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A Characterization of S_β- continuous Fixed Point Property

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Abstract

In this paper, we define and investigate the S_β- continuous retraction and the S_β- continuous fixed point property which apply the S_β- continuity in (6). The study shown that, for the regular and locally indiscrete topological space (X,τ) with the S_β- continuous fixed point property, if a topology σ for X is stronger than τ and for every open subset G in σ with the closure of G in σ and the closure of G in τ are equal, then (X,σ) has the fixed point property.

Keywords: S_β- open sets, S_β- continuous function, fixed point property, S_β- continuous fixed point property

1. Introduction

For the two topological spaces (X,τ) and (X,σ), in 1959 Conell EH. (3) has introduced the concept of the fixed point property. If (X,τ) is a regular space with the fixed point property, σ is stronger than τ, G is an open subset in σ, its closure is the same in both topologies, then (X,σ) has the fixed point property. In 1990, Cammaroto F. and Noiri T. (2) has introduced the concept of the δ- continuous fixed point property. If (X,τ) is a almost-regular space with the δ- continuous fixed point property, σ is stronger than τ, G is an open subset in σ, its closure is the same in both topologies, then (X,σ) has the fixed point property. In 2007, Puturong N. (9) has introduced the concept of the strongly θ- semi-continuous fixed point property. If (X,τ) is a regular space with the strongly θ- semi-continuous fixed point property, σ is stronger than τ, G is an open subset in σ, its closure are the same in both topologies, then (X,σ) has the fixed point property. In 2013, Khalaf BA. and Ahmed KN. (6) has introduced S_β- continuous function $f:(X,\tau)\to(Y,\sigma)$, if for each x in X and for each open subset H in Y containing $f(x)$, there exists an S_β- open subset G in X containing x such that $f(G)\subseteq H$. The purpose of this paper is to apply S_β- continuity to the retraction and fixed point property. We shall offer some statements, (X,τ) is a locally indiscrete space with S_β- continuous fixed point property and the same above conditions implies that (X,σ) has the fixed point property.

2. Preliminaries

Throughout the present note spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Recall the following definitions and characterizations. In a topological space (X,τ) (or simply X) and let A be a subset of X. The intersection of all closed sets containing A is called the closure of A and is denoted by $cl(A)$. The interior of A is defined by the union of all open sets contained in A and is denoted by $\operatorname{int}(A)$. We shall define semi-open set, semi-closed set, β-closed set, S_β-open set, and S_β-closed set as follows.

Definition 2.1. (6-8) Let (X,τ) be a topological space. A subset A of X is called

(i) *semi-open* if $A \subseteq cl(\operatorname{int}(A))$, the complement of a semi-open set is called semi-closed set.

(ii) β- *open* if $A \subseteq cl(\operatorname{int}(cl(A)))$, the complement of a β- open set is called β-closed set.

(iii) S_β-*open* if A is a semi-open set and for each $x \in A$ there exists a β-closed set F such that $x \in F \subseteq A$, and $B \subseteq X$ is called S_β-closed set if the complement of B is S_β-open set.

The set of all semi-open (resp. β-open, S_β- open) sets is denoted by $SO(X,\tau)$ (resp. $\beta O(X,\tau)$, $S_\beta O(X,\tau)$). The set of all semi-closed (resp. β- closed, S_β- closed) sets is denoted by $SC(X,\tau)$ (resp. $\beta C(X,\tau)$, $S_\beta C(X,\tau)$). The semi closure, semi interior, β-closure, β-interior, S_β-closure, and S_β-interior of a set A are defined in similar way of closure and interior of a set A. The closure of G in (X,τ) is denoted by $cl(G)^{(\tau)}$.

Definition 2.2. (4) A topological space (X,τ) is said to be:

(i) *locally indiscrete* if every open subset of X is closed. (A set $A \subseteq X$ is called clopen, if A are both open and closed.)

