Research Article

Solutions of the Pell equation $x^2 - Dy^2 = \pm N$

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Abstract

In this paper, we obtained some formulas for the integer solutions of the Pell equation $x^2 - Dy^2 = N$ and the negative Pell equation $x^2 - Dy^2 = -N$ where $D > 1$ is a non-square integer and N is a positive integer.

Keywords: Pell equation, Fundamental solution.

1. Introduction

The equation $x^2 - Dy^2 = N$ is called Pell equation with D, N are integers and x, y are unknowns. If D is negative, it can have only a finite number of solutions. If D is a square number, for $D = k^2$, the equation reduces to $(x - ky)(x + ky) = N$ and again there is only a finite number of solutions. If D is a non-square integer, it can have infinitely many solutions.

Let $\dfrac{F_n}{L_n}$ denotes the sequence of convergent to the regular continued fraction expansion of $\sqrt{D}$. Then the pair (x_1, y_1) solving Pell equation satisfies $x_1 = F_n$ and $y_1 = L_n$ for some n is positive integer. This pair is called the fundamental solution. Thus, the fundamental solution may be found by performing the continued fraction expansion and testing each successive convergent until a solution to Pell equation is found. Once the fundamental solution is found, all remaining solutions may be calculated algebraically from $\left(x_1 + y_1\sqrt{D}\right)^m = x_m + y_m\sqrt{D}$ expanding the left side, equating coefficients of $\sqrt{D}$ on both sides, and equating the other terms on both sides.

For completeness we recall that there are many papers in which are considered different types of Pell equation. Many authors such as Tekcan (6), Kaplan & Williams (2), Matthews (3) and the others consider some specific Pell equations and their integer solutions. In 2007, Tekcan (7) obtained some formulas for the integer solutions the Pell equation $x^2 - Dy^2 = \pm 4$. In 2008, Shabani (5) proved two conjectures related to Pell equation $x^2 - Dy^2 = \pm 4$. In 2011, Chandoul (1) obtained some formulas for the integer solutions the Pell equation $x^2 - Dy^2 = \pm k^2$. In 2015, Ramya et al. (5) obtained some formulas for the integer solutions the Pell equation $x^2 - Dy^2 = \pm 390625$.

In this paper, we obtain some formulas for the integer solutions of the Pell equation $x^2 - Dy^2 = \pm N$ where $D > 1$ is a non-square integer and N is a positive integer.

2. Preliminaries

In this section, we will prove that (x_{2n+1}, y_{2n+1}) is solution of Pell equation $x^2 - Dy^2 = \pm N$ where $D > 1$ is a non-square integer and N is a positive integer.

Lemma 2.1 Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = N$ and let

$$\begin{pmatrix} u_{2n+1} \\ v_{2n+1} \end{pmatrix} = \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (2.1)$$

for $n \geq 0$. Then the integer solutions of the Pell equation $x^2 - Dy^2 = N$ are (x_{2n+1}, y_{2n+1}) where

$$(x_{2n+1}, y_{2n+1}) = \left(\frac{u_{2n+1}}{N^n}, \frac{v_{2n+1}}{N^n} \right) \qquad (2.2)$$

for u_{2n+1} and v_{2n+1} are multiples of N^n.

Proof. We prove theorem by mathematical induction on n. For $n = 0$, we have from (2.1), $(u_1, v_1) = (x_1, y_1)$ which is the fundamental solution of $x^2 - Dy^2 = N$. With the assumption that the Pell equation $x^2 - Dy^2 = N$ is satisfied for $n \geq 0$, that is

$$x_{2n+1}^2 - Dy_{2n+1}^2 = \frac{u_{2n+1}^2 - Dv_{2n+1}^2}{N^{2n}} = N \qquad (2.3)$$

To prove that the Pell equation $x^2 - Dy^2 = N$ is true for $n + 1$.

