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The Differential Equation in Terms of Jacobsthal and Jacobsthal-Lucas Numbers

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Abstract

In this paper, we study Jacobsthal sine, Jacobsthal-Lucas sine, Jacobsthal cosine, Jacobsthal-Lucas cosine, Jacobsthal tangent, Jacobsthal-Lucas tangent, Jacobsthal cotangent, Jacobsthal-Lucas cotangent, Jacobsthal secant, Jacobsthal-Lucas secant, Jacobsthal cosecant, and Jacobsthal-Lucas cosecant. Furthermore, we establish some identities of Jacobsthal sine, Jacobsthal-Lucas sine, Jacobsthal cosine, Jacobsthal-Lucas cosine, Jacobsthal tangent, Jacobsthal-Lucas tangent, Jacobsthal cotangent, Jacobsthal-Lucas cotangent, Jacobsthal secant, Jacobsthal-Lucas secant, Jacobsthal cosecant, and Jacobsthal-Lucas cosecant.

Keywords: differential equations, Jacobsthal number, Jacobsthal-Lucas number

1. Introduction

The well-known Fibonacci $\{F_n\}$, Lucas $\{L_n\}$, Pell $\{P_n\}$, and Pell-Lucas $\{Q_n\}$ sequences have been found for several years. Their Binet's formulas are $F_n = \frac{a^n - b^n}{a-b}$, $L_n = a^n + b^n$, $J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, and $j_n = \alpha^n + \beta^n$, where n is an integer, $a = \frac{1+\sqrt{5}}{2}$, $b = \frac{1-\sqrt{5}}{2}$ and $\alpha = 2$, $\beta = -1$ are the root of the characteristic equation $r^2 - r - 1 = 0$ and $r^2 - r - 2 = 0$, respectively [1,2,4]. So $a > b$, $a + b = 1$ $a - b = \sqrt{5}$, $ab = -2$ and $\alpha > \beta$, $\alpha + \beta = 1$, $\alpha - \beta = 3$, $\alpha\beta = -2$.

Recently, the general solution of a second-order homogeneous linear differential equation in terms of numbers was studied by many authors in different ways to derive many identities. In 1964, Verner E. Hoggatt, Jr. [3] studied a general solution of a second-order homogeneous linear differential equation $y'' - y' - y = 0$ with an initial value $y(0) = 0$ $y'(0) = 1$, which is defined by

$$y = \frac{e^{ax} - e^{bx}}{a-b} = \sum_{n=0}^{\infty} \frac{a^n - b^n}{a-b} \frac{x^n}{n!}, \tag{1.1}$$

where $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $r^2 - r - 1 = 0$. They obtained some identities of these (5, 7).

In 2016, Prasanta Kumar Ray (6) studied a general solution of a second-order homogeneous linear differential equation $y''-6y'+y=0$ with an initial value $y(0)=0 \quad y'(0)=1$. The author obtained some identities of these.

The inspiration for doing this research due to the direction of this research and development. We present the general solution of a second-order homogeneous linear differential equation in terms of Jacobsthal and Jacobsthal-Lucas numbers, along with finding these identities.

2. Main results

In this section, we begin to give second-order homogeneous linear differential equations

$$y''-y'-2y=0 \tag{2.1}$$

with initial value $y(0)=0 \quad y'(0)=1$ and $y(0)=2 \quad y'(0)=1$, respectively.

Next, we define Jacobsthal sine and Jacobsthal-Lucas sine, which correspond to the following definition.

Definition 2.1 Let $\alpha>\beta$. Then the Jacobsthal sine $\boldsymbol{\sin J(x)}$ and Jacobsthal-Lucas sine $\boldsymbol{\sin j(x)}$ are defined respectively by

$$\sin J(x)=\frac{e^{\alpha x}-e^{\beta x}}{\alpha-\beta}, \tag{2.2}$$

$$\sin j(x)=e^{\alpha x}+e^{\beta x}. \tag{2.3}$$

Note that equations (2.2) and (2.3) are the general solution of (2.1).