(ii) *regular* if given any point $x \in X$ and any closed subset $F \subseteq X-\{x\}$ implies that there exist open sets U and V such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$.

Definition 2.3. (6) Let (X,τ) be a topological space. A subset N of X is called S_β- *neighbourhood* of subset A in X if, there exists a S_β- open set U such that $A \subseteq U \subseteq N$. We say that N is a S_β- *neighbourhood* of an element $x \in X$ if $A=\{x\}$, the set of all S_β-neighbourhood of an element x denoted by $S_\beta\eta(x)$. The S_β- *interior* and S_β-*closure* of a set A denoted by $S_\beta \operatorname{int}(A)=\{x\in X \mid A\in S_\beta\eta(x)\}$ and $S_\beta cl(A)=\{x\in X \mid A\cap W\neq\phi,\ \forall W\in S_\beta\eta(x)\}$ respectively.

Example 2.1 (6) Let $X=\{p,q,r,s\}$, $\tau=\{\phi,\{p\},\{q\},\{p,q\},\{p,q,r\},X\}$ be topology on X. We get that $S_\beta O(X)=\{\phi,\{q\},\{p,r,s\},X\}$. Consider $A=\{r,s\}$. Then S_β- neighbourhood of subset A are $\{p,r,s\}$ and X, $S_\beta\eta(q)=\{\{q\},X\}$, $S_\beta \operatorname{int}(A)=\phi$, and $S_\beta cl(A)=\{p,r,s\}$.

Some properties of S_β- interior and S_β-closure sets are mentioned in the following result. For the routine proofs are omitted.

Theorem 2.1. (6) For any subset A and B of a topological space X. The following statements are holds:

(i) The S_β-interior of A is the union of all S_β-open sets contained in A.

(ii) $S_\beta \operatorname{int}(A)$ is the largest S_β-open set contained in A.

(iii) A set A is S_β- open set if and only if $A=S_\beta \operatorname{int}(A)$.

(iv) $S_\beta \operatorname{int}(\phi)=\phi$ and $S_\beta \operatorname{int}(X)=X$.

(v) $S_\beta \operatorname{int}(A)\subseteq A$.

(vi) If $A\subseteq B$, then $S_\beta \operatorname{int}(A)\subseteq S_\beta \operatorname{int}(B)$.

(*vii*) $S_\beta \operatorname{int}(A) \cup S_\beta \operatorname{int}(B)$
$\subseteq S_\beta \operatorname{int}(A \cup B).$

(*viii*) $S_\beta \operatorname{int}(A \cap B)$
$\subseteq S_\beta \operatorname{int}(A) \cap S_\beta \operatorname{int}(B).$

(ix) $S_\beta \operatorname{int}(A \setminus B)$
$\subseteq S_\beta \operatorname{int}(A) \setminus S_\beta \operatorname{int}(B).$

Here we give some properties of S_β-closure of a set similar forms of closure of a set.

Theorem 2.2. (6) For any subset F and E of a topological space X. The following statements are true:

(i) The S_β-closure of F is the intersection of all S_β-closed sets in X containing F.

(ii) $S_\beta cl(F)$ is the smallest S_β-closed set containing F.

(iii) F is S_β-closed set if and only if $F = S_\beta cl(F)$.

(iv) $S_\beta cl(\phi) = \phi$ and $S_\beta cl(X) = X$.

(v) $F \subseteq S_\beta cl(F)$.

(vi) If $F \subseteq E$, then
$S_\beta cl(F) \subseteq S_\beta cl(E)$.

$(vii) S_\beta cl(F) \cup S_\beta cl(E)$
$\subseteq S_\beta cl(F \cup E).$

$(viii) S_\beta cl(F \cap E)$
$\subseteq S_\beta cl(F) \cap S_\beta cl(E).$

(ix) $X \setminus S_\beta cl(F) = S_\beta \operatorname{int}(X \setminus F)$.

