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On the bi-periodic k-Jacobsthal and k-Jacobsthal-Lucas numbers

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Abstract

This paper introduces and investigates the bi-periodic k-Jacobsthal and k-Jacobsthal–Lucas sequences, extending the classical Jacobsthal framework by incorporating periodicity and a tunable parameter k. We establish recurrence relations, derive generating functions, and present Binet-type formulas for these generalized sequences. Furthermore, we obtain extensions of well-known identities, including Catalan's, Cassini's, and d'Ocagne's identities. The proposed generalization reveals deeper algebraic structures and periodic patterns, offering potential applications in cryptography, coding theory, and recurrence-based modeling. These findings provide a foundation for future research on combinatorial interpretations and connections with other special sequences.

Keywords: Bi-Periodic k-Jacobsthal Numbers, Bi-Periodic k-Jacobsthal-Lucas Numbers

1. Introduction

Recurrence relations have long been a cornerstone of mathematics and computer science due to their ability to model sequential patterns and dynamic systems. Classical sequences such as Fibonacci and Jacobsthal numbers have found applications in diverse areas, including algorithm design, combinatorics, and number theory. However, many real-world phenomena exhibit cyclic or alternating behaviors that cannot be fully captured by traditional recurrence models. To address this limitation, we introduce bi-periodicity and a tunable parameter k into the Jacobsthal framework, creating the bi-periodic k-Jacobsthal and k-Jacobsthal–Lucas sequences.

The motivation for incorporating periodicity and parameterization is twofold. First, periodicity enables the modeling of alternating structures, which are common in cryptographic key generation, coding theory, and signal processing. Second, the parameter k provides flexibility to control growth rates and structural complexity, offering deeper insights into stability and predictability within recurrence systems. This generalization not only enriches the theoretical landscape but also opens pathways for practical applications. For example, in cryptography, sequences with controlled periodicity can enhance pseudorandom number generation, while in coding theory, they can improve error detection and correction schemes. Similarly, recurrence-based models in computational biology and finance can leverage these sequences to represent cyclic behaviors more accurately.

Background on k-Jacobsthal and bi-periodic Jacobsthal families began in 2014, when Falcon S (1) introduced the k-Jacobsthal number. In 2016, Uygun S and Eldogan H (2) studied the properties of the k-Jacobsthal and the k-Jacobsthal-Lucas sequences, which are defined as the following.

For any positive real number k, the k-Jacobsthal sequence $\{C_{k,n}\}_{n=0}^{\infty}$ and the k-Jacobsthal-Lucas sequence $\{c_{k,n}\}_{n=0}^{\infty}$ are defined recurrently by

$$C_{k,n} = kC_{k,n-1} + 2C_{k,n-2} \quad \text{for } n \geq 2,$$

and

$$c_{k,n} = kc_{k,n-1} + 2c_{k,n-2} \quad \text{for } n \geq 2,$$

with initial conditions $C_{k,0} = 0$, $C_{k,1} = 1$ and $c_{k,0} = 2$, $c_{k,1} = k$, respectively.

Moreover, they gave their Binet's formulas which are

$$C_{k,n} = \frac{\varphi^n - \gamma^n}{\varphi - \gamma} \text{ and } c_{k,n} = \varphi^n + \gamma^n,$$

where n is an integer and $\varphi = \frac{k+\sqrt{k^2+8}}{2}$ and $\gamma = \frac{k-\sqrt{k^2+8}}{2}$ are the roots of the characteristic equation $r^2 - kr - 2 = 0$.

In 2016, Uygun S and Owusu E (3) proposed a new generation of the Jacobsthal numbers called the bi-periodic Jacobsthal sequence and then, in 2019, they (4) also introduced a new generation of the Jacobsthal-Lucas numbers called the bi-periodic Jacobsthal-Lucas sequence. These two sequences are defined recursively as below.

