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Some Matrices with Padovan Q-matrix and the Generalized Relations

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Abstract

In this paper, we establish a new Q-matrix for Padovan numbers and the multiplies between the Q-matrix and the A-matrix. Moreover, we investigate the n^{th} of Q_2, the n^{th} of Q_1 multiply the A-matrix, and the n^{th} of Q_2 multiply the A-matrix. Finally, we use these matrices to obtain elementary identities for Padovan, Perrin, and relations between numbers.

Keywords: Q-matrix, Padovan Number, Perrin Number

1. Introduction

The Fibonacci $\{F_n\}$ and Lucas sequences $\{L_n\}$ are well–known sequences. For $n \geq 2$, the Fibonacci and Lucas sequences are defined respectively by $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$, with an initial value $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, and $L_1 = 1$. The Padovan $\{P_n\}$ and Perrin sequences $\{R_n\}$ are the favorable third-order sequences. For $n \geq 3$, the Padovan and Perrin sequences are defined respectively by $P_n = P_{n-2} + P_{n-3}$ and $R_n = R_{n-2} + R_{n-3}$, with an initial value $P_0 = P_1 = P_2 = 1$, $R_0 = 3$, $R_1 = 0$, and $R_2 = 2$. The first few values of P_n and R_n are 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16,… and 3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, …., respectively (see (6)).

In 1963, S. L. Basin and Verner E. Hoggatt, Jr. (1) studied the Fibonacci Q-matrix Q_F, which is defined as $Q_F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. After that, they showed that $Q_F^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$, for all $n \geq 1$. Moreover, they obtained some identities of Fibonacci numbers.

In 2013, Kritsana Sokhuma (3) studied the Padovan Q-matrix Q_1, defined as

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad (1.1)$$

such that the n^{th} of Q-matrix is defined by

$$Q_1^n = \begin{pmatrix} P_{n-5} & P_{n-3} & P_{n-4} \\ P_{n-4} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-1} & P_{n-2} \end{pmatrix}, \text{ for all } n \geq 1. \qquad (1.2)$$

Also, he obtained some identities of Padovan numbers (see (2, 4, 5)).

The direction of this research and development inspired this study. We present the Padovan Q-matrix, along with finding these identities. Now, we define Padovan and Perrin numbers for negative subscripts as follows: $P_{n-3} = P_n - P_{n-2}$ and $R_{n-3} = R_n - R_{n-2}$, (1.3) for all $n < 3$.

2. Main results

In this section, we first give them the new Padovan Q-matrix, which corresponds to the following definition:

Definition 2.1 The Padovan Q-matrix Q_2 can be written as

$$Q_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \tag{2.1}$$

Next, we define the 3×3 matrix of A-matrix, which the component of the matrix consists of R_0, R_1, R_2, R_3, and R_4, as follows:

Definition 2.2 The A-matrix can be written as

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}. \tag{2.2}$$

Now, we find the n^{th} of Q_2, which corresponds to the following theorem.

Theorem 2.3 Let P_n be the Padovan sequences. For $n \geq 1$, we have

$$Q_2^n = \begin{pmatrix} P_{2n-5} & P_{2n-3} & P_{2n-4} \\ P_{2n-4} & P_{2n-2} & P_{2n-3} \\ P_{2n-3} & P_{2n-1} & P_{2n-2} \end{pmatrix}. \tag{2.3}$$

Proof. We will be shown by mathematical induction that $Q_2^n = \begin{pmatrix} P_{2n-5} & P_{2n-3} & P_{2n-4} \\ P_{2n-4} & P_{2n-2} & P_{2n-3} \\ P_{2n-3} & P_{2n-1} & P_{2n-2} \end{pmatrix}$ for $n \geq 1$. Since $Q_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, it follows that $n = 1$ is true. Assume that the result is true for the positive integer $n = k$. Then $Q_2^k = \begin{pmatrix} P_{2k-5} & P_{2k-3} & P_{2k-4} \\ P_{2k-4} & P_{2k-2} & P_{2k-3} \\ P_{2k-3} & P_{2k-1} & P_{2k-2} \end{pmatrix}$. Now, we need to show that (2.3) also holds for $n = k+1$ as follows:

