PROGRESS IN APPLIED SCIENCE AND TECHNOLOGY
Faculty of Science and Technology
Rajamangala University of Technology Thanyaburi
Prog Appl Sci Tech. 2025; 15(2):9-17
ISSN (Online): 2730-3020
https://ph02.tci-thaijo.org/index.php/past

Research Article

Received: June 09, 2025
Revised: August 11, 2025
Accepted: August 13, 2025

DOI: 10.60101/past.2025.259700

Generalized DUS-Bilal Distribution: Properties and Applications

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Abstract

In this article, we proposed a flexible version of the Bilal distribution using the generalized DUS transformation. Its properties are studied. Three methods of parameter estimation, including the maximum likelihood, Anderson-Daring, and Cramer-Von Mises techniques, are used to estimate unknown parameters. Real datasets are used to demonstrate the applicability of the proposed distribution using the three methods of parameter estimation.

Keywords: Generalized DUS Transformation, Lifetime Data, Maximum Likelihood Estimation, Anderson-Daring Estimation, Cramer-Von Mises estimation

1. Introduction

Probability distributions help characterize uncertainty in data sets by identifying variation patterns. They provide a mathematical understanding of data-generating mechanisms, with lifetime distributions largely influenced by device failure mode, as they provide a probabilistic description of the behavior of the length of life. The choice of distribution depends on physical characteristics of the process of observations. The Bilal distribution is a lifetime distribution that represents a class of new, better-than-average renewal failure rates (1). The findings indicate that the density function of the Bilal distribution is consistently unimodal (see Figure 1) and exhibits lower skewness and kurtosis compared to the density of the exponential distribution. In this work, we focus on extending the Bilal distribution to be more flexible for analyzing various lifetime data patterns. In 2017, a generalization of the Bilal distribution with a two-parameter distribution is introduced by (2). The failure rate function may exhibit an inverted bathtub shape. The failure rate can also decrease or increase. The distribution is always unimodal and has less skewness and kurtosis than the density of the exponential distribution (2).

There are many techniques for developing statistical models, one of which is the transformation method. In 2015, Dinesh et al. (3) proposed a transformation technique termed the DUS transformation, utilizing an exponential distribution as the baseline distribution, referred to as the DUS exponential (DUSE) distribution. A novel distribution generated using the DUS transformation does not incorporate any additional parameters beyond those present in the baseline distribution. Maurya et al. (4) subsequently presented a novel generalized form of the DUS transformation, called the generalized DUS (GDUS) transformation. GDUS transformation is an efficient tool for generating new distributions from existing models, providing flexibility and simplicity in modeling various data. The added flexibility allows GDUS-transformed distributions to provide a better fit to observe data that might not be adequately described by simpler, existing distributions. This leads to more accurate statistical inference and predictions. There are lifetime distributions that derive from the GDUS transform, such as the GDUS exponential distribution (5), the GDUS transformed log-normal distribution (6), the GDUS transformed Garima distribution (7).

In this paper, we proposed a flexible version of the Bilal distribution using the GDUS transformation. Some properties of the proposed distribution are studied. Three estimation methods are illustrated. Applications for the proposed distribution are provided. Finally, the conclusion is presented.

2. Methods

2.1 Generalized DUS transformation

The GDUS transformation adjusts the cumulative density function (CDF) of a baseline distribution with CDF of $G(x;\mathbf{v})$ where $\mathbf{v}$ is a parameter vector of the baseline distribution. The CDF and probability density function (PDF) of GDUS distribution are respectively

$$F(x;\mathbf{v},\alpha)=\frac{\exp\left[G^{\alpha}(x;\mathbf{v})\right]-1}{e-1}, \quad (2.1)$$

$$f(x;\mathbf{v},\alpha)=\frac{\alpha g(x;\mathbf{v})G^{\alpha-1}(x;\mathbf{v})\exp\left[G^{\alpha}(x;\mathbf{v})\right]}{e-1}, \quad (2.2)$$

where $-\infty<x<\infty$ and $\alpha>0$.