For any two nonzero real numbers a, b, the bi-periodic Jacobsthal sequence $\{J_n\}_{n=0}^{\infty}$ and the bi-periodic Jacobsthal-Lucas sequence $\{j_n\}_{n=0}^{\infty}$ are respectively defined by

$$J_n = \begin{cases} aJ_{n-1} + 2J_{n-2}, & \text{if } n \text{ is even,} \\ bJ_{n-1} + 2J_{n-2}, & \text{if } n \text{ is odd,} \end{cases} \quad n \geq 2$$

with initial conditions $J_0 = 0, J_1 = 1$ and

$$j_n = \begin{cases} bj_{n-1} + 2j_{n-2}, & \text{if } n \text{ is even,} \\ aj_{n-1} + 2j_{n-2}, & \text{if } n \text{ is odd,} \end{cases} \quad n \geq 2$$

with initial conditions $j_0 = 2,\ j_1 = a$.

They obtained Binet's formulas

$$J_n = \frac{a^{1-\varepsilon(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\nu^n - \xi^n}{\nu - \xi} \right),$$

and

$$j_n = \frac{a^{\varepsilon(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} (\nu^n + \xi^n),$$

where $\lfloor n \rfloor$ is the floor function of n, $\varepsilon(n)$ is the parity function,

$$\nu = \frac{ab+\sqrt{a^2b^2+8ab}}{2} \text{ and } \xi = \frac{ab-\sqrt{a^2b^2+8ab}}{2}$$

are the roots of the nonlinear equation $r^2 - abr - 2ab = 0$.

Furthermore, they provided properties of the floor function, which will be used in the theorems discussed in the following section, as below.

$$\varepsilon(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor, \tag{1.1}$$
$$\varepsilon(n+r) + \left\lfloor \frac{n-r}{2} \right\rfloor + \left\lfloor \frac{n+r}{2} \right\rfloor = n, \tag{1.2}$$
$$\varepsilon(n) - 2\left\lfloor \frac{n+1}{2} \right\rfloor = -n, \tag{1.3}$$
$$\varepsilon(n+r) - \left\lfloor \frac{n-r+1}{2} \right\rfloor - \left\lfloor \frac{n+r+1}{2} \right\rfloor = -n, \tag{1.4}$$
$$\varepsilon(n) + \varepsilon(r+1) - 2\varepsilon(nr+n) = 1 - \varepsilon(n-r) \tag{1.5}$$
$$\varepsilon(n+1) + \varepsilon(r) - 2\varepsilon(nr+r) = 1 - \varepsilon(n-r) \tag{1.6}$$
$$\varepsilon(n-r) = \varepsilon(nr+n) + \varepsilon(nr+r), \tag{1.7}$$
$$\frac{n+r-\varepsilon(n-r)}{2} = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{r+1}{2} \right\rfloor - \varepsilon(nr+r), \tag{1.8}$$
$$\frac{n+r-\varepsilon(n-r)}{2} = \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor - \varepsilon(nr+n), \tag{1.9}$$
$$\frac{n-r-\varepsilon(n-r)}{2} = \left\lfloor \frac{n-r}{2} \right\rfloor. \tag{1.10}$$

Further research on this subject can be found in (5-6).

In this paper, we define these generalized sequences, derive their recurrence relations, generating functions, and Binet-type formulas, and establish extensions of classical identities such as Catalan's, Cassini's, and d'Ocagne's identities. Through this exploration, we aim to uncover new algebraic structures of bi-periodic sequences.

2. Main Results

In this section, we define the bi-periodic k-Jacobsthal and k-Jacobsthal-Lucas sequences, which correspond to the following definition.

Definition 2.1 For any nonzero real numbers a, b and any positive real number k, the bi-periodic k-Jacobsthal sequence $\{J_{k,n}\}_{n=0}^{\infty}$ and the bi-periodic k-Jacobsthal-Lucas sequence $\{j_{k,n}\}_{n=0}^{\infty}$ are defined respectively by

$$J_{k,n}=\begin{cases} akJ_{k,n-1}+2J_{k,n-2}, & \text{if } n \text{ is even,} \\ bkJ_{k,n-1}+2J_{k,n-2}, & \text{if } n \text{ is odd,} \end{cases} \quad n\geq 2$$

with initial conditions $J_{k,0}=0, J_{k,1}=1$, and

$$j_{k,n}=\begin{cases} bkj_{k,n-1}+2j_{k,n-2}, & \text{if } n \text{ is even,} \\ akj_{k,n-1}+2j_{k,n-2}, & \text{if } n \text{ is odd,} \end{cases} \quad n\geq 2$$

with initial conditions $j_{k,0}=2,\ j_{k,1}=ak$.