$$\begin{aligned} Q_2^{k+1} &= Q_2^k Q_2 \\ &= \begin{pmatrix} P_{2k-5} & P_{2k-3} & P_{2k-4} \\ P_{2k-4} & P_{2k-2} & P_{2k-3} \\ P_{2k-3} & P_{2k-1} & P_{2k-2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_{2k-3} & P_{2k-3} + P_{2k-4} & P_{2k-5} + P_{2k-4} \\ P_{2k-2} & P_{2k-2} + P_{2k-3} & P_{2k-4} + P_{2k-3} \\ P_{2k-1} & P_{2k-1} + P_{2k-2} & P_{2k-3} + P_{2k-2} \end{pmatrix} \\ &= \begin{pmatrix} P_{2k-3} & P_{2k-1} & P_{2k-2} \\ P_{2k-2} & P_{2k} & P_{2k-1} \\ P_{2k-1} & P_{2k+1} & P_{2k} \end{pmatrix}. \end{aligned}$$

Also, we find the n^{th} of Q_1 and Q_2, which multiplies the A-matrix, as shown in the following theorem.

Theorem 2.4 Let R_n be the Perrin sequences. For $n \geq 1$, we have

$$AQ_1^n = Q_1^n A = \begin{pmatrix} R_n & R_{n+2} & R_{n+1} \\ R_{n+1} & R_{n+3} & R_{n+2} \\ R_{n+2} & R_{n+4} & R_{n+3} \end{pmatrix}, \tag{2.4}$$

and

$$AQ_2^n = Q_2^n A = \begin{pmatrix} R_{2n} & R_{2n+2} & R_{2n+1} \\ R_{2n+1} & R_{2n+3} & R_{2n+2} \\ R_{2n+2} & R_{2n+4} & R_{2n+3} \end{pmatrix} \tag{2.4}$$

Proof. We will be shown by mathematical induction that $AQ_1^n = \begin{pmatrix} R_n & R_{n+2} & R_{n+1} \\ R_{n+1} & R_{n+3} & R_{n+2} \\ R_{n+2} & R_{n+4} & R_{n+3} \end{pmatrix}$ for $n \geq 1$. Since $AQ_1 = \begin{pmatrix} 0 & 3 & 2 \\ 2 & 2 & 3 \\ 3 & 5 & 2 \end{pmatrix}$, it follows that $n=1$ is true. Assume that the result is true for the positive integer $n=k$. Then $AQ_1^k = \begin{pmatrix} R_k & R_{k+2} & R_{k+1} \\ R_{k+1} & R_{k+3} & R_{k+2} \\ R_{k+2} & R_{k+4} & R_{k+3} \end{pmatrix}$. Now, we need to show that (2.3) also holds for $n=k+1$ as follows:

$$AQ_1^{k+1} = AQ_1^k Q_1$$
$$= \begin{pmatrix} R_k & R_{k+2} & R_{k+1} \\ R_{k+1} & R_{k+3} & R_{k+2} \\ R_{k+2} & R_{k+4} & R_{k+3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} R_{k+1} & R_k + R_{k+1} & R_{k+2} \\ R_{k+2} & R_{k+1} + R_{k+2} & R_{k+3} \\ R_{k+3} & R_{k+2} + R_{k+3} & R_{k+4} \end{pmatrix}$$
$$= \begin{pmatrix} R_{k+1} & R_{k+3} & R_{k+2} \\ R_{k+2} & R_{k+4} & R_{k+3} \\ R_{k+3} & R_{k+5} & R_{k+4} \end{pmatrix}.$$

The proof of $Q_1^n A = \begin{pmatrix} R_n & R_{n+2} & R_{n+1} \\ R_{n+1} & R_{n+3} & R_{n+2} \\ R_{n+2} & R_{n+4} & R_{n+3} \end{pmatrix}$ is similar to the above. Thus,

$$AQ_1^n = Q_1^n A = \begin{pmatrix} R_n & R_{n+2} & R_{n+1} \\ R_{n+1} & R_{n+3} & R_{n+2} \\ R_{n+2} & R_{n+4} & R_{n+3} \end{pmatrix}.$$

The proof of (2.5) is similar to (2.4). The proof is complete.

