}{2}\right\} + L\min\{d(\eta,\mu), D(\mu,Z\eta)\}.$$

Then Z has a fixed point. Moreover, if $s \geq 1$ such a fixed point is unique. On the other hand, Wardowski (7) introduced a generalized version of contraction mappings, called F-contractions, i.e., a mapping $Z: \Omega \to \Omega$ satisfying

$$\tau + Fd(Z\eta, Z\eta) \leq Fd(\eta,\mu), \tag{1.6}$$

for all $\eta,\mu \in \Omega$ with $d(Z\eta, Z\mu) > 0$, where $\tau > 0$ and $F: (0,\infty) \to \mathbb{R}$ is a function verifying the following conditions:

(F1) F is strictly increasing;

(F2) for each $\{\vartheta_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \vartheta_n = 0$ if and only if $\lim_{n\to\infty} F(\vartheta_n) = -\infty$;

(F3) there is $0 < k < 1$ such that $\lim_{\vartheta\to 0^+} \vartheta^k F(\vartheta) = 0$.

He proved that every F-contraction on a complete metric space has a unique fixed point.

Latter, Turinici in (8) relaxed condition (F2) by (F2*) for each $\{\vartheta_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \vartheta_n = 0$, then $\lim_{n\to\infty} F(\vartheta_n) = -\infty$. Then the following

(F2**) $F(\vartheta_n) \to -\infty$ implies $\vartheta_n \to 0$ can be derived from (F1).

Recently, Wardowski (9) considered the class of F-contractions in a generalized way by replacing τ by a function $\varphi: (0,\infty) \to (0,\infty)$ and defined (φ, F)-contractions on a metric space (Ω, d) so that

(H1) F verifies (F1) and (F2*);

(H2) $\liminf_{t\to q^+} \varphi(t) > 0$ for $q \geq 0$;

(H3) $\varphi(d(\eta,\mu)) + F(d(Z\eta, Z\mu)) \leq F(d(\eta,\mu))$ for all $\eta,\mu \in \Omega$ so that $d(Z\eta, Z\mu) > 0$.

It was proved a fixed point result for such nonlinear contractions by omitting (F3).

Altun et al. (10) used an extra condition on F:

(F4) $F(\inf(P)) = \inf F(P)$ for $P \subset (0,\infty)$ such that $\inf(P) > 0$.

Many authors endeavor to reach their goals in real world applications, see (11-15) and the related reference therein.

Motivated and inspired by concept of (F_s, L)-contractive multivalued operators. We will extend the results of Kamran et al. (6) and Popescu (5) as follow define:

(H1*) F satisfies (F1), (F2*), and (F4).

(H3*) There are $s \geq 0$ and $L \geq 0$ such that, for $\eta,\mu \in \Omega$ with $\mathcal{H}(Z\eta, Z\mu) > 0$ we have

$$D(\mu, Z\eta) \leq sd(\mu,\eta) \quad implies \quad \varphi(d(\eta,\mu)) + F(\mathcal{H}(Z\eta, Z\mu)) \leq F(Q(\eta,\mu)), \tag{1.7}$$

where

$$Q(\eta,\mu) = max\left\{d(\eta,\mu), \frac{[1+D(\eta,Z\eta)]D(\mu,Z\mu)}{1+d(\eta,\mu)}\right\} + L\,min\{d(\eta,\mu), D(\mu,Z\eta)\}.$$

Remark. Let $(\boldsymbol{\Omega}, \boldsymbol{d})$ be a metric space. The multivalued operator $\boldsymbol{Z}: \boldsymbol{\Omega} \to \boldsymbol{CB}(\boldsymbol{\Omega})$ is an $(\boldsymbol{F_s}, \boldsymbol{L})$-contraction if conditions (H1*), (H2), and (H3*) are satisfied.

2. Preliminaries

The graph of $Z: \Omega \to 2^\Omega$ is given as

$$Gr(Z) = (\zeta, \theta) \in \Omega^2,\ \theta \in Z\zeta.$$

The mapping Z is said to be upper semi-continuous if the inverse image of closed sets is closed.

Definition 2.1 (16) *A mapping $Z: \Omega \to CB(\Omega)$ is called a multivalued weakly Picard operator if, for all $\eta \in \Omega$ and $\mu \in Z\mu$, there is $\{\eta_n\}$ in Ω such that the following statements hold:*

(i) $\eta_0 = \eta$ *and* $\eta_1 = \mu$,

(ii) $\eta_{n+1} \in Z\eta_n$ *for all* $n \geq 0$,

(iii) $\{\eta_n\}$ *converges to a fixed point of* Z.