2.2 Bilal distribution

The Bilal distribution is introduced by Abd-Elrahman (1) with the PDF and CDF as

$$f(x;\theta)=\frac{6}{\theta}e^{-\frac{2x}{\theta}}\left(1-e^{-\frac{x}{\theta}}\right),\ \text{for } x\geq 0, \theta>0 \quad (2.3)$$

$$F(x;\theta)=1-e^{-\frac{2x}{\theta}}\left(3-2e^{-\frac{x}{\theta}}\right). \quad (2.4)$$

Its PDF plots show in Figure 1. The density function of the Bilal distribution is consistently unimodal. The moment generating function and *r*th moments of Bilal distribution are (1), respectively

$$M_X(t)=\frac{6}{(3-\theta t)(2-\theta t)}, \quad (2.5)$$

$$\mathrm{E}(X^r)=r!\left(\frac{\theta}{6}\right)^r(3^{r+1}-2^{r+1}),\ r=1,2,3,... \quad (2.6)$$

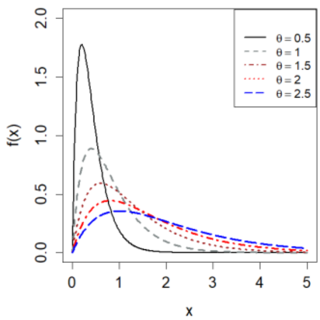

Figure 1 PDF plots of the Bilal distribution with different parameter θ

3. Results and Discussion

3.1 Generalized DUS Bilal distribution

Based on GDUS transformation, if the baseline distribution is the Bilal distribution, then the CDF of extend Bilal distribution with GDUS transformation is

$$F(x;\theta,\alpha)=\left[\exp\left\{\left[1-e^{-\frac{2x}{\theta}}\left(3-2e^{-\frac{x}{\theta}}\right)\right]^{\alpha}\right\}-1\right]\times\frac{1}{e-1}, \quad (3.1)$$

$$f(x;\theta,\alpha)=\frac{\alpha}{e-1}\left\{\left[1-e^{-\frac{2x}{\theta}}\left(3-2e^{-\frac{x}{\theta}}\right)\right]^{\alpha-1}e^{-\frac{2x}{\theta}}\times\frac{6}{\theta}\left(1-e^{-\frac{x}{\theta}}\right)\exp\left\{\left[1-e^{-\frac{2x}{\theta}}\left(3-2e^{-\frac{x}{\theta}}\right)\right]^{\alpha}\right\}\right\}, \quad (3.2)$$

where $0\leq x<\infty, \theta>0$, and $\alpha>0$. We have X distributed the generalized DUS-Bilal (GDUSB) distribution, denoted by X ~GDUSB (θ,α).

For $\alpha=1$, the GDUSB distribution reduces to the DUSB distribution with a parameter θ. The PDF plots of the GDUSB distribution are shown in Figure 2. Its density function is unimodal and decreasing function.

3.2 Properties

Properties of the proposed distribution are derived, including as survival function, hazard function, and order statistics. From the CDF as (3.1) then its survival function is

$$S(x) = 1 - F(x;\theta,\alpha) = \frac{1}{e-1}\left(e - \exp\left\{\left[1-e^{-\frac{2x}{\theta}}\left(3-2e^{-\frac{x}{\theta}}\right)\right]^{\alpha}\right\}\right). \quad (3.3)$$

