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On the Diophantine Equation $\frac{1}{w}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+1}$

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Abstract

In 2023, Wongsanurak and Duangdai found all positive integer solutions of the Diophantine equation $\frac{1}{w}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+1}$,when w, x, y, z and u are positive integers with $w \le x \le y \le z \le 9$ and $u \le 9$. In this work, by using an elementary approach, we solved the Diophantine equation for any positive integer u and $5 \le w \le x \le y \le z$. The results of the research found that the Diophantine equation under the above conditions has twenty-seven positive integer solutions.

Keywords: Diophantine Equation, Positive Integer Solution

1. Introduction

In 2013, Sándor (1) found all positive integer solutions of the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}$. In 2021, Zhao, Lu and Wang (2) discovered some conditions for the non-existing of positive integer solutions for the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{a}{p}$, where a is a positive integer and p is a prime number. In 2021, Sándor and Atanassov (3) proved that the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+1}$ has forty-four positive integer solutions. In 2023, Tadee and Poopra (4) studied and found that the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{3}$ has twenty-one positive integer solutions. In 2024, Tadee (5) proved that the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+2}$ has eighty-seven positive integer solutions. In 2024, by using elementary methods, Yuan (6) gave the general solution expressions for all positive integer solutions of the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{4}{n}$.

Meanwhile, in 2018, Bai (7) provided the positive integer solutions of the Diophantine equation $\frac{1}{w}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}$. After that, in 2022, Atri (8) showed some solutions of the Diophantine equation $\frac{1}{w}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{4}$. In 2023, Wongsanurak and Duangdai (9) gave all positive integer solutions of the Diophantine equation $\frac{1}{w}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+1}$, where w, x, y, z and u are positive integers with $w \le x \le y \le z \le 9$ and $u \le 9$.

From Wongsanurak and Duangdai's research study, it makes us interested in finding the positive integer solutions to the Diophantine equation $\frac{1}{w}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+1}$, where w, x, y, z and u are positive integers with $5 \le w \le x \le y \le z$.

2. Main Results

In this research, we find all positive integer solutions of the Diophantine equation

$$\frac{1}{w}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+1}, \tag{2.1}$$

where w, x, y, z and u are positive integers with $5 \leq w \leq x \leq y \leq z$. Then

$$\frac{4}{w} \geq \frac{u}{u+1} \text{ or } (w-4)u \leq 4. \tag{2.2}$$

Therefore $w \leq 8$. Since $5 \leq w$, it implies that $5 \leq w \leq 8$. We consider the following cases:

Case 1. $w = 5$. From (2.2), we have $u \leq 4$.
Case 1.1 $u = 1$. From (2.1), we get

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{3}{10}. \tag{2.3}$$

Since $x \leq y \leq z$, we obtain that $\frac{3}{x} \geq \frac{3}{10}$ and so $x \leq 10$. Since $w \leq x$, we have $5 \leq x$. Therefore $5 \leq x \leq 10$.

Case 1.1.1 $x = 5$. From (2.3), we get

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{10}. \tag{2.4}$$

Since $y \leq z$, we have $\frac{2}{y} \geq \frac{1}{10}$ and so $y \leq 20$. From (2.4), it implies that $\frac{1}{y} < \frac{1}{10}$ or $11 \leq y$. Substituting $11 \leq y \leq 20$ in (2.4) and to consider the value z, which is a positive integer, we obtain that the positive integer solutions (w, x, y, z, u) are $(5, 5, 11, 110, 1)$, $(5, 5, 12, 60, 1)$, $(5, 5, 14, 35, 1)$, $(5, 5, 15, 30, 1)$ and $(5, 5, 20, 20, 1)$.

