PROGRESS IN APPLIED SCIENCE AND TECHNOLOGY
Faculty of Science and Technology
Rajamangala University of Technology Thanyaburi
Prog Appl Sci Tech. 2024; 14(1): 87-90
ISSN (Print): 2730-3012
ISSN (Online): 2730-3020
https://ph02.tci-thaijo.org/index.php/past

Research Article

Received: February 04, 2024
Revised: April 08, 2024
Accepted: April 18, 2024

DOI: 10.60101/past.2024.252662

A Note on Coregular Elements in Certain Semigroups of Transformations

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Abstract

For a non-empty set X, let $P(X)$ represent the partial transformation semigroup on X. For a non-empty subset Y of X, define $\overline{PT}(X,Y)$ as the semigroup

$$\overline{PT}(X,Y) = \{\alpha \in P(X) : (dom\alpha \cap Y)\alpha \subseteq Y\}.$$

Then $\overline{PT}(X,Y)$ is a generalization of $P(X)$, encompassing all partial transformations on X that maintain Y as an invariant set. This paper delves into exploring the characterization of coregular elements within $\overline{PT}(X,Y)$ and applies the results to obtain a similar characterization in relevant semigroups.

Keywords: Partial Pransformation Semigroups, Regular Elements, Coregular Elements

1. Introduction

Semigroups are a concept in abstract algebra, defined as sets equipped with an associative binary operation. This structure serves as a foundational concept in the study of more complex algebraic structures, such as monoids and groups. Semigroups find applications in various fields, such as computer science for designing algorithms, in the study of abstract structures in mathematics, and in the analysis of certain mathematical systems in physics. Understanding semigroups provides a basis for comprehending higher-level algebraic structures and their properties, contributing to diverse areas of mathematical research and practical applications.

Let a belong to a semigroup S. An element a is considered *idempotent* if $a^2 = a$, and the set of all idempotent elements of S is denoted by $E(S)$. Furthermore, a is termed *regular* if there exists $b \in S$ such that $a = aba$; S itself is termed a *regular semigroup* if every element in S is regular. Moreover, a is termed *coregular* if there exists $b \in S$ such that $a = aba = bab$; S is termed a *coregular semigroup* if every element in S is coregular.

The set of all coregular elements of a semigroup S is denoted by $CoReg(S)$. The definition makes it clear that all idempotents are coregular, and being coregular implies being regular. For more details, we can refer to (2, 8).

Let X be a non-empty set, and denote by $P(X)$ the set of all partial transformations on X. Then $P(X)$ is a semigroup under the composition of functions. For a fixed non-empty subset Y of X, let

$$\overline{PT}(X,Y) = \{\alpha \in P(X) : (dom\alpha \cap Y)\alpha \subseteq Y\},$$

where $dom\alpha$ representing the domain of α. It is evident that $\overline{PT}(X,Y)$ is a subsemigroup of $P(X)$ and also a generalization of $P(X)$ since $\overline{PT}(X,X)$ is equivalent to $P(X)$. The exploration of various algebraic properties in $\overline{PT}(X,Y)$ has been a subject of investigation; for details, refer to (1, 5, 7). In this paper, our focus is on establishing criteria to evaluate the coregularity of elements within $\overline{PT}(X,Y)$ based solely on their individual properties. Subsequently, having derived these criteria, we apply them to determine the coregularity of $\overline{PT}(X,Y)$ and three other semigroups of transformations that are relevant to $\overline{PT}(X,Y)$.

In this study's scope, when discussing the composition $\alpha\beta$, we apply the transformation α initially, followed by β. To elaborate, for any particular transformation α in semigroup $P(X)$, we commonly denote its domain and image as $dom\alpha$ and $im\alpha$, respectively. If certain terms or symbols lack explicit definitions here, readers can find explanations in the resources referenced in (2-4).

