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On Some Identities of the (s,t)-Pell and (s,t)-Pell-Lucas Polynomial Sequences

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Abstract

In this paper, we establish some identities of the relations between the (s,t)-Pell and (s,t)-Pell-Lucas polynomial sequences. Moreover, we obtain some identities of limits for the (s,t)-Pell and (s,t)-Pell-Lucas polynomial sequences.

Keywords: (s,t)-Pell polynomial, (s,t)-Pell-Lucas polynomial, recursive sequence

1. Introduction

For the recursive sequences, there are many forms of the recursive sequences that have been widely studied and appeared in various fields of sciences (1-3). In 1883, the Belgian mathematician Eugene Charles Catalan introduced the polynomials sequence which are defined by

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x), n \geq 1$$

where $F_0(x)$=0 and $F_1(x) = 1$. Subsequently, the Fibonacci polynomials studied by the German mathematician E. Jacobsthal are defined by

$$J_{n+1}(x) = J_n(x) + 2xJ_{n-1}(x), n \geq 1$$

where $J_0(x)$=1 and $J_1(x) = 1$.

In 1970, the Lucas polynomial studied by Bicknell, are defined by.

$$L_{n+1}(x) = xL_n(x) + L_{n-1}(x), n \geq 1$$

where $L_0(x)$=2 and $L_1(x) = x$. For more details can found in (4-5).

In 1985, A.F. Horadam and J.M. Mahon (6) introduced the Pell polynomial sequence and the Pell-Lucas polynomial sequence which are defined by

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x),$$
$$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x)$$

for $n \geq 2$, with initial conditions $P_0(x) = 0$, $P_1(x) = 1$, $Q_0(x) = 2$ and $Q_1(x) = 2x$.

In 2012, Gulec and Taskara (7) introduced the (s,t)-Pell sequence and (s,t)-Pell-Lucas sequence which are defined by

$$P_n(s,t) = 2sP_{n-1}(s,t) + tP_{n-2}(s,t),$$
$$Q_n(s,t) = 2sQ_{n-1}(s,t) + tQ_{n-2}(s,t)$$

for $n \geq 2$, where s and t are any real numbers with $s^2 + t > 0, s > 0$ and $t \neq 0$ with the initial conditions $P_0(s,t) = 0$, $P_1(s,t) = 1$, $Q_0(s,t) = 2$ and $Q_1(s,t) = 2s$. Later, Srisawat and Sripad (8-9) introduced the matrix methods and some more identities for the (s,t)-Pell and (s,t)-Pell-Lucas numbers.

In 2021, S. Srisawat and W. Sriprad (10) introduced the new generalizations of the (s,t)-Pell polynomial sequence and the (s,t)-Pell-Lucas polynomial sequence which are as following definition.

Definition 1.1 Let s and t be any real numbers with $s^2+t>0, s>0$ and $t\neq 0$. Then the (s,t)-Pell polynomial sequence $\{P_n(s,t)(x)\}_{n=0}^{\infty}$ and the (s,t)-Pell-Lucas polynomial sequence $\{Q_n(s,t)(x)\}_{n=0}^{\infty}$ are defined respectively by
$P_n(s,t)(x)=2sxP_{n-1}(s,t)(x)+tP_{n-2}(s,t)(x)$,
$Q_n(s,t)(x)=2sxQ_{n-1}(s,t)(x)+tQ_{n-2}(s,t)(x)$
for $n\geq 2$, with the initial conditions
$P_0(s,t)(x)=0$, $P_1(s,t)(x)=1$, $Q_0(s,t)(x)=2$ and $Q_1(s,t)(x)=2sx$.

The first few terms of the (s,t)-Pell polynomial sequence $\{P_n(s,t)(x)\}_{n=0}^{\infty}$ are 0, 1, $2sx$, $4s^2x^2+t$, $8s^3x^3+4tsx, \ldots$ and so on. Also, the first few terms of the (s,t)-Pell-Lucas polynomial sequence $\{Q_n(s,t)(x)\}_{n=0}^{\infty}$ are 2, $2sx$, $4s^2x^2+2t$, $8s^3x^3+6tsx$, $16s^4x^4+16ts^2x^2+2t^2, \ldots$ and so on.