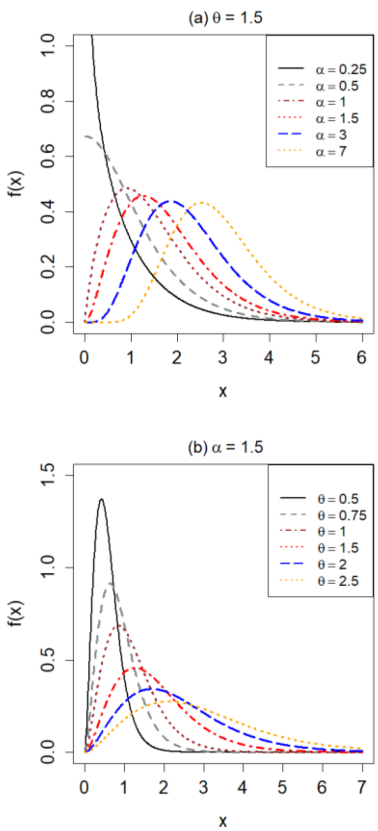

Figure 2 PDF plots of the GDUSB distribution with different parameters θ and α

From PDF and survival function of the GDUSB distribution, the hazard function is

$$\begin{aligned} h(x) &= \frac{f(x;\theta,\alpha)}{S(x)} \\ &= \alpha\left\{\frac{6}{\theta}e^{-\frac{2x}{\theta}}\left(1-e^{-\frac{x}{\theta}}\right)\left[1-e^{-\frac{2x}{\theta}}\left(3-2e^{-\frac{x}{\theta}}\right)\right]^{\alpha-1}\right. \\ &\quad \times \exp\left\{\left[1-e^{-\frac{2x}{\theta}}\left(3-2e^{-\frac{x}{\theta}}\right)\right]^{\alpha}\right\}\Bigg\} \\ &\quad \times \left(e - \exp\left\{\left[1-e^{-\frac{2x}{\theta}}\left(3-2e^{-\frac{x}{\theta}}\right)\right]^{\alpha}\right\}\right)^{-1}. \end{aligned} \quad (3.4)$$

Some plots of the hazard function of the GDUSB are provided in Figure 3.

Let $X_1,\ldots,X_n$ be random samples with PDF and CDF as $f(x;\theta,\alpha)$ and $F(x;\theta,\alpha)$ respectively. If $X_{(1)},\ldots,X_{(n)}$ be ordered random samples, then the PDF of the ith order statistics can be written as

$$\begin{aligned} f_{i:n}(x) &= \frac{f(x)}{B(i,n-i-1)}[F(x)]^{r-1}[1-F(x)]^{n-r} \\ &= \frac{\alpha}{B(i,n-i-1)(e-1)}\left\{\left[1-e^{-\frac{2x}{\theta}}\left(3-2e^{-\frac{x}{\theta}}\right)\right]^{\alpha-1}\right. \\ &\quad \times \frac{6}{\theta}e^{-\frac{2x}{\theta}}\left(1-e^{-\frac{x}{\theta}}\right)\exp\left\{\left[1-e^{-\frac{2x}{\theta}}\left(3-2e^{-\frac{x}{\theta}}\right)\right]^{\alpha}\right\}\Bigg\} \\ &\quad \times \left\{\frac{1}{e-1}\left[\exp\left\{\left[1-e^{-\frac{2x}{\theta}}\left(3-2e^{-\frac{x}{\theta}}\right)\right]^{\alpha}\right\}-1\right]\right\}^{i-1} \\ &\quad \times \left\{\frac{1}{e-1}\left(e-\exp\left\{\left[1-e^{-\frac{2x}{\theta}}\left(3-2e^{-\frac{x}{\theta}}\right)\right]^{\alpha}\right\}\right)\right\}^{n-i}, \end{aligned} \quad (3.5)$$

where $B(a,b)$ is a beta function.