Case 1.1.2 $x = 6$. From (2.3), we get

$$\frac{1}{y}+\frac{1}{z}=\frac{2}{15}. \tag{2.5}$$

Since $y \leq z$, we have $\frac{2}{y} \geq \frac{2}{15}$ and so $y \leq 15$. From (2.5), it implies that $\frac{1}{y} < \frac{2}{15}$ or $8 \leq y$. Substituting $8 \leq y \leq 15$ in (2.5) and to consider the value z, which is a positive integer, we obtain that the positive integer solutions (w, x, y, z, u) are $(5, 6, 8, 120, 1)$, $(5, 6, 9, 45, 1)$, $(5, 6, 10, 30, 1)$, $(5, 6, 12, 20, 1)$ and $(5, 6, 15, 15, 1)$.

Case 1.1.3 $x = 7$. Since $x \leq y$, it implies that $7 \leq y$. From (2.3), we get

$$\frac{1}{y}+\frac{1}{z}=\frac{11}{70}. \tag{2.6}$$

Since $y \leq z$, we have $\frac{2}{y} \geq \frac{11}{70}$ and so $y \leq 12$. Substituting $7 \leq y \leq 12$ in (2.6) and to consider the value z, which is a positive integer, we obtain that the positive integer solution (w, x, y, z, u) is $(5, 7, 7, 70, 1)$.

Case 1.1.4 $x = 8$. Since $x \leq y$, it implies that $8 \leq y$. From (2.3), we get

$$\frac{1}{y}+\frac{1}{z}=\frac{7}{40}. \tag{2.7}$$

Since $y \leq z$, we have $\frac{2}{y} \geq \frac{7}{40}$ and so $y \leq 11$. Substituting $8 \leq y \leq 11$ in (2.7) and to consider the value z, which is a positive integer, we obtain that the positive integer solution (w, x, y, z, u) is $(5, 8, 8, 20, 1)$.

Case 1.1.5 $x = 9$. Since $x \leq y$, it implies that $9 \leq y$. From (2.3), we get

$$\frac{1}{y}+\frac{1}{z}=\frac{17}{90}. \tag{2.8}$$

Since $y \leq z$, we have $\frac{2}{y} \geq \frac{17}{90}$ and so $y \leq 10$. Substituting $9 \leq y \leq 10$ in (2.8), we obtain that the value z is not a positive integer. Thus, in this case, there is no positive integer solution.

Case 1.1.6 $x = 10$. Since $x \leq y$, it implies that $10 \leq y$. From (2.3), we get

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{5}. \tag{2.9}$$

Since $y \leq z$, we have $\frac{2}{y} \geq \frac{1}{5}$ and so $y \leq 10$. Substituting $y = 10$ in (2.9) and to consider the value z, which is a positive integer, we obtain that the positive integer solution (w, x, y, z, u) is $(5, 10, 10, 10, 1)$.

Case 1.2 $u = 2$. From (2.1), we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{7}{15}. \tag{2.10}$$

Since $x \le y \le z$, we obtain that $\frac{3}{x} \ge \frac{7}{15}$ and so $x \le 6$. Since $w \le x$, we have $5 \le x$. Therefore $5 \le x \le 6$.

Case 1.2.1 $x = 5$. Since $x \le y$, it implies that $5 \le y$. From (2.10), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{4}{15}. \tag{2.11}$$

Since $y \le z$, we have $\frac{2}{y} \ge \frac{4}{15}$ and so $y \le 7$. Substituting $5 \le y \le 7$ in (2.11) and to consider the value z, which is a positive integer, we obtain that the positive integer solutions (w, x, y, z, u) are $(5, 5, 5, 15, 2)$ and $(5, 5, 6, 10, 2)$.

Case 1.2.2 $x = 6$. Since $x \le y$, it implies that $6 \le y$. From (2.10), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{3}{10}. \tag{2.12}$$

Since $y \le z$, we have $\frac{2}{y} \ge \frac{3}{10}$ and so $y \le 6$. Substituting $y = 6$ in (2.12), we obtain that the value z is not a positive integer. Thus, in this case, there is no positive integer solution.