(x) $X \setminus S_\beta \operatorname{int}(F) = S_\beta cl(X \setminus F)$.

(xi) $S_\beta \operatorname{int}(F) = X \setminus S_\beta cl(X \setminus F)$.

Definition 2.4. (4, 6) Let $f:(X,\tau)\to(Y,\sigma)$ be a function,

(i) f is *continuous* at $x \in X$ if for each open subset V of $f(x)$, there exists an open set U of x such that $f(U) \subseteq V$.

(ii) f is S_β-*continuous* at $x \in X$ if for each open subset V of $f(x)$, there exists an S_β-open set U of x such that $f(U) \subseteq V$. If f is S_β-continuous at x for all $x \in X$, we say that f is S_β-continuous on X.

(iii) f is *open* if for each open subset U of X, $f(U) \in \sigma$.

Definition 2.5. (1) Let (X,τ) be a topological space.

(i) A topological space (X,τ) is said to be *regular* if for any point $x \in X$ and any closed subset $F \subseteq X \setminus \{x\}$, implies that there exist open sets U and V, such that $x \in U$, and $U \cap V = \phi$.

(ii) A space X has the *fixed point property* if, every continuous function $f: X \to X$, there exists an $x \in X$ such that $f(x) = x$, briefly denoted by *FPP*.

Theorem 2.3. (6) Let (X,τ) be a topological space and $G \subseteq X$. The following statements hold:

(i) If $G \in \tau$ then $G \in SO(X,\tau)$.

(ii) $G \in S_\beta O(X,\tau)$ if and only if $G \in SO(X,\tau)$ and $G = \cup\{B \subseteq G \mid B \in \beta O(X,\tau)\}$.

(iii) If X is a locally indiscrete space and $G \in SO(X,\tau)$ $(G \in SC(X,\tau))$ then G is closed (open) in X.

(iv) If X is a locally indiscrete space and $G \in SO(X,\tau)$ then $G \in S_\beta O(X,\tau)$.

Theorem 2.4. (6) Let (X,τ) and (Y,σ) be a topological spaces, and $f: X \to Y$, following are equivalent:

(i) f is S_β-continuous.

(ii) For each open set B in Y
$f^{-1}(B) \in S_\beta O(X,\tau)$.

(iii) For each closed set B in Y,
$f^{-1}(B) \in S_\beta C(X,\tau)$.

(iv) For each subset A in X,
$f(S_\beta cl(A)) \subseteq cl(f(A))$.

(v) For each subset A in X,
$\operatorname{int}(f(A)) \subseteq f(S_\beta \operatorname{int}(A))$.

(vi) For each subset B in Y,
$S_\beta cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

(vii) For each subset B in Y,
$f^{-1}(\operatorname{int}(B)) \subseteq S_\beta \operatorname{int}(f^{-1}(cl(B)))$.

Theorem 2.5. (6) Let (X,τ) and (Y,σ) be topological spaces, and $f: X \to Y$, $A \subseteq X$. If A is a clopen set then the restriction of a function f is S_β- continuous, as $f_{|A}: A \to Y$ is S_β-continuous.

The following theorem indicates that compositions of S_β- continuous functions are S_β- continuous, when a domain of the second function is locally indiscrete space.

Theorem 2.6. Let (X,τ), (Y,σ) and (Z,υ) be topological spaces, and Y is a locally indiscrete space, if $f: X \to Y$ and $g: Y \to Z$ be the two S_β- continuous then the composite function $g \circ f$ (see in Figure 1) is S_β- continuous.

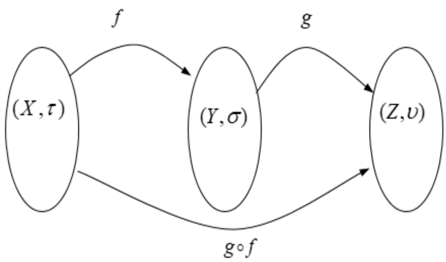

Figure 1 The composite function $g \circ f$.