Also, we find some identities of Jacobsthal and Jacobsthal-Lucas numbers which correspond to the following lemma.

Lemma 2.2 Let $n\geq 0$ and $\alpha=2$, $\beta=-1$. The following results hold.

(i) $J_{n+1}+2J_{n-1}=j_n$

(ii) $j_{n+1}+2j_{n-1}=9J_n$,

(iii) $j_{n+1}+4j_n+2j_{n-1}+8j_{n-2}=9j_n$,

(iv) $\alpha^n=J_n\alpha+2J_{n-1}$,

(v) $\beta^n=J_n\beta+2J_{n-1}$.

Proof. Since Binet's formulas, we have

$$\begin{aligned} J_{n+1}+2J_{n-1} &= \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}+2\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta} \\ &= \alpha^n+\beta^n \\ &= j_n. \end{aligned}$$

$$\begin{aligned} j_{n+1}+2j_{n-1} &= \alpha^{n+1}+\beta^{n+1}+2\left(\alpha^{n-1}+\beta^{n-1}\right) \\ &= (\alpha-\beta)^2\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) \\ &= 9J_n. \end{aligned}$$

$$\begin{aligned} & j_{n+1}+4j_n+2j_{n-1}+8j_{n-2} \\ &= \alpha^{n+1}+\beta^{n+1}+4\left(\alpha^n+\beta^n\right)+2\left(\alpha^{n-1}+\beta^{n-1}\right) \\ &\quad +8\left(\alpha^{n-2}+\beta^{n-2}\right) \\ &= (\alpha-\beta)^2\left(\alpha^n+\beta^n\right) \\ &= 9j_n. \end{aligned}$$

Next, If $n=0$, then the proof is obvious. Next, we will be shown by mathematical induction that $\alpha^n=J_n\alpha+2J_{n-1}$ for $n\in\mathbb{N}$. Since $J_1\alpha+2J_0=\alpha$, it follows that $n=1$ is ture. Assume that the result is true for the positive integer, $n=k$. Then $\alpha^k=J_k\alpha+2J_{k-1}$. Now, we need to show that (iv) also holds for $n=k+1$ as follows:

$$\begin{aligned} \alpha^{k+1} &= \alpha^k\alpha \\ &= \left(J_k\alpha+2J_{k-1}\right)\alpha \\ &= J_k\alpha^2+2J_{k-1}\alpha \\ &= J_k(\alpha+2)+2J_{k-1}\alpha \\ &= J_k\alpha+2J_k+2J_{k-1}\alpha \\ &= J_k\alpha+2J_{k-1}\alpha+2J_k \\ &= \left(J_k+2J_{k-1}\right)\alpha+2J_k \\ &= J_{k+1}\alpha+2J_k. \end{aligned}$$

Thus, $n=k+1$ is ture. The similar proof of (iv) is applied for (v). Therefore, the proof is complete.

After that, we find undetermined coefficients of the Maclaurin series, the general solution of second-order homogeneous linear differential equations, as follows.

Lemma 2.3 Let $n \geq 0$. Then the recurrence relation c_n is given by
$(n+2)(n+1)c_{n+2} - (n+1)c_{n+1} - 2c_n = 0$.

Proof. Let the Maclaurin series

$$y = \sum_{n=0}^{\infty} c_n x^n . \tag{2.4}$$

Since the differentiation of equation (2.4), we have

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \tag{2.5}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} . \tag{2.6}$$

By using equations (2.4), (2.5), and (2.6) in (2.1), we obtain

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^{n-1} - 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n - 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)c_{n+2} - (n+1)c_{n+1} - 2c_n\right]x^n = 0 .$$

Thus,
$$(n+2)(n+1)c_{n+2} - (n+1)c_{n+1} - 2c_n = 0 .$$
Therefore, the proof is complete.