Case 1.3 $u = 3$. From (2.1), we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{11}{20}. \tag{2.13}$$

Since $x \le y \le z$, we obtain that $\frac{3}{x} \ge \frac{11}{20}$ and so $x \le 5$. Since $w \le x$, it implies that $5 \le x$. Thus $x = 5$. Since $x \le y$, we have $5 \le y$. Substituting $x = 5$ in (2.13), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{7}{20}. \tag{2.14}$$

Since $y \le z$, we have $\frac{2}{y} \ge \frac{7}{20}$ and so $y \le 5$. Substituting $y = 5$ in (2.14), we obtain that the value z is not a positive integer. Thus, in this case, there is no positive integer solution.

Case 1.4 $u = 4$. From (2.1), we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{5}. \tag{2.15}$$

Since $x \le y \le z$, we obtain that $\frac{3}{x} \ge \frac{3}{5}$ and so $x \le 5$. Since $w \le x$, it implies that $5 \le x$. Thus $x = 5$. Since $x \le y$, we have $5 \le y$. Substituting $x = 5$ in (2.15), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{2}{5}. \tag{2.16}$$

Since $y \le z$, we have $\frac{2}{y} \ge \frac{2}{5}$ and so $y \le 5$. Substituting $y = 5$ in (2.16), we obtain that the positive integer solution (w, x, y, z, u) is $(5, 5, 5, 5, 4)$.

Case 2. $w = 6$. From (2.2), we have $u \le 2$.
Case 2.1 $u = 1$. From (2.1), we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{3}. \tag{2.17}$$

Since $x \le y \le z$, we obtain that $\frac{3}{x} \ge \frac{1}{3}$ and so $x \le 9$. Since $w \le x$, it implies that $6 \le x$. Thus $6 \le x \le 9$.

Case 2.1.1 $x = 6$. From (2.17), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{6}. \tag{2.18}$$

Since $y \le z$, we have $\frac{2}{y} \ge \frac{1}{6}$ and so $y \le 12$. From (2.18), it implies that $\frac{1}{y} < \frac{1}{6}$ or $7 \le y$. Substituting $7 \le y \le 12$ in (2.18) and to consider the value z, which is a positive integer, we obtain that the positive integer solutions (w, x, y, z, u) are $(6, 6, 7, 42, 1)$, $(6, 6, 8, 24, 1)$, $(6, 6, 9, 18, 1)$, $(6, 6, 10, 15, 1)$ and $(6, 6, 12, 12, 1)$.

Case 2.1.2 $x = 7$. Since $x \leq y$, it implies that $7 \leq y$. From (2.17), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{4}{21}. \tag{2.19}$$

Since $y \leq z$, we have $\frac{2}{y} \geq \frac{4}{21}$ and so $y \leq 10$. Substituting $7 \leq y \leq 10$ in (2.19) and to consider the value z, which is a positive integer, we obtain that the positive integer solution (w, x, y, z, u) is $(6, 7, 7, 21, 1)$.

Case 2.1.3 $x = 8$. Since $x \leq y$, it implies that $8 \leq y$. From (2.17), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{5}{24}. \tag{2.20}$$

Since $y \leq z$, we have $\frac{2}{y} \geq \frac{5}{24}$ and so $y \leq 9$. Substituting $8 \leq y \leq 9$ in (2.20) and to consider the value z, which is a positive integer, we obtain that the positive integer solution (w, x, y, z, u) is $(6, 8, 8, 12, 1)$.

Case 2.1.4 $x = 9$. Since $x \leq y$, it implies that $9 \leq y$. From (2.17), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{2}{9}. \tag{2.21}$$

Since $y \leq z$, we have $\frac{2}{y} \geq \frac{2}{9}$ and so $y \leq 9$. Substituting $y = 9$ in (2.21), we obtain that the positive integer solution (w, x, y, z, u) is $(6, 9, 9, 9, 1)$.