$$\begin{pmatrix} u_{2n+3} \\ v_{2n+3} \end{pmatrix} = \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2n+3} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^2 \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^2 \begin{pmatrix} u_{2n+1} \\ v_{2n+1} \end{pmatrix}$$

$$= \begin{pmatrix} \left(x_1^2 + Dy_1^2\right)u_{2n+1} + 2Dx_1 y_1 v_{2n+1} \\ 2x_1 y_1 u_{2n+1} + \left(x_1^2 + Dy_1^2\right)v_{2n+1} \end{pmatrix}.$$

Next, we will show that (x_{2n+3}, y_{2n+3}) is solution of Pell equation $x^2 - Dy^2 = N$.
Hence, by (2.2) we obtain

$$x_{2n+3}^2 - Dy_{2n+3}^2 = \left(\frac{u_{2n+3}}{N^{n+1}} \right)^2 - D\left(\frac{v_{2n+3}}{N^{n+1}} \right)^2$$

$$= \frac{1}{N^{2n+2}} \left(u_{2n+3}^2 - Dv_{2n+3}^2 \right)$$

$$= \frac{1}{N^{2n+2}} \left[\left(\left(x_1^2 + Dy_1^2\right)u_{2n+1} + 2Dx_1 y_1 v_{2n+1} \right)^2 \right.$$
$$\left. - D\left(2x_1 y_1 u_{2n+1} + \left(x_1^2 + Dy_1^2\right)v_{2n+1} \right)^2 \right]$$

$$= \frac{1}{N^{2n+2}} \left[\left(x_1^2 + Dy_1^2\right)^2 u_{2n+1}^2 \right.$$
$$+ 4\left(x_1^2 + Dy_1^2\right)Dx_1 y_1 u_{2n+1} v_{2n+1} + 4D^2 x_1^2 y_1^2 v_{2n+1}^2$$
$$- 4Dx_1^2 y_1^2 u_{2n+1}^2 - 4\left(x_1^2 + Dy_1^2\right)Dx_1 y_1 u_{2n+1} v_{2n+1}$$
$$\left. - D\left(x_1^2 + Dy_1^2\right)^2 v_{2n+1}^2 \right]$$

$$= \frac{1}{N^{2n+2}} \left[\left(x_1^2 + Dy_1^2\right)^2 u_{2n+1}^2 + 4D^2 x_1^2 y_1^2 v_{2n+1}^2 \right.$$
$$\left. - 4Dx_1^2 y_1^2 u_{2n+1}^2 - D\left(x_1^2 + Dy_1^2\right)^2 v_{2n+1}^2 \right]$$

$$= \frac{1}{N^{2n+2}} \left[\left(\left(x_1^2 + Dy_1^2\right)^2 - 4Dx_1^2 y_1^2 \right) u_{2n+1}^2 \right.$$
$$\left. - \left(\left(x_1^2 + Dy_1^2\right)^2 - 4Dx_1^2 y_1^2 \right) Dv_{2n+1}^2 \right]$$

$$= \frac{\left(\left(x_1^2 + Dy_1^2\right)^2 - 4Dx_1^2 y_1^2 \right)\left(u_{2n+1}^2 - Dv_{2n+1}^2 \right)}{N^{2n+2}}$$

$$= \frac{\left(x_1^2 - Dy_1^2\right)^2 \left(u_{2n+1}^2 - Dv_{2n+1}^2 \right)}{N^{2n+2}}.$$

By (2.3) we have $u_{2n+1}^2 - Dv_{2n+1}^2 = N^{2n+1}$ and since (x_1, y_1) is the fundamental solution of the Pell equation $x^2 - Dy^2 = N$. Hence, we conclude that

$$x_{2n+3}^2 - Dy_{2n+3}^2 = \frac{\left(x_1^2 - Dy_1^2\right)^2 \left(u_{2n+1}^2 - Dv_{2n+1}^2 \right)}{N^{2n+2}}$$

$$= \frac{N^2 N^{2n+1}}{N^{2n+2}} = N.$$

This complete the proof.