Lemma 2.4 Let $n \geq 0$. The following results hold.

$$c_n = \frac{J_n c_1 + 2J_{n-1}c_0}{n!} , \tag{2.7}$$

$$c_n = \frac{(2j_{n-1} + j_{n+1})c_1 + 2(2j_{n-2} + j_n)c_0}{9n!} . \tag{2.8}$$

Proof. If $n = 0$, then $\frac{J_0 c_1 + 2J_{-1}c_0}{0!} = c_0$ the proof is obvious. Next, we will be shown that $c_n = \frac{J_n c_1 + 2J_{n-1}c_0}{n!}$ for $n \in \mathbb{N}$. It is not hard to see that $\frac{J_1 c_1 + 2J_0 c_0}{1!} = c_1$. Thus (2.7) holds $n = 1$. Let us assume that the equality in (2.7) holds for all $n \leq k \in \mathbb{N}$ by iterating this procedure and considering induction steps. To finish the proof. We must show that (2.7) also holds $n = k+1$ by considering Lemma 2.3. Thus

$$\begin{aligned}
c_{k+1} &= \frac{kc_k + 2c_{k-1}}{k(k+1)} \\
&= \frac{k\left(\frac{J_k c_1 + 2J_{k-1}c_0}{k!}\right) + 2\left(\frac{J_{k-1}c_1 + 2J_{k-2}c_0}{(k-1)!}\right)}{k(k+1)} \\
&= \frac{J_k c_1 + 2J_{k-1}c_0 + 2(J_{k-1}c_1 + 2J_{k-2}c_0)}{(k+1)!} \\
&= \frac{(J_k + 2J_{k-1})c_1 + 2(J_{k-1} + 2J_{k-2})c_0}{(k+1)!} \\
&= \frac{J_{k+1}c_1 + 2J_k c_0}{(k+1)!} .
\end{aligned}$$

Thus, $n = k+1$ is true. The similar proof of (2.7) is applied for (2.8). Therefore, the proof is complete.

Now, we find $\sin J(x)$ and $\sin j(x)$ in terms of sums, which corresponds to the following theorem.

Theorem 2.5 Let $n \geq 0$. Then $\sin J(x)$ and $\sin j(x)$ are given respectively by

$$\sin J(x) = \sum_{n=0}^{\infty} J_n \frac{x^n}{n!} , \tag{2.9}$$

$$\sin j(x)=\sum_{n=0}^{\infty} j_n \frac{x^n}{n!}. \quad (2.10)$$

Proof. Let $y=\sum_{n=0}^{\infty} c_n x^n$, we have

$$y=c_0+c_1x+\frac{J_2c_1+2J_1c_0}{2!}x^2+\ldots+\frac{J_nc_1+2J_{n-1}c_0}{n!}x^n+\ldots \quad (2.11)$$

and

$$y'=c_1+(J_2c_1+2J_1c_0)x+\ldots+\frac{J_nc_1+2J_{n-1}c_0}{(n-1)!}x^{n-1}+\ldots. \quad (2.12)$$

By using initial values $y(0)=0$, $y'(0)=1$ in (2.11) and (2.12), we obtain

$$c_0=0 \text{ and } c_1=1. \quad (2.13)$$

By using (2.13) in (2.11), we get

$$y=x+\frac{J_2}{2!}x^2+\ldots+\frac{J_n}{n!}x^n+\ldots=\sum_{n=0}^{\infty} J_n\frac{x^n}{n!}.$$

Thus, $\sin J(x)=\sum_{n=0}^{\infty} J_n\frac{x^n}{n!}$. The similar proof of (2.9) is applied for (2.10). Therefore, the proof is complete.

Then, we define Jacobsthal cosine $\mathbf{\cos J(x)}$ and Jacobsthal-Lucas cosine $\mathbf{\cos j(x)}$ by using derivatives, which correspond to the following definition.

Definition 2.6 Let $\alpha>\beta$. Then the Jacobsthal cosine $\mathbf{\cos J(x)}$ and Jacobsthal-Lucas cosine $\mathbf{\cos j(x)}$ are defined respectively by

$$\cos J(x)=\frac{\alpha e^{\alpha x}-\beta e^{\beta x}}{\alpha-\beta}, \quad (2.14)$$

$$\cos j(x)=\alpha e^{\alpha x}+\beta e^{\beta x}, \quad (2.15)$$

Moreover, we find $\cos J(x)$ and $\cos j(x)$ in terms of sums, which corresponds to the following theorem.