Proof. Let G be an open set in Z. Since g is S_β- continuous, by theorem 2.4. (ii), so that $g^{-1}(G) \in S_\beta O(Y,\sigma)$. Moreover, since Y is a locally indiscrete space, by Theorem 2.3. (i, iv), $g^{-1}(G)$ is open set in Y. For assumption f is S_β- continuous, we obtain $f^{-1}(g^{-1}(G)) \in S_\beta O(X,\tau)$.

But $(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G))$, therefore and hence $g \circ f$ is S_β- continuous. □

3. Main Results

The purpose of this section is to explore properties and characterizations of S_β- continuous retract and S_β- continuous fixed point property. Cammarolo and Nori (2) defined a subset A of a space X to be a δ- continuous retract of X, defined a space X has the δ- continuous fixed point property. We shall similarly defined a S_β- continuous retract and S_β- continuous fixed point property as follows.

Definition 3.1. Let (X,τ) be a topological spaces.

(i) A subset A of space X is called a S_β- *continuous retract of* X if there exists a S_β- continuous $f: X \to A$ such that $f_{|A}$ is the identity on A, that is $f(x) = x$ for every $x \in A$, and f is called a S_β- *continuous retraction*.

(ii) A space X is said to have the S_β- *continuous fixed point property*, briefly denoted by $S_\beta cFPP$, if for every S_β- continuous function $f: X \to X$, there exists an $x \in X$ such that $f(x) = x$.

Example 3.1. Let $X = \{p,q,r,s\}$ and let $\tau = \{\phi,\{r\},\{p,q\},\{p,q,r\},\{p,q,s\},X\}$, $\tau_A = \{\phi,\{p,q\},A\}$, $A = \{p,q,s\}$, and $f:(X,\tau) \to (A,\tau_A)$ defined that $f(p) = p$, $f(q) = q$, $f(r) = r$, $f(s) = s$. Since $S_\beta O(X,\tau) = \{\phi,\{r\},\{p,q\},\{p,q,r\},\{p,q,s\},X\}$ and f is S_β- continuous. Hence A is a S_β- continuous retract of X.

Example 3.2. Let $X = \{p,q,r\}$ and $\tau = \{\phi,\{p\},\{q,r\},X\}$ be a topology for X. We show that X has the S_β- continuous fixed point property.

Solution. Now, we prove by contrapositive. Assume that, there exist a S_β- continuous function $f: X \to X$ such that $f(x) \neq x$, $\forall x \in X$. Suppose that for $p,q,r \in X$ such that $f(p) = q$, $f(q) = r$, and $f(r) = q$.

Step1. To show that, f is S_β- continuous. Because $\phi,\{p\},\{q,r\}$, and X are open sets in X. By definition of f, we have, $f^{-1}(\phi) = \phi$, $f^{-1}(p) = \phi$, and $f^{-1}(\{q,r\}) = X$, such that $\phi, X \in S_\beta O(X)$. By Theorem 2.4. (ii), Therefore f is a S_β- continuous.

Step2. To show that, $f(x) \neq x$, $\forall x \in X$. By defined function of f $f(p) = q$, $f(q) = r$ and $f(r) = q$, such that $p \neq q$ and $q \neq r$. It follows that, $f(x) \neq x$, $\forall x \in X$. Form the both steps, it shows that, X has the S_β- continuous fixed point property.

Next, we investigate some properties of S_β- continuous retract of a space X.

Lemma 3.1. Let (X,τ) be a topological space, $A \subseteq X$ with a locally indiscrete subspace (A,τ_A). Then A is a S_β- continuous retract of X if and only if for every topological space (Y,σ) and for each S_β- continuous function $f: A \to Y$ can be extend to a S_β- continuous over X relative to Y.