Example 2.1 Consider the Pell equation $x^2 - 3y^2 = 6$. The fundamental solution is $(x_1, y_1) = (3, 1)$. By (2.1) we have

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 1 & 3 \end{pmatrix}^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 6^0 \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^0 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 1 & 3 \end{pmatrix}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 6^1 \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 54 \\ 30 \end{pmatrix}$$

$$\begin{pmatrix} u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 1 & 3 \end{pmatrix}^5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 6^2 \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1188 \\ 684 \end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix} u_{2n+1} \\ v_{2n+1} \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 1 & 3 \end{pmatrix}^{2n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 6^n \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

And by (2.2) we obtain

$$(x_1, y_1) = (u_1, v_1) = (3,1)$$

$$(x_3, y_3) = \left(\frac{u_3}{6^1}, \frac{v_3}{6^1} \right) = (9,5)$$

$$(x_5, y_5) = \left(\frac{u_5}{6^2}, \frac{v_5}{6^2} \right) = (33,19)$$

$$\vdots$$

$$\begin{pmatrix} x_{2n+1} \\ y_{2n+1} \end{pmatrix} = \frac{1}{6^n} \begin{pmatrix} u_{2n+1} \\ v_{2n+1} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Hence

$$u_1^2 - 3v_1^2 = 6^1 = 6$$

$$u_3^2 - 3v_3^2 = 6^3 = 216$$

$$u_5^2 - 3v_5^2 = 6^5 = 7776$$

$$\vdots$$

$$u_{2n+1}^2 - 3v_{2n+1}^2 = 6^{2n+1}.$$

And

$$x_{2n+1}^2 - 3y_{2n+1}^2 = 6$$

for $n \geq 0$.

Lemma 2.2 Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = -N$ and let

$$\begin{pmatrix} u_{2n+1} \\ v_{2n+1} \end{pmatrix} = \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for $n \geq 0$. Then the integer solutions of the Pell equation $x^2 - Dy^2 = -N$ are (x_{2n+1}, y_{2n+1}) where

$$(x_{2n+1}, y_{2n+1}) = \left(\frac{u_{2n+1}}{N^n}, \frac{v_{2n+1}}{N^n} \right)$$

for u_{2n+1} and v_{2n+1} are multiples of N^n.

Proof. This lemma can be proved as in the same way that lemma 2.1 was proved.

Example 2.2 Consider the Pell equation $x^2 - 6y^2 = -2$. The fundamental solution is $(x_1, y_1) = (2,1)$. By (2.1) we have

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 1 & 2 \end{pmatrix}^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2^0 \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix}^0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 1 & 2 \end{pmatrix}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2^1 \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix}^1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 44 \\ 18 \end{pmatrix}$$

$$\begin{pmatrix} u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 1 & 2 \end{pmatrix}^5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2^2 \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix}^2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 872 \\ 356 \end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix} u_{2n+1} \\ v_{2n+1} \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 1 & 2 \end{pmatrix}^{2n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2^n \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix}^n \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

And by (2.2) we obtain

$$(x_1, y_1) = (u_1, v_1) = (2,1)$$

$$(x_3, y_3) = \left(\frac{u_3}{2^1}, \frac{v_3}{2^1} \right) = (22,9)$$

$$(x_5, y_5) = \left(\frac{u_5}{2^2}, \frac{v_5}{2^2} \right) = (218,89)$$

$$\vdots$$

$$\begin{pmatrix} x_{2n+1} \\ y_{2n+1} \end{pmatrix} = \frac{1}{2^n} \begin{pmatrix} u_{2n+1} \\ v_{2n+1} \end{pmatrix} = \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix}^n \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Hence

$$u_1^2 - 6v_1^2 = -2^1 = -2$$

$$u_3^2 - 6v_3^2 = -2^3 = -8$$

$$u_5^2 - 6v_5^2 = -2^5 = -32$$

$$\vdots$$

$$u_{2n+1}^2 - 6v_{2n+1}^2 = -2^{2n+1}.$$

And

$$x_{2n+1}^2 - 6y_{2n+1}^2 = -2$$

for $n \geq 0$.

3. Main results

In this section, we will find the solutions of Pell equation $x^2 - Dy^2 = \pm N$ where $D > 1$ is a non-square integer and N is a positive integer.