The characteristic equation for the recurrence relation of the (s,t)-Pell polynomial sequence $\{P_n(s,t)(x)\}_{n=0}^{\infty}$ and the (s,t)-Pell-Lucas polynomial sequence $\{Q_n(s,t)(x)\}_{n=0}^{\infty}$ in Definition 1.1 is

$$r^2-2sxr-t=0, \quad (1.1)$$

where α and β are the roots of the equation (1.1), where $\alpha=sx+\sqrt{s^2x^2+t}$ and $\beta=sx-\sqrt{s^2x^2+t}$. Note that $\alpha+\beta=2sx$, $\alpha-\beta=2\sqrt{s^2x^2+t}$ and $\alpha\beta=-t$.

To convenience, we will use the symbol $P_n(x)$ and $Q_n(x)$ instead of the n^{th} term of (s,t)-Pell polynomial $\{P_n(s,t)(x)\}_{n=0}^{\infty}$ and the n^{th} term of the (s,t)-Pell-Lucas polynomial $\{Q_n(s,t)(x)\}_{n=0}^{\infty}$, respectively.

Theorem 1.2 (Binet's formulas) The n^{th} (s,t)-Pell and the n^{th} (s,t)-Pell-Lucas polynomials are given by

$$P_n(x)=\frac{\alpha^n-\beta^n}{\alpha-\beta}, n\geq 0 \quad (1.2)$$

and

$$Q_n(x)=\alpha^n+\beta^n, n\geq 0, \quad (1.3)$$

respectively, where α and β are the roots of the characteristic equation $r^2-2sxr-t=0$ and $\alpha>\beta$.

2. Main Results

Theorem 2.1 For n and r are positive integers with $n\geq r$. Let $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ be the (s,t)-Pell and (s,t)-Pell-Lucas polynomial sequences, respectively. Then

(i) $P_{n+r}(x)+t^rP_{n-r}(x)$
$$=\begin{cases} P_n(x)Q_r(x), & r \text{ is even} \\ P_r(x)Q_n(x), & r \text{ is odd}\end{cases}$$

(ii) $Q_{n+r}(x)+t^rQ_{n-r}(x)$
$$=\begin{cases} Q_r(x)Q_n(x) & , r \text{ is even} \\ 4(s^2x^2+t)P_r(x)P_n(x), & r \text{ is odd.}\end{cases}$$

Proof. (i) Firstly, we assume that r is an even number. By using Binet's formulas for the (s,t)-Pell and (s,t)-Pell-Lucas polynomial sequences, we have

$$\begin{aligned}
&P_{n+r}(x)+t^rP_{n-r}(x)\\
&=\frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta}+(-\alpha\beta)^r\left(\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\right)\\
&=\frac{1}{\alpha-\beta}(\alpha^{n+r}-\beta^{n+r}+\alpha^n\beta^r-\alpha^r\beta^n)\\
&=\frac{1}{\alpha-\beta}\left(\alpha^r(\alpha^n-\beta^n)+\beta^r(\alpha^n-\beta^n)\right)\\
&=\frac{1}{\alpha-\beta}(\alpha^n-\beta^n)(\alpha^r+\beta^r)\\
&=P_n(x)Q_r(x),
\end{aligned}$$

and we obtain

$$\begin{aligned}
&Q_{n+r}(x)+t^rQ_{n-r}(x)\\
&=(\alpha^{n+r}+\beta^{n+r})+(-\alpha\beta)^r(\alpha^{n-r}+\beta^{n-r})\\
&=(\alpha^{n+r}+\beta^{n+r})+(\alpha\beta)^r(\alpha^{n-r}+\beta^{n-r})\\
&=\alpha^{n+r}+\beta^{n+r}+\alpha^n\beta^r+\alpha^r\beta^n\\
&=\alpha^r(\alpha^n+\beta^n)+\beta^r(\alpha^n+\beta^n)\\
&=(\alpha^r+\beta^r)(\alpha^n+\beta^n)\\
&=Q_r(x)Q_n(x).
\end{aligned}$$