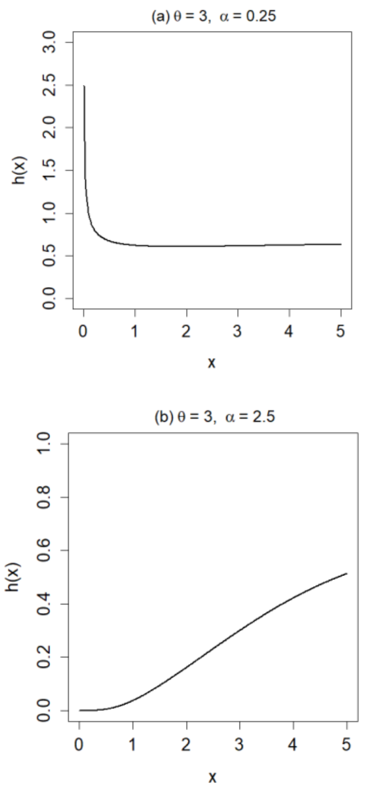

Figure 3 Hazard plots of GDUSB distribution with different parameters θ and α

3.3 Parameter estimation

In this paper, three methods of parameter estimation, including the maximum likelihood (ML), Anderson-Daring (AD), and Cramer-Von Mises (CVM) methods, are used to estimate unknown parameters θ and α of the GDUSB distribution.

3.3.1 Maximum likelihood estimation

Let $x_1,\ldots,x_n$ be random samples from $X \sim \text{GDUSB}(\theta,\alpha)$. The observed samples x_i has the explicit PDF as in (3.2), the likelihood function can be written as

$$L(\theta,\alpha)=\frac{6^n\alpha^n}{\theta^n(e-1)^n}\prod_{i=1}^{n}\left\{\left[1-e^{-\frac{2x_i}{\theta}}\left(3-2e^{-\frac{x_i}{\theta}}\right)\right]^{\alpha-1}\right.$$

$$\left.\times e^{-\frac{2x_i}{\theta}}\left(1-e^{-\frac{x_i}{\theta}}\right)\exp\left\{\left[1-e^{-\frac{2x_i}{\theta}}\left(3-2e^{-\frac{x_i}{\theta}}\right)\right]^{\alpha}\right\}\right\}. \tag{3.6}$$

Its corresponding log-likelihood function is

$$\ell(\theta,\alpha)=n\log 6+n\log\alpha-n\log\theta-n\log(e-1)$$
$$+(\alpha-1)\sum_{i=1}^{n}\log\left[1-e^{-\frac{2x_i}{\theta}}\left(3-2e^{-\frac{x_i}{\theta}}\right)\right]-\frac{2}{\theta}\sum_{i=1}^{n}x_i$$
$$+\sum_{i=1}^{n}\log\left(1-e^{-\frac{x_i}{\theta}}\right)+\sum_{i=1}^{n}\left[1-e^{-\frac{2x_i}{\theta}}\left(3-2e^{-\frac{x_i}{\theta}}\right)\right]^{\alpha}. \tag{3.7}$$

The ML estimators of θ and α can be solving the following equations

$$\frac{\partial\ell(\theta,\alpha)}{\partial\theta}=-\frac{n}{\theta}+\frac{2}{\theta^2}\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log\left(1-e^{-\frac{x_i}{\theta}}\right)$$
$$+(\alpha-1)\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log\left[1-e^{-\frac{2x_i}{\theta}}\left(3-2e^{-\frac{x_i}{\theta}}\right)\right]$$
$$+\sum_{i=1}^{n}\left[1-e^{-\frac{2x_i}{\theta}}\left(3-2e^{-\frac{x_i}{\theta}}\right)\right]^{\alpha}, \tag{3.8}$$

$$\frac{\partial\ell(\theta,\alpha)}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left[1-e^{-\frac{2x_i}{\theta}}\left(3-2e^{-\frac{x_i}{\theta}}\right)\right]$$
$$+\sum_{i=1}^{n}\frac{\partial}{\partial\alpha}\left[1-e^{-\frac{2x_i}{\theta}}\left(3-2e^{-\frac{x_i}{\theta}}\right)\right]^{\alpha}. \tag{3.9}$$

Since equations (3.8) - (3.9) are non-linear function, then we cannot obtain the ML estimates of θ and α directly. In this study, the ML estimates can be obtained numerically by maximizing, and the nlm function of stat package in R (8) is used to estimate each parameter.