Proof. ($\Rightarrow$) Suppose that (X,τ) is a topological space and $A \subseteq X$ is locally indiscrete subspace (A,τ_A). Let $f: A \to Y$ be a S_β- continuous function. Since A is a S_β- continuous retract of X. By Definition 3.1. (i), we have $g: X \to A$ is a S_β- continuous function.
By Theorem 2.6. and property of locally indiscrete subspace (A,τ_A), therefore $f \circ g: X \to Y$ is a S_β- continuous function and implies $f \circ g(x) = f(g(x)) = f(x)$, for every $x \in A$. Hence $f \circ g$ is an extension of f.
($\Leftarrow$) Suppose that $i_A: A \to A$ is an identity function on A. Note that i_A is also a S_β- continuity. By assumption, we have a S_β- continuous function $g: X \to A$ where $g_{|A} = i_A$. Therefore A is a S_β- continuous retract of X. □

A S_β- continuous retract of a space X with a clopen set is necessity for the S_β- continuous fixed point property, as following.

Theorem 3.1. Let (X,τ) be a topological space, A be a clopen subset in X. If X has the S_β- continuous fixed point property and A is a S_β- continuous retract of X, then A has the S_β- continuous fixed point property $(S_\beta cFPP)$.

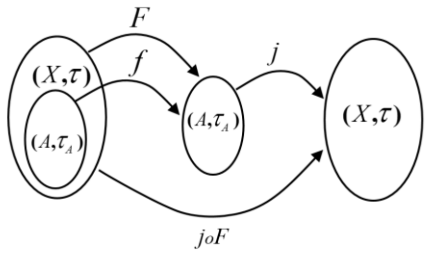

Figure 2 To show functions f, F, j, and $j \circ F$.

Proof. Suppose that $f: A \to A$ is a S_β- continuous function, and A is a clopen (both open and closed) set in X. Then (A,τ_A) is a locally indiscrete subspace. Since A is a S_β- continuous retract of X, by Lemma 3.1., we have a S_β- continuous function $F: X \to A$ which it is an extension of f.
Let $j: A \to X$ be an identity function, as $f(x) = x$, for all $x \in A$. To show that $j \circ F$ is a S_β- continuous function (see in Figure 2). Let U be an open in X, hence $j^{-1}(U) = U \cap A$ is an open in (A,τ_A). Since $F: X \to A$ is a S_β- continuous function, implies that $F^{-1}(j^{-1}(U)) = (j \circ F)^{-1}(U) \in S_\beta O(X)$, and therefore $(j \circ F): X \to X$ is a S_β- continuous function. Consequently to show that A has the S_β- continuous fixed point property. Since X has the S_β- continuous fixed point property. There exists an $x \in A \subseteq X$ such that $x = (j \circ F)(x) = j(F(x)) = j(f(x)) = f(x)$. So, it completes the proof. □

The following theorem is a slight modification of theorem 1 in (3), Theorem 3.2. in (2), and Theorem 4.11. in (9).

Theorem 3.2 Let (X,τ) and (X,σ) be topological spaces where (X,τ) is a locally indiscrete space, and it has the S_β- continuous fixed point property. If σ stronger than τ

(as $\sigma \supseteq \tau$) and $cl(G)^{(\tau)} = cl(G)^{(\sigma)}$, for any $G \in \sigma$ then (X,σ) has the continuous fixed point property.

$$\begin{tikzcd}
(X,\sigma) \arrow[r, "f"] \arrow[d, "g"'] & (X,\sigma) \\
(X,\tau) \arrow[r, "h"'] & (X,\tau) \arrow[u, "i"']
\end{tikzcd}$$

Figure 3 Diagram of functions f, g, h, and i.