2. Results

Recall that $\alpha \in \overline{PT}(X,Y)$ is regular if and only if $im\alpha \cap Y = Y\alpha$, where $Y\alpha$ denote $(dom\alpha \cap Y)\alpha$; for details, (7). It is evident from the definition that an element a in a semigroup S, satisfying $a^3 = a$, is clearly a coregular element. Additionally, on the other hand, if $a = aba = bab$ for some $b \in S$, then $a = bab = b(aba)b = (bab)(aba)b = (bab)a(bab) = a^3$. This implies that a is coregular if and only if $a^3 = a$.

While we understand that all coregular elements are regular elements, it is important to note that there are regular elements that do not qualify as coregular. This is illustrated in the following example.

Example 2.1 Let $X = \{1,2,3,4\}$ and $Y = \{1,2\}$. Define

$$\alpha = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 2 & 3 \end{pmatrix}.$$

Following this, α in $\overline{PT}(X,Y)$. Additionally, we note that $im\alpha \cap Y = \{2\} = Y\alpha$, indicating that α is a regular element. However,

$$\alpha^3 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \neq \alpha.$$

so α does not fulfill the criteria for being coregular.

The following theorem presents a characterization of coregular elements within $\overline{PT}(X,Y)$.

Theorem 2.2 Given $\alpha \in \overline{PT}(X,Y)$. Then, α is coregular if and only if $im\alpha \subseteq dom\alpha$ and $\alpha^2|_{im\alpha} = id_{im\alpha}$.

Proof. Assume that α is coregular. Then, by the previous note, $\alpha^3 = \alpha$. Since $dom\alpha = dom\alpha^3 = dom(\alpha^2\alpha) = (im\alpha^2 \cap dom\alpha)(\alpha^2)^{-1} \subseteq dom\alpha^2 \subseteq dom\alpha$, we obtain $dom\alpha = dom\alpha^2$. Hence, $im\alpha = (dom\alpha)\alpha = (dom\alpha^2)\alpha = [(im\alpha \cap dom\alpha)\alpha^{-1}]\alpha \subseteq im\alpha \cap dom\alpha \subseteq dom\alpha$. Thus $im\alpha \subseteq dom\alpha$. Let $x \in im\alpha$. Then $x \in dom\alpha = dom\alpha^2$ and $x = z\alpha$ for some $z \in dom\alpha$. This implies $x\alpha^2 = (z\alpha)\alpha^2 = z\alpha^3 = z\alpha = x = xid_{im\alpha}$. Therefore, $\alpha^2|_{im\alpha} = id_{im\alpha}$.

Conversely, we assume that the condition hold. Since $im\alpha \subseteq dom\alpha$, we get $dom\alpha^3 = (im\alpha^2 \cap dom\alpha)(\alpha^2)^{-1} = (im\alpha^2)(\alpha^2)^{-1} = dom\alpha^2 = (im\alpha \cap dom\alpha)\alpha^{-1} = dom\alpha$. For each $x \in dom\alpha^3$, we get $x\alpha^3 = (x\alpha)\alpha^2 = x\alpha$ because $\alpha^2|_{im\alpha} = id_{im\alpha}$. Thus $\alpha^3 = \alpha$. Therefore, α is coregular.

Theorem 2.3 $\overline{PT}(X,Y)$ is a coregular semi-group if and only if $|X| = 1$.

Proof. Assume that $|X| = 1$. In this case, $\overline{PT}(X,Y)$ consists of precisely two elements: id_X and $\emptyset$, establishing $\overline{PT}(X,Y)$ as coregular.

On the other hand, we assume that $|X| \geq 2$. Then, there exist $y \in Y$ and $x \in X$ such that $x \neq y$. Consider the mapping α defined as follows:

$$\alpha = \begin{pmatrix} x \\ y \end{pmatrix}.$$

It is evident that $\alpha \in \overline{PT}(X,Y)$ and $im\alpha = \{y\} \nsubseteq \{x\} = dom\alpha$. By applying Theorem 2.2, we deduce that α is not coregular, leading to the conclusion that $\overline{PT}(X,Y)$ is not a coregular semigroup.