Secondly, if r is an odd number, then the result are as follows:

$$\begin{aligned}
&P_{n+r}(x)+t^rP_{n-r}(x)\\
&=\frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta}+(-\alpha\beta)^r\left(\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\right)\\
&=\frac{1}{\alpha-\beta}(\alpha^{n+r}-\beta^{n+r}-\alpha^n\beta^r+\alpha^r\beta^n)\\
&=\frac{1}{\alpha-\beta}\left(\alpha^r(\alpha^n+\beta^n)-\beta^r(\alpha^n+\beta^n)\right)\\
&=\frac{1}{\alpha-\beta}(\alpha^r-\beta^r)(\alpha^n+\beta^n)\\
&=P_r(x)Q_n(x),
\end{aligned}$$

and

$$\begin{aligned}
&Q_{n+r}(x)+t^rQ_{n-r}(x)\\
&=(\alpha^{n+r}+\beta^{n+r})+(-\alpha\beta)^r(\alpha^{n-r}+\beta^{n-r})\\
&=(\alpha^{n+r}+\beta^{n+r})-(\alpha\beta)^r(\alpha^{n-r}+\beta^{n-r})\\
&=\alpha^{n+r}+\beta^{n+r}-\alpha^n\beta^r-\alpha^r\beta^n\\
&=\alpha^r(\alpha^n-\beta^n)-\beta^r(\alpha^n-\beta^n)\\
&=(\alpha^r-\beta^r)(\alpha^n-\beta^n)\\
&=(\alpha-\beta)^2P_r(x)P_n(x)\\
&=4(s^2x^2+t)P_r(x)P_n(x). \qquad \square
\end{aligned}$$

Corollary 2.2 For any positive integer n. Let $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ be the (s,t)-Pell and (s,t)-Pell-Lucas polynomial sequences, respectively. Then

(i) $P_{2n}(x)=P_n(x)Q_n(x)$,

(ii) $$Q_{2n}(x)+2t^n=\begin{cases}Q_n^2(x) & ,n \text{ is even}\\ 4(s^2x^2+t)P_n^2(x), & n \text{ is odd.}\end{cases}$$

Proof. Taking $r=n$ in Theorem 2.1, the proof completed. $\square$

Remark 2.3 For $r=1$ in Theorem 2.1, then we have the following identities (Theorem 2.7 in (10)),

$$P_{n+1}(x)+tP_{n-1}(x)=Q_n(x)$$

and

$$Q_{n+1}(x)+tQ_{n-1}(x)=4(s^2x^2+t)P_n(x).$$

Theorem 2.4 For m,n and r are positive integers with $m\geq n$. Let $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ be the (s,t)-Pell and (s,t)-Pell-Lucas polynomial sequences, respectively. Then

(i) $P_m(x)P_{n+r}(x)-P_{m+r}(x)P_n(x)=(-t)^nP_r(x)P_{m-n}(x)$

(ii) $Q_m(x)Q_{n+r}(x)-Q_{m+r}(x)Q_n(x)=(\alpha^r-\beta^r)(-t)^n(Q_{m-n}(x)-2\alpha^{m-n})$.

Proof. By using Binet's formulas for the (s,t)-Pell and (s,t)-Pell-Lucas polynomial sequences, we have

$$\begin{aligned}
&P_m(x)P_{n+r}(x)-P_{m+r}(x)P_n(x)\\
&=\left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right)\left(\frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta}\right)-\left(\frac{\alpha^{m+r}-\beta^{m+r}}{\alpha-\beta}\right)\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)\\
&=\frac{1}{(\alpha-\beta)^2}(-\beta^m\alpha^{n+r}-\alpha^m\beta^{n+r}+\alpha^n\beta^{m+r}+\alpha^{m+r}\beta^n)\\
&=\frac{1}{(\alpha-\beta)^2}\left(-\alpha^n\beta^m(\alpha^r-\beta^r)+\alpha^m\beta^n(\alpha^r-\beta^r)\right)\\
&=\frac{1}{(\alpha-\beta)^2}(-\alpha^n\beta^m+\alpha^m\beta^n)(\alpha^r-\beta^r)\\
&=\frac{1}{(\alpha-\beta)^2}(\alpha\beta)^n(\alpha^{m-n}-\beta^{m-n})(\alpha^r-\beta^r)\\
&=(-t)^nP_r(x)P_{m-n}(x),
\end{aligned}$$