Proof. We must show that, (X,σ) has the continuous fixed point property. Let $f:(X,\sigma) \to (X,\sigma)$ be a continuous function. We must demonstrate $f(x) = x,\ \forall x \in X$. Defined the three functions, g, h, and i which $g:(X,\sigma) \to (X,\tau)$, $h:(X,\tau) \to (X,\tau)$ are defined by setting $g(x) = h(x) = f(x)$, for all $x \in X$, and $i:(X,\tau) \to (X,\sigma)$ is an identity function. For every $x \in X$, we have $f(x) = g(x) = i(g(x)) = i \circ g(x)$, (see in Figure 3) so that $f = i \circ g$ is a continuous function. To show that g is a continuous function. Let $G \in \tau$.

By identity function property, i is an open, one to one and onto. Therefore $i(G) \in \sigma$. Since $i \circ g$ is continuity, we have $(i \circ g)^{-1}(i(G)) \in \sigma$. By the property of one to one, onto, and inverse function. We have

$$\begin{aligned}(i \circ g)^{-1}(i(G)) &= (g^{-1} \circ i^{-1})(i(G))) \\ &= g^{-1}(i^{-1}(i(G))) \\ &= g^{-1}(G).\end{aligned}$$

So that $g^{-1}(G) \in \sigma$, therefore g is a continuous function. To show that h is a S_β- continuous function. Let $x \in X$ and $h(x) \in V$, for all $V \in \tau$. Since (X,τ) is a regular spaces, there exists an $A \in \tau$ such that $h(x) \in A \subseteq cl(A)^{(\tau)} \subseteq V$. Since g is a continuous function, $h(x) \in A \in \tau$, and the setting h, g, and f, we have $g^{-1}(A) \in \sigma$, and $h^{-1}(A) = f^{-1}(A) = g^{-1}(A)$. So that $h^{-1}(A) = f^{-1}(A) \in \sigma$. By closure properties and our assumption $cl(h^{-1}(A))^{(\tau)} = cl(h^{-1}(A))^{(\sigma)}$,

$$\begin{aligned}x \in h^{-1}(A) &\subseteq cl(h^{-1}(A))^{(\tau)} \\ &= cl(h^{-1}(A))^{(\sigma)} = cl(f^{-1}(A))^{(\sigma)}. \qquad (3.1)\end{aligned}$$

Since f is a continuous function and $cl(A)^{(\tau)} \subseteq V$, so that

$cl(f^{-1}(A))^{(\sigma)} \subseteq f^{-1}(cl(A)^{(\sigma)}) = f^{-1}(cl(A)^{(\tau)})$,

and $f^{-1}(cl(A)^{(\sigma)}) \subseteq f^{-1}(V)$. (3.2)

By (3.1) and (3.2), we obtain $cl(h^{-1}(A))^{(\tau)} \subseteq f^{-1}(V)$. From defined h, $h^{-1}(V) = f^{-1}(V)$, implies that $cl(h^{-1}(A))^{(\tau)} \subseteq h^{-1}(V)$. (3.3)

Since i is an open function with $h^{-1}(A) \in \sigma$, and $\tau \subseteq \sigma$, we have $h^{-1}(A)$ is an open in the space (X,τ). Let $U = h^{-1}(A)$. Since (X,τ) is a locally indiscrete space, implies that U be the both open and closed, therefore U is a semi open set in the space (X,τ). By Theorem 2.3. (iv), U is a S_β-open set in the space (X,τ). which $x \in U$. By (3.1) and (3.3), we have $U \subseteq h^{-1}(V)$, so that $h(U) \subseteq h(h^{-1}(V)) \subseteq V$. Hence h is a S_β-continuous function, by Definition 2.5. Finality to show that (X,σ) has the continuous fixed point property. By the assumption, (X,τ) has the S_β- continuous fixed point property. So there exists an $x \in X$ such that $h(x) = x$. But we have $f(x) = h(x)$, for all $x \in X$, implies that $f(x) = x$, so we have the result. □

Next, Lemma is to explore properties of an one to one, onto, open functions and the composition which it is S_β- continuous function.