From Definition 2.1, we obtain the nonlinear quadratic equation for the bi-periodic k-Jacobsthal and k-Jacobsthal-Lucas sequences which are given as

$$r^2-abkr-2ab=0,$$

with the roots α and β defined by

$$\alpha=\frac{abk+\sqrt{a^2b^2k^2+8ab}}{2},$$
$$\beta=\frac{abk-\sqrt{a^2b^2k^2+8ab}}{2}.$$

So $\alpha+\beta=abk$, $\alpha\beta=-2ab$, and $\alpha-\beta=\sqrt{a^2b^2k^2+8ab}$. Moreover, we also derive these identities:

$$\alpha(2+k\beta)=-2\beta\,,\ \ \beta(2+k\alpha)=-2\alpha\,,$$
$$2+k\alpha=\frac{\alpha^2}{ab},\ \ 2+k\beta=\frac{\beta^2}{ab},$$

and

$$(2+k\alpha)(2+k\beta)=4.$$

The first four terms of the bi-periodic k-Jacobsthal and k-Jacobsthal-Lucas sequences are shown in the table below.

Table 1 The first few terms of the bi-periodic k-Jacobsthal and bi-periodic k-Jacobsthal-Lucas numbers for $0\leq n\leq 3$

n	0	1	2	3
$J_{k,n}$	0	1	ak	abk^2+2
$j_{k,n}$	2	ak	abk^2+4	a^2bk^3+6ak

These sequences are generalizations of the classical Jacobsthal and Jacobsthal–Lucas numbers, incorporating a parameter k that influences their expansion and introducing a periodic pattern into their recurrence relations.

Next, we obtain the identities of the bi-periodic k-Jacobsthal and k-Jacobsthal-Lucas numbers, both even and odd, which will be applied in the next theorem.

Lemma 2.2 For any nonzero real numbers a, b, any positive real number k, and $n\geq 2$. The following results hold.

(i) $J_{k,2n}=(abk^2+4)J_{k,2n-2}-4J_{k,2n-4}$,
(ii) $J_{k,2n+1}=(abk^2+4)J_{k,2n-1}-4J_{k,2n-3}$,
(iii) $j_{k,2n}=(abk^2+4)j_{k,2n-2}-4j_{k,2n-4}$,
(iv) $j_{k,2n+1}=(abk^2+4)j_{k,2n-1}-4j_{k,2n-3}$.

Proof. (i) From Definition 2.1, we have

$$\begin{aligned} J_{k,2n}&=ak\,J_{k,2n-1}+2J_{k,2n-2}\\ &=ak(bk\,J_{k,2n-2}+2J_{k,2n-3})+2J_{k,2n-2}\\ &=abk^2J_{k,2n-2}+2ak\,J_{k,2n-3}+2J_{k,2n-2}. \end{aligned}$$

Next, using the recurrence relation for $J_{k,n}$, we substitute $akJ_{k,2n-3}=J_{k,2n-2}-2J_{k,2n-4}$, giving

$$\begin{aligned} J_{k,2n}&=abk^2J_{k,2n-2}+2\big(J_{k,2n-2}-2J_{k,2n-4}\big)\\ &\quad+2J_{k,2n-2}\\ &=abk^2J_{k,2n-2}+4J_{k,2n-2}-4J_{k,2n-4}\\ &=(abk^2+4)J_{k,2n-2}-4J_{k,2n-4}. \end{aligned}$$

Thus, identity (i) is proved.

Identities (ii), (iii), and (iv) follow by applying the same method. □

Next, we derive the generating functions for these sequences, which serve as a foundation for obtaining their Binet's formulas.

Theorem 2.3 The generating function for the bi-periodic k-Jacobsthal and k-Jacobsthal-Lucas sequences are given respectively by

$$J(x)=\frac{x+akx^2-2x^3}{1-(abk^2+4)x^2+4x^4},$$
$$j(x)=\frac{2+akx-(abk^2+4)x^2+2akx^3}{1-(abk^2+4)x^2+4x^4}.$$