Theorem 2.4 $CoReg(\overline{PT}(X,Y))$ is a sub-semigroup of $\overline{PT}(X,Y)$ if and only if the set X contains exactly one element.

Proof. For the case where $|X| = 1$, we get that $CoReg(S) = \overline{PT}(X,Y)$, and it is directly obtained that $CoReg(S)$ is a subsemigroup of $\overline{PT}(X,Y)$.

On the other hand, we assume that $|X| \geq 2$. Then, there exist $y \in Y$ and $x \in X$ such that $x \neq y$. Consider the mappings α and β defined as follow:

$$\alpha = \begin{pmatrix} x \\ x \end{pmatrix} \text{ and } \beta = \begin{pmatrix} x & y \\ y & y \end{pmatrix}.$$

It is evident that both α and β are idempotent elements. Hence, they are coregular elements. However,

$$\alpha\beta = \begin{pmatrix} x \\ y \end{pmatrix},$$

which leads us to conclude that $im(\alpha\beta) = \{y\} \nsubseteq \{x\} = dom(\alpha\beta)$. According to Theorem 2.2, we conclude that $\alpha\beta$ is not coregular, meaning $\alpha\beta \notin CoReg(\overline{PT}(X,Y))$.

The following corollary directly follows as a consequence of the combined implications of both Theorems 2.3 and 2.4.

Corollary 2.5 The following statements are equivalent.

1. $|X| = 1$.
2. $\overline{PT}(X,Y)$ is coregular.
3. $CoReg(\overline{PT}(X,Y))$ is a subsemigroup of $\overline{PT}(X,Y)$.

If $X = Y$, then we get $\overline{PT}(X,Y) = P(X)$. This enables us to establish the coregularity in $P(X)$ directly through the use of Theorem 2.2 and Corollary 2.5. The results are presented in the following two corollaries.

Corollary 2.6 Given $\alpha \in P(X)$. Then, α is coregular if and only if $im\alpha \subseteq dom\alpha$ and $\alpha^2|_{im\alpha} = id_{im\alpha}$.

Corollary 2.7 The following statements are equivalent.

1. $|X| = 1$.
2. $P(X)$ is coregular.
3. $CoReg(P(X))$ is a subsemigroup of $P(X)$.

We now consider two well-known full transformation semigroups defined as follows:

$$T(X) = \{\alpha \mid \alpha : X \to X\}$$

and

$$\bar{T}(X,Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\},$$

where $\emptyset \neq Y \subseteq X$. It is clear that $\bar{T}(X,Y)$ is a subsemigroup of $T(X)$, and it is also a generalization of $T(X)$ since $\bar{T}(X,X) = T(X)$. Additionally, both $T(X)$ and $\bar{T}(X,Y)$ are strictly contained in $P(X)$, while $\bar{T}(X,Y)$ is also contained in $\overline{PT}(X,Y)$.

Considering the fact that element α of a semigroup S is coregular if and only if $\alpha^3 = \alpha$ and $im\alpha \subseteq X = dom\alpha$ for all $\alpha \in \bar{T}(X,Y)$, we can infer the characterization of coregular elements in $\bar{T}(X,Y)$ as originally proposed in [6]. Furthermore, due to the relationship $\bar{T}(X,X) = T(X)$, we can further extend this characterization to apply to the entire set $T(X)$.

Corollary 2.8 Given $\alpha \in \bar{T}(X,Y)$. Then, α is coregular if and only if $\alpha^2|_{im\alpha} = id_{im\alpha}$.

Corollary 2.9 Given $\alpha \in T(X)$. Then, α is coregular if and only if $\alpha^2|_{im\alpha} = id_{im\alpha}$.

Acknowledgments
The authors are thankful to reviewer for their valuable suggestion to improve our paper.

Declaration of conflicting interests
The authors declared no conflicts of interest in this article's research, authorship, and publication.

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