3.3.2 Anderson-Daring estimation

The AD method is an estimator that finds the smallest distance by using an AD statistic (9), (10). Let $x_{(1)},\ldots,x_{(n)}$ be ordered random samples from the observed samples x_i

has the CDF as in (3.1), then the AD estimators θ and α, which are obtained by minimizing the following equation

$$AD(\theta,\alpha)=-n-\sum_{i=1}^{n}\frac{2i-1}{n}\Big\{\log\big(F(x_{(i)};\theta,\alpha)\big)+\log\big[1-F(x_{(n+1-i)};\theta,\alpha)\big]\Big\}, \quad (3.10)$$

The AD estimators derived from (3.10) can be acquired by solving the following

$$\frac{\partial AD(\theta,\alpha)}{\partial\theta}=0,\frac{\partial AD(\theta,\alpha)}{\partial\alpha}=0.$$

The nlm function from the stat package in R (8) is used calculated the AD estimates.

3.3.3 Cramer-Von Mises estimation

Let $x_{(1)},\dots,x_{(n)}$ be ordered random samples from the observed samples x_i has the CDF as in (3.1), then the CVM estimators (11), (8) are obtained by minimizing the following

$$T(\theta,\alpha)=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{2i-1}{2n}-F(x_{(i)};\theta,\alpha)\right]^2. \quad (3.11)$$

The CVM estimators derived from (3.11) can be solved with the following

$$\frac{\partial T(\theta,\alpha)}{\partial\theta}=0,\frac{\partial T(\theta,\alpha)}{\partial\alpha}=0.$$

The nlm function from the stat package in R (8) is used calculated the CVM estimates.

3.4 Applications

In this Section, we have explored the comparative performance of the GDUSB distribution to fit three real datasets with compare the distribution as follows:

(i) DUSB distribution:

$$F_{\text{DUSB}}(x)=\frac{1}{e-1}\left[\exp\left\{1-e^{-\frac{2x}{\theta}}\left[3-2e^{-\frac{x}{\theta}}\right]\right\}-1\right],$$

$$f_{\text{DUSB}}(x)=\left\{\frac{6}{\theta}e^{-\frac{2x}{\theta}}\left[1-e^{-\frac{2x}{\theta}}\left(3-2e^{-\frac{x}{\theta}}\right)\right]^{-1}\times\left(1-e^{-\frac{x}{\theta}}\right)\exp\left\{1-e^{-\frac{2x}{\theta}}\left(3-2e^{-\frac{x}{\theta}}\right)\right\}\right\}\frac{1}{e-1}.$$

(ii) Bilal distribution (1) has the CDF and PDF in (2.1) and (2.2) respectively.

(iii) GDUSE distribution (3):

$$F_{\text{GDUSE}}(x)=\frac{\exp\left[(1-e^{-\lambda x})^{\alpha}\right]-1}{e-1},$$

$$f_{\text{GDUSE}}(x)=\frac{\alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}}{e-1}.$$

$$f_{\text{GDUSE}}(x)=\frac{\alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}\exp\left[(1-e^{-\lambda x})^{\alpha}\right]}{e-1}.$$

(iv) DUSE distribution (2):

$$F_{\text{DUSE}}(x)=\frac{\exp(1-e^{-\lambda x})-1}{e-1},$$

$$f_{\text{DUSE}}(x)=\frac{\lambda e^{-\lambda x}\exp(1-e^{-\lambda x})}{e-1}.$$

(v) Exponential (Exp) distribution: $F_{\text{Exp}}(x)=1-e^{-\lambda x}$, and $f_{\text{Exp}}(x)=\lambda e^{-\lambda x}$.

A criterion for comparing distributions is the Kolmogorov-Smirnov (KS) statistic. A distribution exhibiting lower values in the KS test is the most appropriate distribution. This research examines a real dataset comprising with three datasets to assess the flexibility of the suggested distribution. Analysis data of three real data sets is provided.