Theorem 3.1 Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = N$ then

$$x_{2n+1} = \frac{\left(x_1^2 + Dy_1^2 \right) x_{2n-1} + 2Dx_1 y_1 y_{2n-1}}{N}$$

$$y_{2n+1} = \frac{2x_1y_1x_{2n-1} + \left(x_1^2 + Dy_1^2\right)y_{2n-1}}{N}$$

and

$$\begin{vmatrix} x_{2n+1} & x_{2n-1} \\ y_{2n+1} & y_{2n-1} \end{vmatrix} = -2x_1y_1$$

for $n \ge 1$, where (2.1) and (2.2) hold.

Proof. By (2.1) we have

$$u_{2n+1} = \left(x_1^2 + Dy_1^2\right)u_{2n-1} + 2Dx_1y_1v_{2n-1}$$

$$v_{2n+1} = 2x_1y_1u_{2n-1} + \left(x_1^2 + Dy_1^2\right)v_{2n-1}$$

and by (2.2) we have

$$u_{2n+1} = N^n x_{2n+1}$$

$$v_{2n+1} = N^n y_{2n+1}.$$

We get

$$N^n x_{2n+1} = u_{2n+1}$$

$$= \left(x_1^2 + Dy_1^2\right)u_{2n-1} + 2Dx_1y_1v_{2n-1}$$

$$= \left(x_1^2 + Dy_1^2\right)N^{n-1}x_{2n-1} + 2Dx_1y_1N^{n-1}y_{2n-1}$$

$$x_{2n+1} = \frac{\left(x_1^2 + Dy_1^2\right)x_{2n-1} + 2Dx_1y_1y_{2n-1}}{N}$$

and

$$N^n y_{2n+1} = v_{2n+1}$$

$$= 2x_1y_1u_{2n-1} + \left(x_1^2 + Dy_1^2\right)v_{2n-1}$$

$$= 2Dx_1y_1N^{n-1}x_{2n-1} + \left(x_1^2 + Dy_1^2\right)N^{n-1}y_{2n-1}$$

$$y_{2n+1} = \frac{2x_1y_1x_{2n-1} + \left(x_1^2 + Dy_1^2\right)y_{2n-1}}{N}.$$

Hence,

$$\begin{vmatrix} x_{2n+1} & x_{2n-1} \\ y_{2n+1} & y_{2n-1} \end{vmatrix} = x_{2n+1}y_{2n-1} - x_{2n-1}y_{2n+1}$$

$$= \left(\frac{\left(x_1^2 + Dy_1^2\right)x_{2n-1} + 2Dx_1y_1y_{2n-1}}{N}\right)y_{2n-1}$$

$$- x_{2n-1}\left(\frac{2x_1y_1x_{2n-1} + \left(x_1^2 + Dy_1^2\right)y_{2n-1}}{N}\right)$$

$$= \frac{2Dx_1y_1y_{2n-1}^2 - 2x_1y_1x_{2n-1}^2}{N}$$

$$= -\frac{2x_1y_1\left(x_{2n-1}^2 - Dy_{2n-1}^2\right)}{N}$$

$$= -\frac{2x_1y_1(N)}{N}$$

$$= -2x_1y_1.$$

This complete the proof.

Theorem 3.2 Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = -N$ then

$$x_{2n+1} = \frac{\left(x_1^2 + Dy_1^2\right)x_{2n-1} + 2Dx_1y_1y_{2n-1}}{N}$$

$$y_{2n+1} = \frac{2x_1y_1x_{2n-1} + \left(x_1^2 + Dy_1^2\right)y_{2n-1}}{N}$$

and

$$\begin{vmatrix} x_{2n+1} & x_{2n-1} \\ y_{2n+1} & y_{2n-1} \end{vmatrix} = 2x_1y_1$$

for $n \ge 1$, where (2.1) and (2.2) hold.

Proof. This theorem can be proved as in the same way that theorem 3.1 was proved.