Lemma 3.2. Let $(X,\tau), (Y,\sigma)$, and (Z,υ) be topological spaces, and $g:(Y,\sigma) \to (Z,\upsilon)$ be an one to one, onto, and open. If $g \circ f:(X,\tau) \to (Z,\upsilon)$ is a S_β- continuous function then $f:(X,\tau) \to (Y,\sigma)$ is a S_β-continuous function.

Proof. Let V be an open set in space (Y,σ). Since g is an open function, implies that $g(V)$ is an open set in space (Z,υ). by our assumption, $g \circ f$ is a S_β- continuous function, so that $(g \circ f)^{-1}(g(V)) \in S_\beta(X,\tau)$.

By property of inverse function, one to one, and onto function, we have

$$
\begin{aligned}
(g\circ f)^{-1}(g(V)) &= (f^{-1}\circ g^{-1})(g(V)) \\
&= f^{-1}(g^{-1}(g(V))) \\
&= f^{-1}(V)\in S_\beta(X,\tau).
\end{aligned}
$$

And by Theorem 2.4. (ii), hence f is a S_β- continuous function. □

In the following example, we shown that the converse of the Lemma 3.2 is not true. Recall a topological space (X,τ) of Example 3.2 in (5).

Example 3.3. Let $(X,\tau),(Y,\sigma)$, and (Z,υ) be topological spaces, such that $X=\{a,b,c,d\}$, $\tau=\{\phi,X,\{a\},\{b\},\{a,b\},\{a,b,c\}\}$, $Y=\{p,q,r\}$, $\sigma=\{\phi,Y\}$, $Z=\{l,m,n\}$, $\upsilon=\{\phi,Z,\{m\},\{n\},\{m,n\}\}$. Function $f:(X,\tau)\to(Y,\sigma)$ defined that $f(a)=p, f(b)=q, f(c)=r=f(d)$, and function $g:(Y,\sigma)\to(Z,\upsilon)$ defined that $g(p)=l, g(q)=m$, and $g(r)=n$. Then g is a one to one, onto, and open. Since $S_\beta O(X,\tau)=\{\phi,\{b\},\{a,c,d\},X\}$, and

$f^{-1}(\phi)=\phi\in S_\beta O(X,\tau)$, and
$f^{-1}(Y)=X\in S_\beta O(X,\tau)$.

So that f is S_β- continuous. Because

$g\circ f(a)=l, g\circ f(b)=m, g\circ f(c)=n$,
$g\circ f(d)=n$, and
$(g\circ f)^{-1}(\{m,n\})=\{b,c,d\}\notin S_\beta O(X,\tau)$.

Therefore, $g\circ f$ is not S_β- continuous.

In Theorem 3.2., we replace $cl(G)^{(\tau)}=cl(G)^{(\sigma)}$ with $S_\beta cl(G)^{(\tau)}=S_\beta cl(G)^{(\sigma)}$. We have the following Theorem.

Theorem 3.3. Let (X,τ) and (X,σ) be topological spaces with regular space (X,τ) has the continuous fixed point property. If σ stronger than τ and $S_\beta cl(G)^{(\tau)}=S_\beta cl(G)^{(\sigma)}$ where $G\in\sigma$, for all $G\in S_\beta O(X,\sigma)$ then (X,σ) has the S_β- continuous fixed point property.

Proof. Suppose that $f:(X,\sigma)\to(X,\sigma)$ is a S_β- continuous, we need to establish that $f(x)=x$, $\forall x\in X$. The three functions were defined, $g:(X,\sigma)\to(X,\tau)$, $h:(X,\tau)\to(X,\tau)$, and $i:(X,\tau)\to(X,\sigma)$, as $g(x)=h(x)=f(x)$, for all $x\in X$, and i be an identity. It follows that $f=i\circ g$ is a S_β- continuous function.