Finally, we find some identities of P_n and R_n. We also find some identities of the relations between P_n and R_n, as shown in the following theorem.

Theorem 2.5 For $n, r \geq 1$, Then

(i) $\det(Q_1^n) = 1$,

(ii) $P_{n-3}^3 + P_{n-5}P_{n-2}^2 + P_{n-4}^2 P_{n-1} - P_{n-3}(2P_{n-4}P_{n-2} + P_{n-5}P_{n-1}) = 1$,

(iii) $P_{n+r-4} = P_{n-4}P_{r-5} + P_{n-2}P_{r-4} + P_{n-3}P_{r-3}$,

(iv) $P_{n-r-4} = P_{n-2}(P_{r-3}^2 - P_{r-4}P_{r-2}) + P_{n-3}(P_{r-4}P_{r-1} - P_{r-3}P_{r-2}) + P_{n-4}(P_{r-2}^2 - P_{r-3}P_{r-1})$.

Proof. (i) Since $\det(Q_1) = 1$, we have

$$\det(Q_1^n) = (\det Q_1)^n = (1)^n = 1. \qquad (2.6)$$

(ii) Since (1.2), we get

$$\det(Q_1^n)$$
$$= P_{n-3}^3 + P_{n-5}P_{n-2}^2 + P_{n-4}^2 P_{n-1} - P_{n-3}(2P_{n-4}P_{n-2} + P_{n-5}P_{n-1}). \qquad (2.7)$$

By using (2.6) and (2.7), we get

$$P_{n-3}^3 + P_{n-5}P_{n-2}^2 + P_{n-4}^2 P_{n-1} - P_{n-3}(2P_{n-4}P_{n-2} + P_{n-5}P_{n-1}) = 1.$$

(iii) Since $Q_1^{n+r} = Q_1^n Q_1^r$, we have

$$\begin{pmatrix} P_{n+r-5} & P_{n+r-3} & P_{n+r-4} \\ P_{n+r-4} & P_{n+r-2} & P_{n+r-3} \\ P_{n+r-3} & P_{n+r-1} & P_{n+r-2} \end{pmatrix} = \begin{pmatrix} P_{n-5} & P_{n-3} & P_{n-4} \\ P_{n-4} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-1} & P_{n-2} \end{pmatrix} \begin{pmatrix} P_{r-5} & P_{r-3} & P_{r-4} \\ P_{r-4} & P_{r-2} & P_{r-3} \\ P_{r-3} & P_{r-1} & P_{r-2} \end{pmatrix}. \qquad (2.8)$$

By using Matrix multiplication in (2.8), we get $P_{n+r-4} = P_{n-4}P_{r-5} + P_{n-2}P_{r-4} + P_{n-3}P_{r-3}$.

(iv) Since $Q_1^{n-r} = Q_1^n (Q_1^r)^{-1}$, we have

$$\begin{pmatrix} P_{n-r-5} & P_{n-r-3} & P_{n-r-4} \\ P_{n-r-4} & P_{n-r-2} & P_{n-r-3} \\ P_{n-r-3} & P_{n-r-1} & P_{n-r-2} \end{pmatrix}$$

$$= \frac{1}{\det\left(Q_1^r\right)} \begin{pmatrix} P_{n-5} & P_{n-3} & P_{n-4} \\ P_{n-4} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-1} & P_{n-2} \end{pmatrix}$$

$$\begin{pmatrix} P_{r-2}^2 - P_{r-3}P_{r-1} & P_{r-4}P_{r-1} - P_{r-3}P_{r-2} & P_{r-3}^2 - P_{r-4}P_{r-2} \\ P_{r-3}^2 - P_{r-4}P_{r-2} & P_{r-5}P_{r-2} - P_{r-4}P_{r-3} & P_{r-4}^2 - P_{r-5}P_{r-3} \\ P_{r-4}P_{r-1} - P_{r-3}P_{r-2} & P_{r-3}^2 - P_{r-5}P_{r-1} & P_{r-5}P_{r-2} - P_{r-4}P_{r-3} \end{pmatrix}. \tag{2.9}$$

By using Matrix multiplication in (2.9), we get

$$P_{n-r-4} = P_{n-2}\left(P_{r-3}^2 - P_{r-4}P_{r-2}\right) + P_{n-3}\left(P_{r-4}P_{r-1} - P_{r-3}P_{r-2}\right) + P_{n-4}\left(P_{r-2}^2 - P_{r-3}P_{r-1}\right).$$

Thus, the identities of (i), (ii), (iii), and (iv) are easily seen.