Theorem 3.3 Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = N$ then (x_{2n+1}, y_{2n+1}) satisfy the following recurrence relations

$$x_{2n+1} = \left(\frac{4}{N}x_1^2 + s\right)x_{2n-1}$$

$$+ \left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)x_{2n-3} + (s+2)x_{2n-5}$$

$$y_{2n+1} = \left(\frac{4}{N}x_1^2 + s\right)y_{2n-1}$$

$$+ \left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)y_{2n-3} + (s+2)y_{2n-5}$$

$$(3.4)$$

for $n \ge 3$ and s is an integer where (2.1) and (2.2) hold,

$$x_3 = \frac{x_1}{N}\left(4x_1^2 - 3N\right)$$

$$y_3 = \frac{y_1}{N}\left(4x_1^2 - N\right)$$

$$x_5 = \frac{x_1}{N^2}\left(16x_1^4 - 20Nx_1^2 + 5N^2\right)$$

$$y_5 = \frac{y_1}{N^2}\left(16x_1^4 - 12Nx_1^2 + N^2\right).$$

Proof. We prove theorem by mathematical induction on n. By theorem 3.1, we have

$$x_3 = \frac{\left(x_1^2 + Dy_1^2\right)x_1 + 2Dx_1y_1^2}{N}$$

$$= \frac{x_1\left(x_1^2 + 3Dy_1^2\right)}{N}$$

$$= \frac{x_1}{N}\left(4x_1^2 - 3N\right)$$

$$y_3 = \frac{2x_1^2y_1 + \left(x_1^2 + Dy_1^2\right)y_1}{N}$$

$$= \frac{y_1\left(3x_1^2 + Dy_1^2\right)}{N}$$

$$= \frac{y_1}{N}\left(4x_1^2 - N\right)$$

$$x_5 = \frac{\left(x_1^2 + Dy_1^2\right)x_3 + 2Dx_1y_1y_3}{N}$$

$$= \frac{1}{N}\left[\left(x_1^2 + Dy_1^2\right)\left(\frac{x_1}{N}\left(4x_1^2 - 3N\right)\right) \right.$$
$$\left. + 2Dx_1y_1\left(\frac{y_1}{N}\left(4x_1^2 - N\right)\right)\right]$$

$$= \frac{x_1}{N^2}\left[\left(2x_1^2 - N\right)\left(4x_1^2 - 3N\right) \right.$$
$$\left. + 2\left(x_1^2 - N\right)\left(4x_1^2 - N\right)\right]$$

$$= \frac{x_1}{N^2}\left(16x_1^4 - 20Nx_1^2 + 5N^2\right)$$

$$y_5 = \frac{2x_1y_1x_3 + \left(x_1^2 + Dy_1^2\right)y_3}{N}$$

$$= \frac{1}{N}\left[2x_1y_1\left(\frac{x_1}{N}\left(4x_1^2 - 3N\right)\right) \right.$$
$$\left. + \left(x_1^2 + Dy_1^2\right)\left(\frac{y_1}{N}\left(4x_1^2 - N\right)\right)\right]$$

$$= \frac{y_1}{N^2}\left[2x_1^2\left(4x_1^2 - 3N\right) \right.$$
$$\left. + \left(2x_1^2 - N\right)\left(4x_1^2 - N\right)\right]$$

$$= \frac{y_1}{N^2}\left(16x_1^4 - 12Nx_1^2 + N^2\right).$$

And

$$x_7 = \frac{\left(x_1^2 + Dy_1^2\right)x_5 + 2Dx_1y_1y_5}{N}$$

$$= \frac{1}{N}\left[\left(x_1^2 + Dy_1^2\right)\left(\frac{x_1}{N^2}\left(16x_1^4 - 20Nx_1^2 + 5N^2\right)\right) \right.$$

$$\left. + 2Dx_1y_1\left(\frac{y_1}{N^2}\left(16x_1^4 - 12Nx_1^2 + N^2\right)\right)\right]$$

$$= \frac{x_1}{N^3}\left[\left(2x_1^2 - N\right)\left(16x_1^4 - 20Nx_1^2 + 5N^2\right) \right.$$