Lemma 2.1 (1) *Given a metric space* (Ω, d)*. Let* $B \subseteq \Omega$ *and* $\alpha > 1$*. For* $\eta \in \Omega$*, there is* $\mu \in B$ *such that* $d(\eta, \mu) \leq \alpha D(\eta, B)$*.*

Theorem 2.1 (4) *Let* $Z: \Omega \to CB(\Omega)$ *be an almost contraction mapping on a complete metric space. Then* Z *has a fixed point.*

3. Main results

Theorem 3.1 Let (Ω, d) be a complete metric space and $Z: \Omega \to CB(\Omega)$ be an (F_s, L)-contractive multivalued operator on a complete metric space. Assume that $\mathrm{Gr}(Z)$ is a closed subset of Ω^2. Then Z is a multivalued weakly Picard operator.

Proof. Let $\eta_0 \in \Omega$ and $\eta_1 \in Z\eta_0$, then $D(\eta_1, Z\eta_0) = 0$.In the case that $\eta_0 = \eta_1$, then η_1 is a fixed point of Z and the proof is completed.

Suppose that $\eta_0 \neq \eta_1$. If $\eta_1 \in Z\eta_1$, the proof is done. Otherwise, if $\eta_1 \notin Z\eta_1$, then since $Z\eta_1$ is closed, we have $D(\eta_1, Z\eta_1) > 0$. Therefore, $\mathcal{H}(Z\eta_0, Z\eta_1) \geq D(\eta_1, Z\eta_1) > 0$, so $D(\eta_1, Z\eta_0) \leq sd(\eta_1, \eta_0)$. Since Z is an (F_s, L)-contractive multivalued operator, we get

$$\varphi\big(d(\eta_0, \eta_1)\big) + F\big(\mathcal{H}(Z\eta_0, Z\eta_1)\big) \leq F\big(Q(\eta_0, \eta_1)\big), \tag{3.1}$$

where

$$Q(\eta_0, \eta_1) = \max\left\{d(\eta_0, \eta_1), \frac{[1 + D(\eta_0, Z\eta_0)]D(\eta_1, Z\eta_1)}{1 + d(\eta_0, \eta_1)}\right\} + L\min\{d(\eta_0, \eta_1), D(\eta_1, Z\eta_0)\} \leq \max\{d(\eta_0, \eta_1), D(\eta_1, Z\eta_1)\}.$$

Thus,

$$\varphi\big(d(\eta_0, \eta_1)\big) + F\big(\mathcal{H}(Z\eta_0, Z\eta_1)\big) \leq F(\max\{d(\eta_0, \eta_1), D(\eta_1, Z\eta_1)\}).$$

Since $D(\eta_1, Z\eta_1) \leq \mathcal{H}(Z\eta_0, Z\eta_1)$, from (F1) and inequality

$$F\big(D(\eta_1, Z\eta_1)\big) \leq F\big(\mathcal{H}(Z\eta_0, Z\eta_1)\big) \leq F\big(d(\eta_0, \eta_1)\big) - \varphi\big(d(\eta_0, \eta_1)\big). \tag{3.2}$$

From $D(\eta_1, Z\eta_1) > 0$ and (F4), we obtain

$$F\big(D(\eta_1, Z\eta_1)\big) = \inf_{g \in Z\eta_1} F\big(d(\eta_1, g)\big). \tag{3.3}$$

Using inequality (3.2), we get

$$\inf_{g \in Z\eta_1} F\big(d(\eta_1, g)\big) \leq F\big(d(\eta_0, \eta_1)\big) - \varphi\big(d(\eta_0, \eta_1)\big). \tag{3.4}$$

There is $\eta_2 \in Z\eta_1$ such that

$$F\big(d(\eta_1, \eta_2)\big) \leq F\big(d(\eta_0, \eta_1)\big) - \varphi\big(d(\eta_0, \eta_1)\big). \tag{3.5}$$

Continuing in this way, we get $\{\eta_n\}$ such that $\eta_{n+1} \in Z\eta_n$ and

$$F\big(d(\eta_n, \eta_{n+1})\big) \leq F\big(d(\eta_{n-1}, \eta_n)\big) - \varphi\big(d(\eta_{n-1}, \eta_n)\big), \tag{3.6}$$

for all $n \geq 1$. Let $\vartheta_n = \mathrm{d}(\eta_{n-1}, \eta_n)$ for all $n \geq 0$. We assume that $\vartheta_n > 0$ for each $n \in \mathrm{N}$. From inequality (Eq. 3.6), there is $a > 0$ such that

$$F(\vartheta_{n+1}) \leq F(\vartheta_n) - \varphi(\vartheta_n), \tag{3.7}$$

for each $n \in N$.