and

$$\begin{aligned}
&Q_m(x)Q_{n+r}(x)-Q_{m+r}(x)Q_n(x)\\
&=(\alpha^m+\beta^m)(\alpha^{n+r}+\beta^{n+r})-(\alpha^{m+r}+\beta^{m+r})(\alpha^n+\beta^n)\\
&=\alpha^{n+r}\beta^m-\alpha^n\beta^{m+r}+\alpha^m\beta^{n+r}-\alpha^{m+r}\beta^n\\
&=\alpha^n\beta^m(\alpha^r-\beta^r)-\alpha^m\beta^n(\alpha^r-\beta^r)\\
&=(\alpha^r-\beta^r)(\alpha^n\beta^m-\alpha^m\beta^n)\\
&=(\alpha^r-\beta^r)(\alpha\beta)^n(\beta^{m-n}-\alpha^{m-n})\\
&=(\alpha^r-\beta^r)(\alpha\beta)^n(\alpha^{m-n}+\beta^{m-n}-2\alpha^{m-n})\\
&=(\alpha^r-\beta^r)(-t)^n(Q_{m-n}(x)-2\alpha^{m-n}). \qquad \square
\end{aligned}$$

Remark 2.5 If we take $r=1$ in Theorem 2.4. Then we have (Theorem 2.6 in (10))

$$P_m(x)P_{n+1}(x)-P_{m+1}(x)P_n(x)=(-t)^nP_{m-n}(x)$$

and

$$Q_m(x)Q_{n+1}(x)-Q_{m+1}(x)Q_n(x)=2(-t)^n\sqrt{s^2x^2+t}\left(Q_{m-n}(x)-2\left(sx+\sqrt{s^2x^2+t}\right)^{m-n}\right).$$

Remark 2.6 In Theorem 2.4 (ii), we have the relation between the (s,t)-Pell and (s,t)-Pell-Lucas polynomial sequences as follow,

$$\begin{aligned}
&Q_m(x)Q_{n+r}(x)-Q_{m+r}(x)Q_n(x)\\
&=(\alpha^r-\beta^r)(-t)^n(Q_{m-n}(x)-2\alpha^{m-n})\\
&=(\alpha^r-\beta^r)(-t)^n(\beta^{m-n}-\alpha^{m-n})\\
&=-(\alpha^r-\beta^r)(-t)^n(\alpha^{m-n}-\beta^{m-n})\\
&=-(\alpha-\beta)^2(-t)^nP_r(x)P_{m-n}(x)\\
&=-4(-t)^n(s^2x^2+t)P_r(x)P_{m-n}(x).
\end{aligned}$$

Therefore,

$$Q_m(x)Q_{n+r}(x)-Q_{m+r}(x)Q_n(x)=-4(-t)^n(s^2x^2+t)P_r(x)P_{m-n}(x).$$

If $r=1$, we have

$$Q_m(x)Q_{n+1}(x)-Q_{m+1}(x)Q_n(x)=-4(-t)^n(s^2x^2+t)P_{m-n}(x).$$

Theorem 2.7 Let sequences $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ be the (s,t)-Pell and (s,t)-Pell-Lucas polynomial sequences, respectively. If $sx>0, s^2+t>0$ and $t\neq 0$. Then

(i) $\lim_{n\to\infty}\frac{P_{n+r}(x)}{P_n(x)}=\alpha^r$,

(ii) $\lim_{n\to\infty}\frac{Q_{n+r}(x)}{Q_n(x)}=\alpha^r$,

(iii) $\lim_{n\to\infty}\frac{P_n(x)}{Q_n(x)}=\frac{1}{\alpha-\beta}$,

(iv) $\lim_{n\to\infty}\frac{P_{n+r}(x)}{Q_n(x)}=\frac{\alpha^r}{\alpha-\beta}$

and

(v) $\lim_{n\to\infty} \frac{P_n(x)}{Q_{n+r}(x)} = \frac{1}{(\alpha-\beta)\alpha^r}$,

where r is a non-negative integer.