Case 2.2 $u = 2$. From (2.1), we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}. \tag{2.22}$$

Since $x \leq y \leq z$, we obtain that $\frac{3}{x} \geq \frac{1}{2}$ and so $x \leq 6$. Since $w \leq x$, it implies that $6 \leq x$. Thus $x = 6$. Since $x \leq y$, it implies that $6 \leq y$. Substituting $x = 6$ in (2.22), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{3}. \tag{2.23}$$

Since $y \leq z$, we have $\frac{2}{y} \geq \frac{1}{3}$ and so $y \leq 6$. Substituting $y = 6$ in (2.23), we obtain that the positive integer solution (w, x, y, z, u) is $(6, 6, 6, 6, 2)$.

Case 3. $w = 7$. From (2.2), we have $u = 1$. From (2.1), we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{14}. \tag{2.24}$$

Since $x \leq y \leq z$, we obtain that $\frac{3}{x} \geq \frac{5}{14}$ and so $x \leq 8$. Since $w \leq x$, it implies that $7 \leq x$. Thus $7 \leq x \leq 8$.

Case 3.1 $x = 7$. Since $x \leq y$, it implies that $7 \leq y$. From (2.24), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{3}{14}. \tag{2.25}$$

Since $y \leq z$, we have $\frac{2}{y} \geq \frac{3}{14}$ and so $y \leq 9$. Substituting $7 \leq y \leq 9$ in (2.25) and to consider the value z, which is a positive integer, we obtain that the positive integer solution (w, x, y, z, u) is $(7, 7, 7, 14, 1)$.

Case 3.2 $x = 8$. Since $x \leq y$, it implies that $8 \leq y$. From (2.24), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{13}{56}. \tag{2.26}$$

Since $y \leq z$, we have $\frac{2}{y} \geq \frac{13}{56}$ and so $y \leq 8$. Substituting $y = 8$ in (2.26), we obtain that the value z is not a positive integer. Thus, in this case, there is no positive integer solution.

Case 4. $w = 8$. From (2.2), we have $u = 1$. From (2.1), we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{8}. \tag{2.27}$$

Since $x \leq y \leq z$, we obtain that $\frac{3}{x} \geq \frac{3}{8}$ and so $x \leq 8$. Since $w \leq x$, it implies that $8 \leq x$. Then $x = 8$. Since $x \leq y$, it implies that $8 \leq y$. From (2.27), we get

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{4}. \quad (2.28)$$

Since $y \le z$, we have $\frac{2}{y} \ge \frac{1}{4}$ and so $y \le 8$. Substituting $y = 8$ in (2.28), we obtain that the positive integer solution (w, x, y, z, u) is $(8, 8, 8, 8, 1)$.

3. Conclusions

In this paper, we give all positive integer solutions of the Diophantine equation $\frac{1}{w}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+1}$, where w, x, y, z and u are positive integers with $5 \le w \le x \le y \le z$. The research found that the positive integer solutions (w, x, y, z, u) of this equation are $(5,5,11,110,1)$, $(5,5,12,60,1)$, $(5,5,14,35,1)$, $(5,5,15,30,1)$, $(5,5,20,20,1)$, $(5,6,8,120,1)$, $(5,6,9,45,1)$, $(5,6,10,30,1)$, $(5,6,12,20,1)$, $(5,6,15,15,1)$, $(5,7,7,70,1)$, $(5,8,8,20,1)$, $(5,10,10,10,1)$, $(5,5,5,15,2)$, $(5,5,6,10,2)$, $(5,5,5,5,4)$, $(6,6,7,42,1)$, $(6,6,8,24,1)$, $(6,6,9,18,1)$, $(6,6,10,15,1)$, $(6,6,12,12,1)$, $(6,7,7,21,1)$, $(6,8,8,12,1)$, $(6,9,9,9,1)$, $(6,6,6,6,2)$, $(7,7,7,14,1)$ and $(8,8,8,8,1)$.

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Declaration of Conflicting Interests

The authors declare they have no conflicts of interest for this article, and they alone are responsible for the content.

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