From (F1), $\{\vartheta_n\}$ is decreasing and hence $\vartheta_n \to t$, $t \geq 0$. By (H2) there are $a > 0$ and $n_0 \in \mathrm{N}$ such that $\varphi(\vartheta_n) > 0$ for each $n \geq n_0$. Therefore,

$$\begin{aligned} F(\vartheta_n) &\leq F(\vartheta_{n-1}) - \varphi(\vartheta_{n-1}) \\ &\leq F(\vartheta_{n-2}) - \varphi(\vartheta_{n-2}) \\ &\;\vdots \\ &\leq F(\vartheta_1) - \textstyle\sum_{i=1}^{n-1} \varphi(\vartheta_i) \\ &= F(\vartheta_1) - \textstyle\sum_{i=1}^{n_0-1} \varphi(\vartheta_i) - \textstyle\sum_{i=1}^{n-1} \varphi(\vartheta_i) \\ &< F(\vartheta_1) - (n - n_0)a, \quad \forall n > n_0. \end{aligned} \tag{3.8}$$

Letting $n \to \infty$, $F(\vartheta_n) \to -\infty$ and using (F**), $\vartheta_n \to 0$.

Next, we prove that $\{\eta_n\}$ is a Cauchy.

Assume that $\{\eta_n\}$ is not a Cauchy sequence. Using (F1), the set Ξ of all discontinuity elements of F is at most countable. There is $\lambda > 0, \lambda \notin \Xi$ in order that for each $k \geq 0$ there are $m_k, n_k \in \mathrm{N}$ such that

$$k \leq m_k < n_k \text{ and } d\big(\eta_{m_k}, \eta_{n_k}\big) > \lambda, \quad d\big(\eta_{m_k}, \eta_{n_k-1}\big) < \lambda, \; d\big(\eta_{n_k}, \eta_{m_k+1}\big) < \lambda. \tag{3.9}$$

Denote by $\widetilde{m_k}$ is the least of m_k satisfying (3.9) and by $\widetilde{n_k}$ is the least of n_k are satisfying (3.9), so that $\widetilde{m_k} < n_k$ and

$$d(\eta_{\widetilde{m_k}}, \eta_{n_k}) > \lambda. \text{ Naturally, one writes that}$$
$$d(\eta_{\widetilde{m_k}}, \eta_{\widetilde{n_k}}) > \lambda, \quad d(\eta_{\widetilde{m_k}}, \eta_{\widetilde{n_k}-1}) < \lambda,$$
$$d(\eta_{\widetilde{n_k}}, \eta_{\widetilde{m_k}+1}) < \lambda. \tag{3.10}$$

Letting $k_0 \in \mathrm{N}$ such that for $\gamma_k < \lambda$ for each $k \geq k_0$, we get

$$\begin{aligned} \lambda &< d(\eta_{\widetilde{m_k}}, \eta_{\widetilde{n_k}}) \\ &\leq d(\eta_{\widetilde{m_k}}, \eta_{\widetilde{n_k}-1}) + d(\eta_{\widetilde{n_k}-1}, \eta_{\widetilde{n_k}}) \\ &\leq \lambda + \gamma_{\widetilde{n_k}}, \quad \text{for each } k \geq k_0. \end{aligned}$$

Thus,

$$\lim_{k\to\infty} d(\eta_{\widetilde{m_k}}, \eta_{\widetilde{n_k}}) = \lambda. \tag{3.11}$$

Observe that

$$\begin{aligned} D(\eta_{\widetilde{n_k}}, Z\eta_{\widetilde{m_k}}) &\leq d(\eta_{\widetilde{n_k}}, \eta_{\widetilde{m_k}+1}) \\ &< \lambda < d(\eta_{\widetilde{n_k}}, \eta_{\widetilde{m_k}}) \\ &\leq sd(\eta_{\widetilde{n_k}}, \eta_{\widetilde{m_k}}). \end{aligned} \tag{3.12}$$

Using (H3*), we obtain

$$\begin{aligned} &\varphi\left(d(\eta_{\widetilde{m_k}}, \eta_{\widetilde{n_k}})\right) \\ &\leq F\left(d(\eta_{\widetilde{m_k}}, \eta_{\widetilde{n_k}})\right) - F\left(d(\eta_{\widetilde{m_k}+1}, \eta_{\widetilde{n_k}+1})\right), \end{aligned} \tag{3.13}$$

for all $k \geq 0$.