Corollary 2.6 For $n, r \geq 1$, Then

(i) $\det\left(Q_2^n\right) = 1$,

(ii) $P_{2n-3}^3 + P_{2n-5}P_{2n-2}^2 + P_{2n-4}^2 P_{2n-1} - P_{2n-3}\left(2P_{2n-4}P_{2n-2} + P_{2n-5}P_{2n-1}\right) = 1$,

(iii) $P_{2n+2r-4} = P_{2n-2}P_{2r-4} + P_{2n-4}P_{2r-5} + P_{2n-3}P_{2r-3}$,

(iv) $P_{2n-2r-4} = P_{2n-2}\left(P_{2r-3}^2 - P_{2r-4}P_{2r-2}\right) + P_{2n-3}\left(P_{2r-4}P_{2r-1} - P_{2r-3}P_{2r-2}\right) + P_{2n-4}\left(P_{2r-2}^2 - P_{2r-3}P_{2r-1}\right)$.

Proof. By using Theorem 2.3 and the property of a determinant $\det\left(Q_2^n\right) = \det\left(Q_2\right)^n$, then we obtained (i) and (ii). Similarly, by using Theorem 2.3 and the properties of the power matrix $Q_2^{n+r} = Q_2^n Q_2^r$ and the power matrix $Q_2^{n-r} = Q_2^n\left(Q_2^r\right)^{-1}$, then we obtained (iii) and (iv).

Corollary 2.7 For $n, r \geq 1$, Then

(i) $\det\left(AQ_1^n\right) = 23$,

(ii) $R_{n+r+1} = \left(3P_{n-4} + 2P_{n-3}\right)P_{r-5} + \left(3P_{n-2} + 2P_{n-1}\right)P_{r-4} + \left(3P_{n-3} + 2P_{n-2}\right)P_{r-3}$,

(iii) $23R_{n-r+1} = 3P_{n-4}P_{r-2}^2 - 3P_{n-3}P_{r-3}P_{r-2} + 2P_{n-3}P_{r-2}^2 + 2P_{n-1}\left(P_{r-3}^2 - P_{r-4}P_{r-2}\right) + 3P_{n-3}P_{r-4}P_{r-1} - 3P_{n-4}P_{r-3}P_{r-1} - 2P_{n-3}P_{r-3}P_{r-1} + P_{n-2}\left(3P_{r-3}^2 - 2P_{r-3}P_{r-2} + P_{r-4}\left(-3P_{r-2} + 2P_{r-1}\right)\right)$.

Proof. By using Theorem 2.4 and the property of a determinant $\det\left(AQ_1^n\right) = \det\left(A\right)\det\left(Q_1\right)^n$, then we obtained (i). Similarly, by using Theorem 2.4 and the properties of the power matrix $AQ_1^{n+r} = AQ_1^n Q_1^r$ and the power matrix $AQ_1^{n-r} = AQ_1^n\left(Q_1^r\right)^{-1}$, then we obtained (ii) and (iii).

Corollary 2.8 For $n, r \geq 1$, Then

(i) $\det\left(AQ_2^n\right) = 23$,

(ii) $R_{2n+2r+1} = \left(3P_{2n-2} + 2P_{2n-1}\right)P_{2r-4} + \left(3P_{2n-4} + 2P_{2n-3}\right)P_{2r-5} + \left(3P_{2n-3} + 2P_{2n-2}\right)P_{2r-3}$,

(iii) $23R_{2n-2r+1} = 3P_{2n-4}P_{2r-2}^2 - 3P_{2n-3}P_{2r-3}P_{2r-2} + 2P_{2n-3}P_{2r-2}^2 + 2P_{2n-1}(P_{2r-3}^2 - P_{2r-4}P_{2r-2}) + 3P_{2n-3}P_{2r-4}P_{2r-1} - 3P_{2n-4}P_{2r-3}P_{2r-1} - 2P_{2n-3}P_{2r-3}P_{2r-1} + P_{2n-2}(3P_{2r-3}^2 - 2P_{2r-3}P_{2r-2} + P_{2r-4}(2P_{2r-1} - 3P_{2r-2}))$.