Theorem 2.7 Let $n\geq 0$. Then $\cos J(x)$ and $\cos j(x)$ are given respectively by

$$\cos J(x)=\sum_{n=0}^{\infty} J_{n+1}\frac{x^n}{n!}, \quad (2.16)$$

$$\cos j(x)=\sum_{n=0}^{\infty} j_{n+1}\frac{x^n}{n!}. \quad (2.17)$$

Proof. Since Theorem 2.4, we have

$$\frac{d}{dx}\sin J(x)=\frac{d}{dx}\sum_{n=0}^{\infty} J_n\frac{x^n}{n!}=\sum_{n=0}^{\infty} J_n\frac{d}{dx}\frac{x^n}{n!}=\sum_{n=1}^{\infty} J_n\frac{x^{n-1}}{(n-1)!}=\sum_{n=0}^{\infty} J_{n+1}\frac{x^n}{n!}.$$

Thus, $\cos J(x)=\sum_{n=0}^{\infty} J_{n+1}\frac{x^n}{n!}$. The similar proof of (2.9) is applied for (2.10). Therefore, the proof is complete.

Furthermore, we find Jacobsthal tangent, Jacobsthal-Lucas tangent, Jacobsthal cotangent, and Jacobsthal-Lucas cotangent, which corresponds to the following lemma definition and theorem.

Lemma 2.8 For all real numbers x. The following results hold.

(i) $\cos J(x)\neq 0$,

(ii) $\cos j(x)\neq 0$'

(iii) $\sin J(x)\neq 0$,

(iv) $\sin j(x)\neq 0$.

Proof. Suppose that $\cos J(x)=0$, then $\frac{\alpha e^{\alpha x}-\beta e^{\beta x}}{\alpha-\beta}=0$. It follows that $\alpha e^{\alpha x}-\beta e^{\beta x}=0$. So $\alpha e^{\alpha x}=\beta e^{\beta x}$. Therefore $\alpha=\beta$. But $\alpha>\beta$, we have a contradiction.

Thus $\cos J(x) \neq 0$, for all real numbers x. The similar proof of (i) is applied for (ii), (iii), and (iv). Therefore, the proof is complete.

Definition 2.9 Let $\alpha > \beta$. Then the Jacobsthal tangent $\tan J(x)$, Jacobsthal-Lucas tangent $\tan j(x)$, Jacobsthal cotangent $\cot J(x)$, and Jacobsthal-Lucas cotangent $\cot j(x)$ are defined respectively by

$$\tan J(x) = \frac{\sin J(x)}{\cos J(x)} = \frac{e^{\alpha x} - e^{\beta x}}{\alpha e^{\alpha x} - \beta e^{\beta x}}, \quad (2.18)$$

$$\tan j(x) = \frac{\sin j(x)}{\cos j(x)} = \frac{e^{\alpha x} + e^{\beta x}}{\alpha e^{\alpha x} + \beta e^{\beta x}}, \quad (2.19)$$

$$\cot J(x) = \frac{\cos J(x)}{\sin J(x)} = \frac{\alpha e^{\alpha x} - \beta e^{\beta x}}{e^{\alpha x} - e^{\beta x}}, \quad (2.20)$$

$$\cot j(x) = \frac{\cos j(x)}{\sin j(x)} = \frac{\alpha e^{\alpha x} + \beta e^{\beta x}}{e^{\alpha x} + e^{\beta x}}. \quad (2.21)$$