$$\left. + 2\left(x_1^2 - N\right)\left(16x_1^4 - 12Nx_1^2 + N^2\right)\right]$$

$$= \frac{x_1}{N^3}\left(64x_1^6 - 112Nx_1^4 + 56N^2x_1^2 - 7N^3\right)$$

$$y_7 = \frac{2x_1y_1x_5 + \left(x_1^2 + Dy_1^2\right)y_5}{N}$$

$$= \frac{1}{N}\left[2x_1y_1\left(\frac{x_1}{N^2}\left(16x_1^4 - 20Nx_1^2 + 5N^2\right)\right) \right.$$

$$\left. + \left(x_1^2 + Dy_1^2\right)\left(\frac{y_1}{N^2}\left(16x_1^4 - 12Nx_1^2 + N^2\right)\right)\right]$$

$$= \frac{y_1}{N^3}\left[2x_1^2\left(16x_1^4 - 20Nx_1^2 + 5N^2\right) \right.$$

$$\left. + \left(2x_1^2 - N\right)\left(16x_1^4 - 12Nx_1^2 + N^2\right)\right]$$

$$= \frac{y_1}{N^3}\left(64x_1^6 - 80Nx_1^4 + 24N^2x_1^2 - N^3\right).$$

For $n = 3$, we have

$$\left(\frac{4}{N}x_1^2 + s\right)x_5 + \left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)x_3$$

$$+ (s+2)x_1$$

$$= \left(\frac{4}{N}x_1^2 + s\right)\left(\frac{x_1}{N^2}\left(16x_1^4 - 20Nx_1^2 + 5N^2\right)\right)$$

$$+ \left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)\left(\frac{x_1}{N}\left(4x_1^2 - 3N\right)\right)$$

$$+ (s+2)x_1$$

$$= x_1\left(\frac{64}{N^3}x_1^6 - \frac{112}{N^2}x_1^4 + \frac{56}{N}x_1^2 - 7\right)$$

$$= \frac{x_1}{N^3}\left(64x_1^6 - 112Nx_1^4 + 56N^2x_1^2 - 7N^3\right)$$

$$= x_7$$

$$\left(\frac{4}{N}x_1^2 + s\right)y_5 + \left(-\frac{4}{N}(s+2)x_1^2 + 2s + 3\right)y_3$$

$$+ (s+2)y_1$$

$$= \left(\frac{4}{N}x_1^2 + s\right)\left(\frac{y_1}{N^2}\left(16x_1^4 - 12Nx_1^2 + N^2\right)\right)$$

$$+\left(-\frac{4}{N}(s+2)x_1^2+2s+3\right)\left(\frac{y_1}{N}\left(4x_1^2-N\right)\right)$$

$$+(s+2)y_1$$

$$=y_1\left(\frac{64}{N^3}x_1^6-\frac{80}{N^2}x_1^4+\frac{24}{N}x_1^2-1\right)$$

$$=\frac{y_1}{N^3}\left(64x_1^6-80Nx_1^4+24N^2x_1^2-N^3\right)$$

$$=y_7\,.$$

Therefore (3.4) is true for $n=3$. With the assumption that (3.4) is satisfied for $n\ge3$. To prove that (3.4) is true for $n+1$.

$$\left(\frac{4}{N}x_1^2+s\right)x_{2n+1}+\left(-\frac{4}{N}(s+2)x_1^2+2s+3\right)x_{2n-1}$$

$$+(s+2)x_{2n-3}$$

$$=\left(\frac{4}{N}x_1^2+s\right)\left(\frac{\left(x_1^2+Dy_1^2\right)x_{2n-1}+2Dx_1y_1y_{2n-1}}{N}\right)$$

$$+\left(-\frac{4}{N}(s+2)x_1^2+2s+3\right)\left(\frac{\left(x_1^2+Dy_1^2\right)x_{2n-3}+2Dx_1y_1y_{2n-3}}{N}\right)$$

$$+(s+2)\left(\frac{\left(x_1^2+Dy_1^2\right)x_{2n-5}+2Dx_1y_1y_{2n-5}}{N}\right)$$

$$=\frac{1}{N}\left[\left(x_1^2+Dy_1^2\right)\left(\left(\frac{4}{N}x_1^2+s\right)x_{2n-1}\right.\right.$$