Proof. By using Binet's fomulas, we have:

(i) $\lim_{n\to\infty} \frac{P_{n+r}(x)}{P_n(x)} = \lim_{n\to\infty} \frac{\alpha^{n+r}-\beta^{n+r}}{\alpha^n-\beta^n}$

$$= \lim_{n\to\infty} \frac{1-\left(\frac{\beta}{\alpha}\right)^{n+r}}{\frac{1}{\alpha^r}-\frac{1}{\beta^r}\left(\frac{\beta}{\alpha}\right)^{n+r}}$$

Since $\left|\frac{\beta}{\alpha}\right| < 1$, then $\lim_{n\to\infty} \left(\frac{\beta}{\alpha}\right)^{n+r} = 0$.

Therefore, $\lim_{n\to\infty} \frac{P_{n+r}(x)}{P_n(x)} = \alpha^r$.

Next for (ii), we consider

$$\lim_{n\to\infty} \frac{Q_{n+r}(x)}{Q_n(x)} = \lim_{n\to\infty} \frac{\alpha^{n+r}+\beta^{n+r}}{\alpha^n+\beta^n}$$

$$= \lim_{n\to\infty} \frac{1+\left(\frac{\beta}{\alpha}\right)^{n+r}}{\frac{1}{\alpha^r}+\frac{1}{\beta^r}\left(\frac{\beta}{\alpha}\right)^{n+r}}$$

$$= \alpha^r.$$

Similarly,

$$\lim_{n\to\infty} \frac{P_n(x)}{Q_n(x)} = \lim_{n\to\infty} \frac{\alpha^n-\beta^n}{(\alpha-\beta)(\alpha^n+\beta^n)}$$

$$= \lim_{n\to\infty} \frac{1-\left(\frac{\beta}{\alpha}\right)^n}{(\alpha-\beta)\left(1+\left(\frac{\beta}{\alpha}\right)^n\right)}$$

$$= \frac{1}{\alpha-\beta}.$$

Then, we have

$$\lim_{n\to\infty} \frac{P_{n+r}(x)}{Q_n(x)}$$

$$= \frac{1}{\alpha-\beta} \lim_{n\to\infty} \frac{\alpha^{n+r}-\beta^{n+r}}{\alpha^n+\beta^n}$$

$$= \frac{1}{\alpha-\beta} \lim_{n\to\infty} \alpha^r \left(\frac{1-\left(\frac{\beta}{\alpha}\right)^{n+r}}{1+\left(\frac{\beta}{\alpha}\right)^n}\right)$$

$$= \frac{\alpha^r}{\alpha-\beta}.$$

Finally,

$$\lim_{n\to\infty} \frac{P_n(x)}{Q_{n+r}(x)} = \frac{1}{\alpha-\beta} \lim_{n\to\infty} \frac{\alpha^n-\beta^n}{\alpha^{n+r}+\beta^{n+r}}$$

$$= \frac{1}{\alpha-\beta} \lim_{n\to\infty} \frac{1}{\alpha^r} \left(\frac{1-\left(\frac{\beta}{\alpha}\right)^n}{1+\left(\frac{\beta}{\alpha}\right)^{n+r}}\right)$$

$$= \frac{1}{(\alpha-\beta)\alpha^r}.$$

3. Conclusions

In this paper, we obtain some more identities of relations between the (s,t)-Pell and (s,t)-Pell-Lucas polynomial sequences by using the Binet formulars. Furthermore, some identities of limits for the (s,t)-Pell and (s,t)-Pell-Lucas polynomial sequences are obtained.

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