Data I: The first data set consists of thirty successive values of March precipitation (in inches) recorded in Minneapolis/St. Paul, which is given by Hinkley (12). Abd-Elrahman (1) uses this data to fit the Bilal distribution. The data are: 0.32, 0.47, 0.52, 0.59, 0.77, 0.81, 0.81, 0.9, 0.96, 1.18, 1.20, 1.20, 1.31, 1.35, 1.43, 1.51, 1.62, 1.74, 1.87, 1.89, 1.95, 2.05, 2.10, 2.20, 2.48, 2.81, 3.0, 3.09, 3.37, 4.75.

Table 1 Parameter estimates of each distribution (Dist.) and KS test for Data I.

Dist.	Parameter Estimates	KS test (p-value)
	ML method	
GDUSB	$\hat{\theta} = 1.4457$	0.0565 (1.0000)
	$\hat{\alpha} = 1.4234$	
DUSB	$\hat{\theta} = 1.6714$	0.0868 (0.9776)
Bilal	$\hat{\theta} = 2.0186$	0.1144 (0.8276)
GDUSE	$\hat{\lambda} = 1.2928$	**0.0558** (1.0000)
	$\hat{\alpha} = 3.0612$	
DUSE	$\hat{\lambda} = 0.7799$	0.1988 (0.1868)
Exp	$\hat{\lambda} = 0.5970$	0.2352 (0.0724)
	AD method	
GDUSB	$\hat{\theta} = 1.4977$	**0.0575** (1.0000)
	$\hat{\alpha} = 1.300$	
DUSB	$\hat{\theta} = 1.7061$	0.0765 (0.9947)
Bilal	$\hat{\theta} = 2.1037$	0.0947 (0.9508)
GDUSE	$\hat{\lambda} = 1.2418$	0.0594 (0.9999)
	$\hat{\alpha} = 2.7867$	
DUSE	$\hat{\lambda} = 0.6805$	0.1597 (0.4283)
Exp	$\hat{\lambda} = 0.4966$	0.1844 (0.2592)
	CMV method	
GDUSB	$\hat{\theta} = 1.5028$	**0.0594** (0.9999)
	$\hat{\alpha} = 1.2853$	
DUSB	$\hat{\theta} = 1.7071$	0.0762 (0.9950)
Bilal	$\hat{\theta} = 2.1150$	0.0922 (0.9608)
GDUSE	$\hat{\lambda} = 1.2349$	0.0605 (0.9999)
	$\hat{\alpha} = 2.7524$	
DUSE	$\hat{\lambda} = 0.6710$	0.1559 (0.4591)
Exp	$\hat{\lambda} = 0.4885$	0.1802 (0.2844)

Data II: The second data is the private health insurance premiums data about the average annual percent change in 1969–2007 (see (13), (14), (15)). The data are: 14.4, 14.0, 15.4, 9.4, 11.7, 15.0, 24.9, 20.7, 12.5, 14.9, 12.6, 16.7, 13.8, 11.0, 12.9, 10.1,1.9, 8.5, 16.5, 15.3, 13.3, 9.8, 8.4, 7.9, 3.7, 5.1, 4.6, 4.4, 5.4, 6.1, 8.0, 10.0, 11.2, 10.1, 6.4, 6.7, 5.7, 5.8.

Data III: The third data is the breaking stress of carbon fibres of 50 mm length, which have been previously used by Nichols and Padgettcit (15). The data are: 0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Table 2 Parameter estimates of each distribution (Dist.) and KS test for Data II.