Proof. Consider the generating function for the bi-periodic k-Jacobsthal sequence

$$J(x)=\sum_{n=0}^{\infty}J_{k,n}x^n.$$

Multiplying both sides by $(abk^2+4)x^2$ and $4x^4$, we obtain

$$(abk^2+4)x^2J(x)=\sum_{n=0}^{\infty}(abk^2+4)J_{k,n}x^{n+2},$$

$$4x^4J(x)=\sum_{n=0}^{\infty}4J_{k,n}x^{n+4}.$$

Thus

$$[1-(abk^2+4)x^2+4x^4]J(x)$$
$$=\sum_{n=0}^{\infty}J_{k,n}x^n-\sum_{n=0}^{\infty}(abk^2+4)J_{k,n}x^{n+2}+\sum_{n=0}^{\infty}4J_{k,n}x^{n+4}.$$

Reindexing and separating terms, we have

$$[1-(abk^2+4)x^2+4x^4]J(x)$$
$$=J_{k,0}+J_{k,1}x+J_{k,2}x^2+J_{k,3}x^3-(abk^2+4)(J_{k,0}x^2+J_{k,1}x^3)+\sum_{n=0}^{\infty}(J_{k,n+4}-(abk^2+4)J_{k,n+2}-4J_{k,n})x^{n+4}$$

Using initial conditions $J_{k,0}=0, J_{k,1}=1$, $J_{k,2}=ak$, and $J_{k,3}=abk^2+2$, this simplifies to

$$[1-(abk^2+4)x^2+4x^4]J(x)=x+akx^2-2x^3.$$

By Lemma 2.2 (i) and (ii), the summation terms vanish, yielding

$$J(x)=\frac{x+akx^2-2x^3}{1-(abk^2+4)x^2+4x^4}.$$

The generating function for the bi-periodic k-Jacobsthal–Lucas sequence can be derived similarly, giving

$$j(x)=\frac{2+akx-(abk^2+4)x^2+2akx^3}{1-(abk^2+4)x^2+4x^4}. \qquad \square$$

From now on, we note that $\lfloor n\rfloor$ is the floor function of n, and $\varepsilon(n)$ is the parity function of n, which is

$$\varepsilon(n)=\begin{cases}0, & \text{if } n \text{ is even}\\ 1, & \text{if } n \text{ is odd}\end{cases}.$$

Theorem 2.4 The Binet's formulas for the bi-periodic k-Jacobsthal and k-Jacobsthal-Lucas sequences are given respectively by

$$J_{k,n}=\frac{a^{1-\varepsilon(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$$

and

$$j_{k,n}=\frac{a^{\varepsilon(n)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}(\alpha^n+\beta^n).$$

Proof. From Theorem 2.3, the generating function for the bi-periodic k-Jacobsthal sequence is

$$J(x)=\frac{x+akx^2-2x^3}{1-(abk^2+4)x^2+4x^4}.$$

The denominator can be factored as

$$1-(abk^2+4)x^2+4x^4=\left(2x^2-\frac{2+k\alpha}{2}\right)\left(2x^2-\frac{2+k\beta}{2}\right).$$

Expressing $J(x)$ in partial fraction form, we have

$$J(x)=\frac{\frac{-\alpha}{(\alpha-\beta)}x+\frac{a(2+k\alpha)}{2(\alpha-\beta)}}{2x^2-\frac{2+k\alpha}{2}}+\frac{\frac{\beta}{(\alpha-\beta)}x-\frac{a(2+k\beta)}{2(\alpha-\beta)}}{2x^2-\frac{2+k\beta}{2}}$$
$$=\frac{1}{4(\alpha-\beta)}\left[\frac{a(2+k\alpha)-2\alpha x}{x^2-\frac{2+k\alpha}{4}}-\frac{a(2+k\beta)-2\beta x}{x^2-\frac{2+k\beta}{4}}\right].$$

By the Macluarin series expression

$$\frac{A-Bx}{x^2-C}=\sum_{n=0}^{\infty}BC^{-n-1}x^{2n+1}-\sum_{n=0}^{\infty}AC^{-n-1}x^{2n},$$