$$+\left(-\frac{4}{N}(s+2)x_1^2+2s+3\right)x_{2n-3}+(s+2)x_{2n-5}\right)$$

$$+2Dx_1y_1\left(\left(\frac{4}{N}x_1^2+s\right)y_{2n-1}\right.$$

$$+\left.\left(-\frac{4}{N}(s+2)x_1^2+2s+3\right)y_{2n-3}+(s+2)y_{2n-5}\right)\right]$$

$$=\frac{\left(x_1^2+Dy_1^2\right)x_{2n+1}+\left(2Dx_1y_1\right)y_{2n+1}}{N}$$

$$=x_{2n+3}$$

$$\left(\frac{4}{N}x_1^2+s\right)y_{2n+1}+\left(-\frac{4}{N}(s+2)x_1^2+2s+3\right)y_{2n-1}$$

$$+(s+2)y_{2n-3}$$

$$=\left(\frac{4}{N}x_1^2+s\right)\left(\frac{2x_1y_1x_{2n-1}+\left(x_1^2+Dy_1^2\right)y_{2n-1}}{N}\right)$$

$$+\left(-\frac{4}{N}(s+2)x_1^2+2s+3\right)\left(\frac{2x_1y_1x_{2n-3}+\left(x_1^2+Dy_1^2\right)y_{2n-3}}{N}\right)$$

$$+(s+2)\left(\frac{2x_1y_1x_{2n-5}+\left(x_1^2+Dy_1^2\right)y_{2n-5}}{N}\right)$$

$$=\frac{1}{N}\left[2x_1y_1\left(\left(\frac{4}{N}x_1^2+s\right)x_{2n-1}\right.\right.$$

$$+\left(-\frac{4}{N}(s+2)x_1^2+2s+3\right)x_{2n-3}+(s+2)x_{2n-5}\right)$$

$$+\left(x_1^2+Dy_1^2\right)\left(\left(\frac{4}{N}x_1^2+s\right)y_{2n-1}\right.$$

$$+\left.\left(-\frac{4}{N}(s+2)x_1^2+2s+3\right)y_{2n-3}+(s+2)y_{2n-5}\right)\right]$$

$$=\frac{\left(2x_1y_1\right)x_{2n+1}+\left(x_1^2+Dy_1^2\right)y_{2n+1}}{N}$$

$$=y_{2n+3}\,.$$

This complete the proof.

Example 3.1 From example 2.1, by theorem 3.3 with $s=-2$ we have

$$x_1=3,y_1=1$$

$$x_3=\frac{x_1}{N}\left(4x_1^2-3N\right)$$

$$=\frac{3}{6}\left(4(3)^2-3(6)\right)$$

$$=9$$

$$y_3=\frac{y_1}{N}\left(4x_1^2-N\right)$$

$$=\frac{1}{6}\left(4(3)^2-6\right)$$

$$=5$$

$$x_5=\frac{x_1}{N^2}\left(16x_1^4-20Nx_1^2+5N^2\right)$$

$$=\frac{3}{6^2}\left(16(3)^4-20(6)(3)^2+5(6)^2\right)$$

$$=33$$

$$y_5=\frac{y_1}{N^2}\left(16x_1^4-12Nx_1^2+N^2\right)$$

$$=\frac{1}{6^2}\left(16(3)^4-12(6)(3)^2+(6)^2\right)$$

$$=19$$

$$x_7=\left(\frac{4}{N}x_1^2-2\right)x_5-x_3$$

$$= \left(\frac{4}{6}(3)^2 - 2 \right)(33) - 9$$

$$= 123$$

$$y_7 = \left(\frac{4}{N} x_1^2 - 2 \right) y_5 - y_3$$

$$= \left(\frac{4}{6}(3)^2 - 2 \right)(19) - 5$$

$$= 71$$

$$x_9 = \left(\frac{4}{N} x_1^2 - 2 \right) x_7 - x_5$$

$$= \left(\frac{4}{6}(3)^2 - 2 \right)(123) - 33$$

$$= 459$$

$$y_9 = \left(\frac{4}{N} x_1^2 - 2 \right) y_7 - y_5$$

$$= \left(\frac{4}{6}(3)^2 - 2 \right)(71) - 19$$

$$= 265$$

Hence

$$x_{2n+1} = 4x_{2n-1} - x_{2n-3}$$
$$y_{2n+1} = 4y_{2n-1} - y_{2n-3}$$

for $n \geq 3$.