Theorem 2.10 Let $n \geq 0$. Then $\tan J(x)$, $\tan j(x)$, $\cot J(x)$, and $\cot j(x)$ are given respectively by

$$\tan J(x) = -\frac{\beta}{2} + \frac{\beta^2 + 4\beta}{4} \sum_{n=0}^{\infty} (-1)^n \frac{(J_n\beta + 2J_{n-1})^2}{2^n} e^{-(n+1)(\alpha-\beta)x}, \quad (2.22)$$

$$\tan j(x) = -\frac{\beta}{2} - \frac{\beta^2 + 4\beta}{324} \sum_{n=0}^{\infty} \frac{(2j_{n-1}\beta + j_{n+1}\beta + 4j_{n-2} + 2j_n)^2}{2^n} e^{-(n+1)(\alpha-\beta)x}, \quad (2.23)$$

$$\cot J(x) = -\frac{t}{\beta} - \frac{\beta + 4}{\beta} \sum_{n=0}^{\infty} e^{-(n+1)(\alpha-\beta)x}, \quad (2.24)$$

$$\cot j(x) = -\frac{t}{\beta} + \frac{\beta + 4}{\beta} \sum_{n=0}^{\infty} (-1)^n e^{-(n+1)(\alpha-\beta)x}. \quad (2.25)$$

Proof. Since (2.18), we have

$$\tan J(x) = \frac{e^{\alpha x} - e^{\beta x}}{\alpha e^{\alpha x} - \beta e^{\beta x}}$$

$$= \frac{\beta}{2}\left(\frac{-1 + e^{(\beta-\alpha)x}}{1 + \frac{\beta^2}{2} e^{(\beta-\alpha)x}} \right)$$

$$= \frac{\beta}{2}\left(-1 + \frac{\beta+4}{2} e^{(\beta-\alpha)x} - \frac{\beta+4}{2}\frac{\beta^2}{2} e^{2(\beta-\alpha)x} + \ldots \right)$$

$$= -\frac{\beta}{2} + \frac{\beta^2 + 4\beta}{4} e^{(\beta-\alpha)x} - \frac{\beta^2 + 4\beta}{4}\frac{\beta^2}{2} e^{2(\beta-\alpha)x} + \ldots$$

$$= -\frac{\beta}{2} + \frac{\beta^2 + 4\beta}{4} \sum_{n=0}^{\infty} (-1)^n \frac{\beta^{2n}}{2^n} e^{-(n+1)(\alpha-\beta)x}.$$

Thus, $\tan J(x) =$

$$-\frac{\beta}{2} + \frac{\beta^2 + 4\beta}{4} \sum_{n=0}^{\infty} (-1)^n \frac{(J_n\beta + 2J_{n-1})^2}{2^n} e^{-(n+1)(\alpha-\beta)x}.$$

The similar proof of (2.18) is applied for (2.19), (2.20), and (2.21). Therefore, the proof is complete.

Next, we find Jacobsthal secant, Jacobsthal-Lucas secant, Jacobsthal cosecant, and Jacobsthal-Lucas cosecant, which corresponds to the following definition and theorem.

Definition 2.11 Let $\alpha > \beta$. Then the Jacobsthal secant $\sec J(x)$, Jacobsthal-Lucas secant $\sec j(x)$, Jacobsthal cosecant $\operatorname{cosec} J(x)$, and Jacobsthal-Lucas cosecant $\operatorname{cosec} j(x)$ are defined respectively by

$$\sec J(x) = \frac{1}{\cos J(x)} = \frac{\alpha - \beta}{\alpha e^{\alpha x} - \beta e^{\beta x}}, \quad (2.26)$$

$$\sec j(x) = \frac{1}{\cos j(x)} = \frac{1}{\alpha e^{\alpha x} + \beta e^{\beta x}}, \quad (2.27)$$

$$\operatorname{cosec} J(x) = \frac{1}{\sin J(x)} = \frac{\alpha - \beta}{e^{\alpha x} - e^{\beta x}}, \quad (2.28)$$

$$\operatorname{cosec} j(x) = \frac{1}{\sin j(x)} = \frac{1}{e^{\alpha x} + e^{\beta x}}. \quad (2.29)$$