we have

$$J(x)=\frac{1}{4(\alpha-\beta)}\Big[\sum_{n=0}^{\infty}2\alpha\left(\frac{2+k\alpha}{4}\right)^{-n-1}x^{2n+1}-\sum_{n=0}^{\infty}a(2+k\alpha)\left(\frac{2+k\alpha}{4}\right)^{-n-1}x^{2n}-\sum_{n=0}^{\infty}2\beta\left(\frac{2+k\beta}{4}\right)^{-n-1}x^{2n+1}+\sum_{n=0}^{\infty}a(2+k\beta)\left(\frac{2+k\beta}{4}\right)^{-n-1}x^{2n}\Big]$$
$$=\frac{1}{4(\alpha-\beta)}\Big[\sum_{n=0}^{\infty}\left(\frac{2\alpha\cdot 4^{n+1}}{(2+k\alpha)^{n+1}}-\frac{2\beta\cdot 4^{n+1}}{(2+k\beta)^{n+1}}\right)x^{2n+1}+\sum_{n=0}^{\infty}\left(\frac{a\cdot 4^{n+1}}{(2+k\beta)^{n}}-\frac{a\cdot 4^{n+1}}{(2+k\alpha)^{n}}\right)x^{2n}\Big]$$
$$=\frac{1}{4(\alpha-\beta)}\sum_{n=0}^{\infty}\left(\frac{2\alpha(2+k\beta)^{n+1}-2\beta(2+k\alpha)^{n+1}}{(2+k\alpha)^{n+1}(2+k\beta)^{n+1}}\right)4^{n+1}x^{2n+1}+\frac{a}{4(\alpha-\beta)}\sum_{n=0}^{\infty}\left(\frac{(2+k\alpha)^n-(2+k\beta)^n}{(2+k\alpha)^n(2+k\beta)^n}\right)4^{n+1}x^{2n}$$

By $\alpha(2+k\beta)=-2\beta, \beta(2+k\alpha)=-2\alpha$, and $(2+k\alpha)(2+k\beta)=4$, we have

$$J(x)=\frac{1}{4(\alpha-\beta)}\sum_{n=0}^{\infty}\left(\frac{-4\beta(2+k\beta)^n+4\alpha(2+k\alpha)^n}{4^{n+1}}\right)4^{n+1}x^{2n+1}+\frac{a}{4(\alpha-\beta)}\sum_{n=0}^{\infty}\left(\frac{(2+k\alpha)^n-(2+k\beta)^n}{4^n}\right)4^{n+1}x^{2n}.$$

By $2+k\alpha=\frac{\alpha^2}{ab}$ and $2+k\beta=\frac{\beta^2}{ab}$, we obtain

$$J(x)=\sum_{n=0}^{\infty}\frac{1}{(ab)^n}\left(\frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta}\right)x^{2n+1}+\sum_{n=0}^{\infty}\frac{a}{(ab)^n}\left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right)x^{2n}.$$

Using the parity function, we get that

$$J(x)=\sum_{n=0}^{\infty}\frac{a^{1-\varepsilon(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)x^n.$$

Thus, $J_{k,n}=\frac{a^{1-\varepsilon(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$.

The Binet's formula for the bi-periodic k-Jacobsthal-Lucas sequence can be derived similarly. Begin with expressing $j(x)$ in partial fraction form, we have

$$j(x)=\frac{\frac{\alpha(4+k\alpha)}{(\alpha-\beta)}x-\frac{\alpha(4+k\alpha)}{2(\alpha-\beta)}}{2x^2-\frac{2+k\alpha}{2}}+\frac{\frac{-\alpha(4+k\alpha)}{(\alpha-\beta)}x+\frac{\beta(4+k\beta)}{2(\alpha-\beta)}}{2x^2-\frac{2+k\beta}{2}}$$
$$=\frac{1}{4(\alpha-\beta)}\left[-\frac{\alpha(4+k\alpha)-2\alpha(4+k\alpha)x}{x^2-\frac{2+k\alpha}{4}}-\frac{\beta(4+k\beta)-2\alpha(4+k\beta)}{x^2-\frac{2+k\beta}{4}}\right].$$

Using these identities:

$$(4+k\alpha)(2+k\beta)=2(4+k\beta),$$
$$(4+k\beta)(2+k\alpha)=2(4+k\alpha),$$
$$\beta(4+k\alpha)=-2(\alpha-\beta),$$
$$\alpha(4+k\beta)=2(\alpha-\beta),$$
$$a(4+k\alpha)=\frac{\alpha(\alpha-\beta)}{b},$$
$$a(4+k\beta)=-\frac{\beta(\alpha-\beta)}{b},$$

and applying the parity function, we obtain

$$j_{k,n}=\frac{a^{\varepsilon(n)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}(\alpha^n+\beta^n).$$ □

Next, we extend the sequences to negative indices, as stated in the following theorem.