Proof. By using Theorem 2.4 and the property of a determinant $\det\left(AQ_2^n\right) = \det\left(A\right)\det\left(Q_2\right)^n$, then we obtained (i). Similarly, by using Theorem 2.4 and the properties of the power matrix $AQ_2^{n+r} = AQ_2^n Q_2^r$ and the power matrix $AQ_2^{n-r} = AQ_2^n\left(Q_2^r\right)^{-1}$, then we obtained (ii) and (iii).

Theorem 2.9 For $n, r \geq 1$, Then

(i) $P_{n-4} = P_{n-r-5}P_{r-4} + P_{n-r-3}P_{r-3} + P_{n-r-4}P_{r-2}$,

(ii) $P_{2n-4} = P_{2n-2r-5}P_{2r-4} + P_{2n-2r-3}P_{2r-3} + P_{2n-2r-4}P_{2r-2}$,

(iii) R_{n+1}

$$= P_{n-r-5}\left(3P_{r-4}+2P_{r-3}\right) + P_{n-r-3}\left(3P_{r-3}+2P_{r-2}\right)+P_{n-r-4}\left(3P_{r-2}+2P_{r-1}\right),$$

(iv) R_{2n+1}

$$= P_{2n-2r-5}\left(3P_{2r-4}+2P_{2r-3}\right) + P_{2n-2r-3}\left(3P_{2r-3}+2P_{2r-2}\right)+P_{2n-2r-4}\left(3P_{2r-2}+2P_{2r-1}\right).$$

Proof. Since $Q_1^n = Q_1^r Q_1^{n-r}$, we have

$$\begin{pmatrix} P_{n-5} & P_{n-3} & P_{n-4} \\ P_{n-4} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-1} & P_{n-2} \end{pmatrix} = \begin{pmatrix} P_{r-5} & P_{r-3} & P_{r-4} \\ P_{r-4} & P_{r-2} & P_{r-3} \\ P_{r-3} & P_{r-1} & P_{r-2} \end{pmatrix}\begin{pmatrix} P_{n-r-5} & P_{n-r-3} & P_{n-r-4} \\ P_{n-r-4} & P_{n-r-2} & P_{n-r-3} \\ P_{n-r-3} & P_{n-r-1} & P_{n-r-2} \end{pmatrix}.$$

Thus,

$P_{n-4} = P_{n-r-5}P_{r-4} + P_{n-r-3}P_{r-3} + P_{n-r-4}P_{r-2}$.

The proof of (ii), (iii), and (iv) are similar to (i). Therefore, the identities of (i), (ii), (iii), and (iv) are easily seen.

Corollary 2.10 For $n \geq 1$, Then

(i) $P_{2n-4} = P_{2n-11} + 2P_{2n-7}$,

(ii) $R_{n+1} = 2P_{n-8} + 5P_{n-4}$,

(iii) $R_{2n+1} = 7P_{2n-11} + 10P_{2n-9} + 12P_{2n-10}$.

Proof. Taking $r = 3$ in Theorem 2.9 (ii), (iii), and (iv), then we have (i), (ii), and (iii). Therefore, the identities of (i), (ii), and (iii) are easily seen.

3. Conclusions

In this paper, we establish the Q-matrix and the multiplies between the Q-matrix and the A-matrix, which consist of Padovan and Perrin numbers. After that, we prove that the n^{th} of Q_2, the n^{th} of Q_1 multiply the A-matrix, and the n^{th} of Q_2 multiply the A-matrix to help find the elementary identities of the Padovan, Perrin, and the relations between numbers. Also, we get a particular case of some identities.

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