To show that g is a S_β continuous function. Let $G\in\tau$. The identity function imply open and bijection, we get $i(G)\in\sigma$. Since $i\circ g$ is a S_β- continuous, by Lemma 3.2, we have g is a S_β- continuous function. To show that h is a continuous function. Let $x\in X$ and $h(x)\in V$, for all open set V in space (X,τ). Since (X,τ). is a regular space, there exists an open subset A in space (X,τ). such that $h(x)\in A\subseteq cl(A)^{(\tau)}\subseteq V$. By the S_β-continuous function g, $h(x)\in A$, and the setting h, g, and f, we have $g^{-1}(A)\in S_\beta O(X,\sigma)$, and $h^{-1}(A)=f^{-1}(A)=g^{-1}(A)$, implies that $h^{-1}(A)=f^{-1}(A)\in S_\beta O(X,\sigma)$. Utilize S_β-closure, S_β-interior properties, and our assumption, we have $S_\beta cl(h^{-1}(A))^{(\tau)}=S_\beta cl(h^{-1}(A))^{(\sigma)}$, implies that

$$
\begin{aligned}
x\in h^{-1}(A) &= S_\beta \operatorname{int}(h^{-1}(A))^{(\sigma)} \\
&\subseteq S_\beta \operatorname{int}(S_\beta cl(h^{-1}(A))^{(\sigma)})^{(\sigma)} \\
&\subseteq S_\beta cl(h^{-1}(A))^{(\sigma)} \\
&= S_\beta cl(h^{-1}(A))^{(\tau)} \\
&= S_\beta cl(f^{-1}(A))^{(\sigma)} = S_\beta cl(f^{-1}(A))^{(\tau)}. \quad (3.4)
\end{aligned}
$$

Since $\tau\subseteq\sigma$, we have $cl(A)^{(\sigma)}\subseteq cl(A)^{(\tau)}$. Since f is a S_β- continuous function and $cl(A)^{(\tau)}\subseteq V$, we obtain that

$$
\begin{aligned}
S_\beta cl(f^{-1}(A))^{(\sigma)} &\subseteq f^{-1}(cl(A)^{(\sigma)}) \\
&\subseteq f^{-1}(cl(A)^{(\sigma)}) \\
&\subseteq f^{-1}(V). \qquad (3.5)
\end{aligned}
$$

By the (3.4) and (3.5), $h^{-1}(A)\subseteq f^{-1}(V)$. Since $h(x)=f(x)$, for every $x\in X$. So that $h^{-1}(V)=f^{-1}(V)$, we get $h^{-1}(A)\subseteq h^{-1}(V)$. Since $h^{-1}(A)=g^{-1}(A)\in\sigma$, $\tau\subseteq\sigma$, and using identity function, we have $i(h^{-1}(A))=h^{-1}(A)$, so that $h^{-1}(A)$ is an open set in space (X,τ) and $x\in h^{-1}(A)$. Let $U=h^{-1}(A)$, implies that $h(U)=h(h^{-1}(A))\subseteq h(h^{-1}(V))\subseteq V$. Hence h is a continuous function on (X,τ). Since (X,τ) has the continuous fixed point property, so there exists an $x \in X$ with $h(x) = x$, but $h(x) = f(x)$, for every $x \in X$, there exists an $x \in X$ with $f(x) = x$, hence (X, σ) has the S_β- continuous fixed point property. □

4. Conclusion

From the above study it is clear that for regular and locally indiscrete space (X, τ). If (X, τ) has the S_β- continuous fixed point property (the continuous fixed point property), σ is a topology for X stronger than τ and $cl(G)^{(\tau)} = cl(G)^{(\sigma)}$ ($S_\beta cl(G)^{(\tau)} = S_\beta cl(G)^{(\sigma)}$) for every $G \in \sigma$, then (X, σ) has the fixed point property (the S_β- continuous fixed point property). The results from this paper are based for further studies on the concept of other types of weak the continuous fixed point property. In the future we will study on minimal S_β- open sets and the continuous fixed point property in generalized topological spaces.

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