Theorem 2.5 Let $n\geq 1$. The negative terms of the bi-periodic k-Jacobsthal and k-Jacobsthal-Lucas sequences are defined respectively by

$$J_{k,-n}=\frac{(-1)^{n+1}}{2^n}J_{k,n},$$

and

$$j_{k,-n}=\frac{(-1)^n}{2^n}j_{k,n}.$$

Proof. Using Theorem 2.4, $\varepsilon(-n)=\varepsilon(n)$ and the identity $\left\lfloor\frac{-n}{2}\right\rfloor+n=\left\lceil\frac{n}{2}\right\rceil$, we have

$$J_{k,-n}=\frac{a^{1-\varepsilon(-n)}}{(ab)^{\lfloor\frac{-n}{2}\rfloor}}\left(\frac{\alpha^{-n}-\beta^{-n}}{\alpha-\beta}\right)$$
$$=-\frac{a^{1-\varepsilon(-n)}}{(ab)^{\lfloor\frac{-n}{2}\rfloor}(\alpha\beta)^n}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$$
$$=-\frac{a^{1-\varepsilon(n)}}{(ab)^{\lfloor\frac{-n}{2}\rfloor}(-2ab)^n}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$$
$$=\frac{(-1)^{n+1}}{2^n}\frac{a^{1-\varepsilon(n)}}{(ab)^{\lceil\frac{n}{2}\rceil}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$$
$$=\frac{(-1)^{n+1}}{2^n}J_{k,n}.$$

Using Theorem 2.4, $\varepsilon(-n)=\varepsilon(n)$ and the identity $\left\lfloor\frac{-n+1}{2}\right\rfloor+n=\left\lceil\frac{n+1}{2}\right\rceil$, the proof for the bi-periodic k-Jacobsthal–Lucas sequence is analogous. □

Theorem 2.6 (Catalan's identity) For all integers n, r with $n\geq r$, the following identities hold:

$$\left(\frac{a}{b}\right)^{\varepsilon(n-r)}J_{k,n-r}J_{k,n+r}-\left(\frac{a}{b}\right)^{\varepsilon(n)}J_{k,n}^2=-(-2)^{n-r}\left(\frac{a}{b}\right)^{\varepsilon(r)}J_{k,r}^2$$

and

$$\left(\frac{b}{a}\right)^{\varepsilon(n+r)}j_{k,n-r}j_{k,n+r}-\left(\frac{b}{a}\right)^{\varepsilon(n)}j_{k,n}^2=(-2)^{n-r}\left(\frac{b}{a}\right)^{1-\varepsilon(r)}(abk^2+8)J_{k,r}^2.$$

Proof. Using Theorem 2.4 and identities (1.1) and (1.2), we have

$$\left(\frac{a}{b}\right)^{\varepsilon(n-r)}J_{k,n-r}J_{k,n+r}-\left(\frac{a}{b}\right)^{\varepsilon(n)}J_{k,n}^2$$

$$= \frac{a^{\varepsilon(n-r)}a^{1-\varepsilon(n-r)}a^{1-\varepsilon(n+r)}}{b^{\varepsilon(n-r)}(ab)^{\lfloor\frac{n-r}{2}\rfloor}(ab)^{\lfloor\frac{n+r}{2}\rfloor}}\frac{(\alpha^{n-r}-\beta^{n-r})(\alpha^{n+r}-\beta^{n+r})}{(\alpha-\beta)^2}$$
$$-\frac{a^{\varepsilon(n)}a^{1-\varepsilon(n)}a^{1-\varepsilon(n)}}{b^{\varepsilon(n)}(ab)^{\lfloor\frac{n}{2}\rfloor}(ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2$$
$$= \frac{a^{2-\varepsilon(n+r)-\lfloor\frac{n-r}{2}\rfloor-\lfloor\frac{n+r}{2}\rfloor}}{b^{\varepsilon(n+r)+\lfloor\frac{n-r}{2}\rfloor+\lfloor\frac{n+r}{2}\rfloor}}\frac{(\alpha^{n-r}-\beta^{n-r})(\alpha^{n+r}-\beta^{n+r})}{(\alpha-\beta)^2}$$
$$-\frac{a^{2-\varepsilon(n)-2\lfloor\frac{n}{2}\rfloor}}{b^{\varepsilon(n)+2\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2$$