Theorem 2.12 Let $n \geq 0$. Then $\sec J(x)$, $\sec j(x)$, $\operatorname{cosec} J(x)$, and $\operatorname{cosec} j(x)$ are given respectively by

$$\sec J(x)=\left(\frac{3}{\alpha e^{\alpha x}-\beta e^{\beta x}}\right)^2\sum_{n=0}^{\infty}J_{n+1}\frac{x^n}{n!}, \tag{2.30}$$

$$\sec j(x)=\frac{1}{\left(\alpha e^{\alpha x}+\beta e^{\beta x}\right)^2}\sum_{n=0}^{\infty}j_{n+1}\frac{x^n}{n!}, \tag{2.31}$$

$$\cos ecJ(x)=\left(\frac{3}{\alpha e^{\alpha x}-\beta e^{\beta x}}\right)^2\sum_{n=0}^{\infty}J_{n}\frac{x^n}{n!}, \tag{2.32}$$

$$\cos ecj(x)=\frac{1}{\left(e^{\alpha x}+e^{\beta x}\right)^2}\sum_{n=0}^{\infty}j_{n}\frac{x^n}{n!}. \tag{2.33}$$

Proof. The proof of Theorem 2.10 is applied for (2.30), (2.31), (2.32), and (2.33).

Finally, we find some identities of the Jacobsthal sine, Jacobsthal-Lucas sine, Jacobsthal cosine, Jacobsthal-Lucas cosine, Jacobsthal tangent, Jacobsthal-Lucas tangent, Jacobsthal cotangent, Jacobsthal-Lucas cotangent, Jacobsthal secant, Jacobsthal-Lucas secant, Jacobsthal cosecant, and Jacobsthal - Lucas cosecant, which corresponds to the following definition and theorem.

Theorem 2.13 Let $\alpha>\beta$. The following results hold.

(i) $\cos J^2(x)-\sin J(x)\cos J(x)-2\sin J^2(x)=e^x$,

(ii) $\cos j^2(x)-\sin j(x)\cos j(x)-2\sin j^2(x)=-9e^x$,

(iii) $e^x\sec J^2(x)+\tan J(x)+2\tan J^2(x)=1$,

(iv) $-9e^x\sec j^2(x)+\tan j(x)+2\tan j^2(x)=1$.

Proof. Since (2.2) and (2.14), we have

$$\cos J^2(x)-\sin J(x)\cos J(x)-2\sin J^2(x)$$

$$=\left(\frac{\alpha e^{\alpha x}-\beta e^{\beta x}}{e^{\alpha x}-e^{\beta x}}\right)^2$$

$$-\left(\frac{e^{\alpha x}-e^{\beta x}}{e^{\alpha x}-e^{\beta x}}\right)\left(\frac{\alpha e^{\alpha x}-\beta e^{\beta x}}{e^{\alpha x}-e^{\beta x}}\right)-2\left(\frac{e^{\alpha x}-e^{\beta x}}{e^{\alpha x}-e^{\beta x}}\right)^2$$

$=e^x$.

Thus,

$$\cos J^2(x)-\sin J(x)\cos J(x)-2\sin J^2(x)$$

$=e^x$, The proof of (i) is applied for (ii), (iii), and (iv). by using (2.3), (2.15), (2.18), (2.26), and (2.27). Therefore, the proof is complete.

3. Conclusions

In this paper, we investigate Jacobsthal sine, Jacobsthal-Lucas sine, Jacobsthal cosine, Jacobsthal-Lucas cosine, Jacobsthal tangent, Jacobsthal-Lucas tangent, Jacobsthal cotangent, Jacobsthal-Lucas cotangent, Jacobsthal secant, Jacobsthal-Lucas secant, Jacobsthal cosecant, and Jacobsthal-Lucas cosecant. Furthermore, we obtain some identities of Jacobsthal sine, Jacobsthal-Lucas sine, Jacobsthal cosine, Jacobsthal-Lucas cosine, Jacobsthal tangent, Jacobsthal-Lucas tangent, Jacobsthal cotangent, Jacobsthal-Lucas cotangent, Jacobsthal secant, Jacobsthal-Lucas secant, Jacobsthal cosecant, and Jacobsthal-Lucas cosecant.

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