Using (3.10)-(3.13) and by the continuity of F at λ, we obtain

$$\begin{aligned} &\liminf_{s\to\lambda^+} \varphi(s) \\ &\leq \liminf_{k\to\infty} \varphi\left(d(\eta_{\widetilde{m_k}}, \eta_{\widetilde{n_k}})\right) \\ &\leq \lim_{k\to\infty}\Big(F\left(d(\eta_{\widetilde{m_k}}, \eta_{\widetilde{n_k}})\right) \\ &\quad - F\left(d(\eta_{\widetilde{m_k}+1}, \eta_{\tilde{n}+1})\right)\Big) \\ &= 0, \end{aligned} \tag{3.14}$$

which is a contradiction to (H2). Hence $\{\eta_n\}$ is a Cauchy sequence. Therefore, $\eta_n \to \mathrm{x} \in \Omega$ as $n \to \infty$. Since $\mathrm{Gr}(Z)$ is closed, at the limit $n \to \infty$, $(\eta_n, \eta_{n+1}) \to (\mathrm{x}, \mathrm{x})$ with $(\mathrm{x}, \mathrm{x}) \in \mathrm{Gr}(Z)$. Hence, $\mathrm{x} \in Z\mathrm{x}$, i.e., x is a fixed point of Z.

4. Application

Theorem 4.1 Let (Ω, d) be a complete metric space and $Z: \Omega \to CB(\Omega)$ be an (F_s, L)-*contractive multivalued operator on a complete metric space. Assume that Z is upper semi-continuous. Then Z is a multivalued weakly Picard operator.*

Proof. Using the upper semi-continuity condition is stronger than the closedness of $\mathrm{Gr}(Z)$ and follow prove in Theorem 3.1.

Theorem 4.2 Let (Ω, d) be a complete metric space and $Z: \Omega \to K(\Omega)$ be an (F_s, L)-*contractive multivalued operator on a complete metric space and $s \geq 1$. Assume that $Gr(Z)$ is a closed subset of Ω^2. Then Z is a multivalued weakly Picard operator.*

Proof. Let $\eta_0 \in \Omega$ and $\eta_1 \in Z\eta_0$. If $\eta_1 \in Z\eta_1$, then the proof is complete. Suppose $\eta_1 \notin Z\eta_1$. Then, from $Z\eta_1$ is closed, $\mathrm{D}(\eta_1, Z\eta_1) > 0$. On the other hand, from

$D(\eta_1, Z\eta_1) \leq \mathcal{H}(Z\eta_0, Z\eta_1)$, using (F1),

we obtain

$$F(D(\eta_1, Z\eta_1)) \leq F(\mathcal{H}(Z\eta_0, Z\eta_1)). \tag{4.1}$$

Thus, we have

$D(\eta_1, Z\eta_0) \leq sd(\eta_1, \eta_0)$.

Using (H3*), we get

$$\begin{aligned} &F(D(\eta_1, Z\eta_1)) \\ &\leq F(\mathcal{H}(Z\eta_0, Z\eta_1)) \\ &\leq F(Q(\eta_0, \eta_1)) - \varphi(d(\eta_0, \eta_1)) \\ &\leq F(\max\{d(\eta_0, \eta_1), D(\eta_1, Z\eta_1)\}) \\ &\quad - \varphi(d(\eta_0, \eta_1)) \\ &\leq F(d(\eta_0, \eta_1)) - \varphi(d(\eta_0, \eta_1)). \end{aligned} \tag{4.2}$$

Since $Z\eta_1$ is compact, there exists $\eta_2 \in Z\eta_1$ such that $d(\eta_1, \eta_2) = D(\eta_1, Z\eta_1)$. Then from inequality (4.2), we get

$$F(d(\eta_1, \eta_2)) \leq F(d(\eta_0, \eta_1)) - \varphi(d(\eta_0, \eta_1)).$$

The rest of the proof is similar to that of the proof of Theorem 3.1.