$$= \frac{a^{2-n}}{b^n}\frac{(\alpha^{2n}-\alpha^{n-r}\beta^{n+r}-\alpha^{n+r}\beta^{n-r}+\beta^{2n})}{(\alpha-\beta)^2}$$
$$-\frac{a^{2-n}}{b^n}\frac{(\alpha^{2n}-2\alpha^n\beta^n+\beta^{2n})}{(\alpha-\beta)^2}$$
$$= \frac{a^2(-\alpha^{n-r}\beta^{n+r}-\alpha^{n+r}\beta^{n-r}+2\alpha^n\beta^n)}{(ab)^n(\alpha-\beta)^2}$$
$$= -\frac{a^2(\alpha\beta)^{n-r}}{(ab)^n(\alpha-\beta)^2}[\beta^{2r}+\alpha^{2r}-2\alpha^r\beta^r]$$
$$= -\frac{a^2(-2ab)^{n-r}}{(ab)^n}\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)^2$$
$$= -\frac{(-2)^{n-r}a^2}{(ab)^r}\frac{(ab)^{2\lfloor\frac{r}{2}\rfloor}}{a^{2-2\varepsilon(r)}}\left(\frac{a^{1-\varepsilon(r)}}{(ab)^{\lfloor\frac{r}{2}\rfloor}}\right)^2\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)^2$$
$$= -\frac{(-2)^{n-r}a^{2\varepsilon(r)}}{(ab)^{r-2\lfloor\frac{r}{2}\rfloor}}J_{k,r}^2$$
$$= -\frac{(-2)^{n-r}a^{2\varepsilon(r)}}{(ab)^{\varepsilon(r)}}J_{k,r}^2$$
$$= -(-2)^{n-r}\left(\frac{a}{b}\right)^{\varepsilon(r)}J_{k,r}^2\,.$$

The proof for the second identity follows similarly by using Theorem 2.4 and identities (1.3) and (1.4). □

After that, we derive the Cassini's identities as a special case of Theorem 2.6 as follows.

Theorem 2.7 (Cassini's identity or Simpson identity) For a positive integer n, the following identities hold:

$$\left(\frac{a}{b}\right)^{\varepsilon(n-1)}J_{k,n-1}J_{k,n+1}-\left(\frac{a}{b}\right)^{\varepsilon(n)}J_{k,n}^2$$
$$= -(-2)^{n-1}\frac{a}{b},$$
$$\left(\frac{b}{a}\right)^{\varepsilon(n+1)}j_{k,n-1}j_{k,n+1}-\left(\frac{b}{a}\right)^{\varepsilon(n)}j_{k,n}^2$$
$$= (-2)^{n-1}(abk^2+8).$$

Proof. Setting $r=1$ in Theorem 2.6 yields these identities immediately. □

Theorem 2.8 (D'ocagne's identity) For all integers n, r with $n \ge r$, the following identities hold:

$$a^{\varepsilon(nr+n)}b^{\varepsilon(nr+r)}J_{k,n}J_{k,r+1}$$
$$-a^{\varepsilon(nr+r)}b^{\varepsilon(nr+n)}J_{k,n+1}J_{k,r}$$
$$= -(-2)^r a^{\varepsilon(n-r)}J_{k,n-r},$$
$$a^{\varepsilon(nr+n)}b^{\varepsilon(nr+r)}j_{k,n+1}j_{k,r}$$
$$-a^{\varepsilon(nr+r)}b^{\varepsilon(nr+n)}j_{k,n}j_{k,r+1}$$
$$= (-2)^r a^{\varepsilon(n-r)}(abk^2+8)J_{k,n-r}.$$