Theorem 3.4 Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = -N$ then (x_{2n+1}, y_{2n+1}) satisfy the following recurrence relations

$$x_{2n+1} = \left(\frac{4}{N} x_1^2 + s \right) x_{2n-1}$$

$$+ \left(-\frac{4}{N}(s-2)x_1^2 - 2s + 3 \right) x_{2n-3} + (s-2)x_{2n-5}$$

$$y_{2n+1} = \left(\frac{4}{N} x_1^2 + s \right) y_{2n-1}$$

$$+ \left(-\frac{4}{N}(s-2)x_1^2 - 2s + 3 \right) y_{2n-3} + (s-2)y_{2n-5}$$

for $n \geq 3$ and s is an integer
where (2.1) and (2.2) hold,

$$x_3 = \frac{x_1}{N} \left(4x_1^2 + 3N \right)$$

$$y_3 = \frac{y_1}{N} \left(4x_1^2 + N \right)$$

$$x_5 = \frac{x_1}{N^2} \left(16x_1^4 + 20Nx_1^2 + 5N^2 \right)$$

$$y_5 = \frac{y_1}{N^2} \left(16x_1^4 + 12Nx_1^2 + N^2 \right) .$$

Proof. This theorem can be proved as in the same way that theorem 3.3 was proved.

Example 3.2 From example 2.2, by theorem 3.4 with $s = 2$ we have

$$x_1 = 2, y_1 = 1$$

$$x_3 = \frac{x_1}{N} \left(4x_1^2 + 3N \right)$$

$$= \frac{2}{2} \left(4(2)^2 + 3(2) \right)$$

$$= 22$$

$$y_3 = \frac{y_1}{N} \left(4x_1^2 + N \right)$$

$$= \frac{1}{2} \left(4(2)^2 + 2 \right)$$

$$= 9$$

$$x_5 = \frac{x_1}{N^2} \left(16x_1^4 + 20Nx_1^2 + 5N^2 \right)$$

$$= \frac{2}{2^2} \left(16(2)^4 + 20(2)(2)^2 + 5(2)^2 \right)$$

$$= 218$$

$$y_5 = \frac{y_1}{N^2} \left(16x_1^4 + 12Nx_1^2 + N^2 \right)$$

$$= \frac{1}{2^2} \left(16(2)^4 + 12(2)(2)^2 + (2)^2 \right)$$

$$= 89$$

$$x_7 = \left(\frac{4}{N} x_1^2 + 2 \right) x_5 - x_3$$

$$= \left(\frac{4}{2}(2)^2 + 2 \right)(218) - 22$$

$$= 2158$$

$$y_7 = \left(\frac{4}{N} x_1^2 + 2 \right) y_5 - y_3$$

$$= \left(\frac{4}{2}(2)^2 + 2 \right)(89) - 9$$

$$= 881$$

$$x_9 = \left(\frac{4}{N} x_1^2 + 2 \right) x_5 - x_3$$

$$= \left(\frac{4}{2}(2)^2 + 2 \right)(2158) - 218$$

$$= 21362$$

$$y_9 = \left(\frac{4}{N}x_1^2 + 2\right)y_5 - y_3$$

$$= \left(\frac{4}{2}(2)^2 + 2\right)(881) - 89$$

$$= 8721$$

Hence

$$x_{2n+1} = 10x_{2n-1} - x_{2n-3}$$

$$y_{2n+1} = 10y_{2n-1} - y_{2n-3}$$

for $n \geq 3$.

4. Conclusions

In this paper, we considered the Pell equation $x^2 - Dy^2 = \pm N$ where $D > 1$ is a non-square integer and N is a positive integer, and obtained some formulas its integer solutions.

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