Theorem 4.3 Let (Ω, d) be a complete metric space and $Z:\Omega \to CB(\Omega)$ be an $(F_{r,s}, L)$-*contraction on a complete metric space. Assume that Gr(Z) is a closed subset of* Ω^2. *Then Z is a multivalued weakly Picard operator.*

Proof. Consider $t < 1$ which $0 \le r < t < s$. Since $1 - t > 1$, by Lemma 2.1, $\eta_1 \in \Omega$ and there $1 - s$ is $\eta_2 \in Z\eta_1$ such that

$$d(\eta_1, \eta_2) \le \frac{1-t}{1-s} D(\eta_1, Z\eta_1),$$

then

$$\frac{1}{1+r} D(\eta_1, Z\eta_1) \le D(\eta_1, Z\eta_1) \le d(\eta_1, \eta_2) \le \frac{1}{1-s} D(\eta_1, Z\eta_1).$$

Since Z is an $(F_{r,s}, L)$-contraction, we get

$$\varphi(d(\eta_1, \eta_2)) + F(\mathcal{H}(Z\eta_1, Z\eta_2)) \le F(Q(\eta_1, \eta_2)),$$

where

$$Q(\eta_1, \eta_2) = \max\left\{d(\eta_1, \eta_2), \frac{[1 + D(\eta_1, Z\eta_1)]D(\eta_2, Z\eta_2)}{1 + d(\eta_1, \eta_2)}\right\} + L\min\{d(\eta_1, \eta_2), D(\eta_2, Z\eta_1)\} \le \max\{d(\eta_1, \eta_2), D(\eta_2, Z\eta_2)\}.$$

Thus,

$$\varphi(d(\eta_1, \eta_2)) + F(\mathcal{H}(Z\eta_1, Z\eta_2)) \le F(max\{d(\eta_1, \eta_2), D(\eta_2, Z\eta_2)\}). \quad (4.3)$$

Since $D(\eta_2, Z\eta_2) \le \mathcal{H}(Z\eta_1, Z\eta_2)$, from (F1) and (4.3), we obtain

$$F(D(\eta_2, Z\eta_2) \le F(\mathcal{H}(Z\eta_1, Z\eta_2)) \le F(d(\eta_1, \eta_2)) - \varphi(d(\eta_1, \eta_2)). \quad (4.4)$$

From $Z\eta_2$ is closed, $D(\eta_2, Z\eta_2) > 0$ and using (F4)

$$F(D(\eta_2, Z\eta_2)) = \inf_{g \in Z\eta_2} F(d(\eta_2, g)). \quad (4.5)$$

Using (4.4), we obtain

$$\inf_{g \in Z\eta_2} F(d(\eta_2, g)) \le F(d(\eta_1, \eta_2)) - \varphi(d(\eta_1, \eta_2)). \quad (4.6)$$

There is $\eta_3 \in Z\eta_2$ such that
$F(d(\eta_2, \eta_3)) \le F(d(\eta_1, \eta_2)) - \varphi(d(\eta_1, \eta_2))$.
From the proof of Theorem 3.1, $\{\eta_n\}$ is a Cauchy sequence and hence $\eta_n \to x \in \Omega$. By the same way to those given in Theorem 3.1, we have that $D(x, Zx) = 0$.

Example. Let $\Omega = \{0,2,4,6\}$ and $d(\eta, \mu) = |\eta - \mu|$. Consider $Z:\Omega \to CB(\Omega)$ as

$$Z\eta = \begin{cases} \{2,6\}, & \text{if } \eta = 6, \\ \{4\}, & \text{if } not. \end{cases}$$

For $(\eta, \mu) \in \{(0,0), (0,2), (0,4), (2,0), (2,2), (2,4), (4,0), (4,2), (4,4), (6,6)\}$, then $\mathcal{H}(Z\eta, Z\mu) = 0$.
For $(\eta, \mu) \in \{(0,6), (2,6), (4,6), (6,0), (6,2), (6,4)\}$, then $\mathcal{H}(Z\eta, Z\mu) = 2$.
Choosing $s = 0.5$ and $(\eta, \mu) \in \{(4,6), (6,4)\}$, we get

$$D(\mu, Z\eta) = 2 = d(\mu, \eta),$$

which yields

$$D(\mu, Z\eta) > sd(\mu, \eta),$$

Now, for $(\eta, \mu) \in \{(0,6), (2,6), (6,0), (6,2)\}$, we get

$$D(\mu, Z\eta) \le sd(\mu, \eta),$$

Thus, for any $L \ge 0$, choosing $\varphi(t) = 1$ and $F(t) = t + \ln(t)$, we obtain

$$\varphi(d(\eta, \mu)) + F(\mathcal{H}(Z\eta, Z\mu)) < F(Q(\eta, \mu)).$$

That is, Z is an (F_s, L)-contraction. So, $Gr(Z)$ is a closed subset of Ω^2. Using Theorem 3.1, Z has 4 and 6 as fixed points.

Acknowledgements

The authors were financially supported by Rajamangala University of Technology Phra Nakhon (RMUTP) Research Scholarship.

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