Proof. Using Theorem 2.4 and identities (1.5) – (1.10), we have

$$a^{\varepsilon(nr+n)}b^{\varepsilon(nr+r)}J_{k,n}J_{k,r+1}$$
$$-a^{\varepsilon(nr+r)}b^{\varepsilon(nr+n)}J_{k,n+1}J_{k,r}$$
$$= a^{\varepsilon(nr+n)}b^{\varepsilon(nr+r)}\frac{a^{1-\varepsilon(n)}a^{1-\varepsilon(r+1)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}(ab)^{\lfloor\frac{r+1}{2}\rfloor}}$$
$$\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)\left(\frac{\alpha^{r+1}-\beta^{r+1}}{\alpha-\beta}\right)$$
$$-a^{\varepsilon(nr+r)}b^{\varepsilon(nr+n)}\frac{a^{1-\varepsilon(n+1)}a^{1-\varepsilon(r)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}(ab)^{\lfloor\frac{r}{2}\rfloor}}$$
$$\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)$$
$$= \frac{a^{\varepsilon(nr+n)+1-\varepsilon(n)+1-\varepsilon(r+1)-\lfloor\frac{n}{2}\rfloor-\lfloor\frac{r+1}{2}\rfloor}}{b^{-\varepsilon(nr+r)+\lfloor\frac{n}{2}\rfloor+\lfloor\frac{r+1}{2}\rfloor}}$$
$$\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)\left(\frac{\alpha^{r+1}-\beta^{r+1}}{\alpha-\beta}\right)$$
$$-\frac{a^{\varepsilon(nr+r)+1-\varepsilon(n+1)+1-\varepsilon(r)-\lfloor\frac{n+1}{2}\rfloor-\lfloor\frac{r}{2}\rfloor}}{b^{-\varepsilon(nr+n)+\lfloor\frac{n+1}{2}\rfloor+\lfloor\frac{r}{2}\rfloor}}$$
$$\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)$$
$$= \frac{a^{1-\frac{n+r-\varepsilon(n-r)}{2}}}{b^{\frac{n+r-\varepsilon(n-r)}{2}}}\cdot\frac{(\alpha^{n+r+1}-\alpha^n\beta^{r+1}-\alpha^{r+1}\beta^n+\beta^{n+r+1})}{(\alpha-\beta)^2}$$
$$-\frac{a^{1-\frac{n+r-\varepsilon(n-r)}{2}}}{b^{\frac{n+r-\varepsilon(n-r)}{2}}}\cdot\frac{(\alpha^{n+r+1}-\alpha^{n+1}\beta^r-\alpha^r\beta^{n+1}+\beta^{n+r+1})}{(\alpha-\beta)^2}$$
$$= \frac{a}{(ab)^{\frac{n+r-\varepsilon(n-r)}{2}}}\cdot\frac{(\alpha^{n+1}\beta^r-\alpha^n\beta^{r+1}-\alpha^{r+1}\beta^n+\alpha^r\beta^{n+1})}{(\alpha-\beta)^2}$$
$$= \frac{a}{(ab)^{\frac{n+r-\varepsilon(n-r)}{2}}}\cdot\frac{(\alpha\beta)^r(\alpha-\beta)(\alpha^{n-r}-\beta^{n-r})}{(\alpha-\beta)^2}$$
$$= \frac{a(-2ab)^r}{(ab)^{\frac{n+r-\varepsilon(n-r)}{2}}}\left(\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\right)$$
$$= \frac{a(-2)^r}{(ab)^{\frac{n-r-\varepsilon(n-r)}{2}}}\left(\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\right)$$
$$= \frac{a(-2)^r}{(ab)^{\lfloor\frac{n-r}{2}\rfloor}}\cdot\frac{(ab)^{\lfloor\frac{n-r}{2}\rfloor}}{a^{1-\varepsilon(n-r)}}\cdot\frac{a^{1-\varepsilon(n-r)}}{(ab)^{\lfloor\frac{n-r}{2}\rfloor}}\left(\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\right)$$
$$= (-2)^r a^{\varepsilon(n-r)}J_{k,n-r}\,.$$

The proof of d'ocagne's identity for the bi-periodic k-Jacobsthal-Lucas sequence follows similarly by applying the same method to $j_{k,n}$. □

3. Conclusions

This paper extends the classical Jacobsthal framework by introducing bi-periodicity and a tunable parameter k, defining the bi-periodic k-Jacobsthal and k-Jacobsthal–Lucas sequences. We derived their recurrence relations, generating functions, and Binet-type formulas, and established generalized forms of Catalan's, Cassini's, and d'Ocagne's identities. These results reveal deeper algebraic structures and patterns, with potential applications in computer science and discrete mathematics. Future work may explore combinatorial interpretations, algorithmic implementations, and links to other special sequences.

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The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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