Dist.	Parameter Estimates	KS test (p-value)
	ML method	
GDUSB	$\hat{\theta} = 7.7444$	**0.0848 (0.9474)**
	$\hat{\alpha} = 2.2599$	
DUSB	$\hat{\theta} = 10.5487$	0.1276 (0.5638)
Bilal	$\hat{\theta} = 12.8825$	0.1501 (0.3591)
GDUSE	$\hat{\lambda} = 0.2416$	0.0863 (0.9398)
	$\hat{\alpha} = 4.8865$	
DUSE	$\hat{\lambda} = 0.1238$	0.2511 (0.0166)
Exp	$\hat{\lambda} = 0.0939$	0.2858 (0.0040)
	AD method	
GDUSB	$\hat{\theta} = 7.9899$	**0.0816 (0.9620)**
	$\hat{\alpha} = 2.1586$	
DUSB	$\hat{\theta} = 11.2714$	0.1423 (0.4253)
Bilal	$\hat{\theta} = 13.9732$	0.1667 (0.2414)
GDUSE	$\hat{\lambda} = 0.2344$	0.0841 (0.9509)
	$\hat{\alpha} = 4.7013$	
DUSE	$\hat{\lambda} = 0.1028$	0.2075 (0.0758)
Exp	$\hat{\lambda} = 0.0745$	0.2356 (0.0295)
	CVM method	
GDUSB	$\hat{\theta} = 8.1526$	**0.0886 (0.9265)**
	$\hat{\alpha} = 2.1045$	
DUSB	$\hat{\theta} = 11.3780$	0.1469 (0.3849)
Bilal	$\hat{\theta} = 14.1049$	0.1710 (0.2162)
GDUSE	$\hat{\lambda} = 0.2299$	0.0911 (0.9106)
	$\hat{\alpha} = 4.6044$	
DUSE	$\hat{\lambda} = 0.1014$	0.2131 (0.0634)
Exp	$\hat{\lambda} = 0.0736$	0.2399 (0.0252)

The result of parameter estimates and KS test of each data sets with fitting distributions are shown in Tables 1-3. Figures 4-6 shows graphs of the empirical and fitted distribution functions for the datasets with different estimation methods. Results show that each estimation method in the parameter estimates makes the KS value close to each other, which indicates that each estimation method has no different efficiency. The GDUSB gives lower KS values than the other distributions, indicating that the proposed distribution has a better fit to all three data sets than the other distributions.

Table 3 Parameter estimates of each distribution (Dist.) and KS test for Data III.

Dist.	Parameter Estimates	KS test (p-value)
ML method		
GDUSB	$\hat{\theta}=1.6511$	**0.1449 (0.1250)**
	$\hat{\alpha}=4.1754$	
DUSB	$\hat{\theta}=2.7041$	0.2422 (0.0009)
Bilal	$\hat{\theta}=3.3466$	0.2492 (0.0006)
GDUSE	$\hat{\lambda}=1.1331$	0.1488 (0.1077)
	$\hat{\alpha}=9.0129$	
DUSE	$\hat{\lambda}=0.4825$	0.3322 (<0.0001)
Exp	$\hat{\lambda}=0.3624$	0.3581 (<0.0001)
AD method		
GDUSB	$\hat{\theta}=1.4840$	**0.1000 (0.5236)**
	$\hat{\alpha}=6.5938$	
DUSB	$\hat{\theta}=3.0110$	0.2121 (0.0053)
Bilal	$\hat{\theta}=3.7599$	0.2320 (0.0016)
GDUSE	$\hat{\lambda}=1.2818$	0.1025 (0.4921)
	$\hat{\alpha}=14.9857$	
DUSE	$\hat{\lambda}=0.3812$	0.2611 (0.0002)
Exp	$\hat{\lambda}=0.2744$	0.2816 (<0.0001)
CVM method		
GDUSB	$\hat{\theta}=1.3173$	**0.0834 (0.7481)**
	$\hat{\alpha}=11.1618$	
DUSB	$\hat{\theta}=3.0365$	0.2169 (0.0040)
Bilal	$\hat{\theta}=3.7853$	0.2355 (0.0013)
GDUSE	$\hat{\lambda}=1.4682$	0.0854 (0.7214)
	$\hat{\alpha}=27.0384$	
DUSE	$\hat{\lambda}=0.3779$	0.2648 (0.0002)
Exp	$\hat{\lambda}=0.2727$	0.2839 (<0.0001)

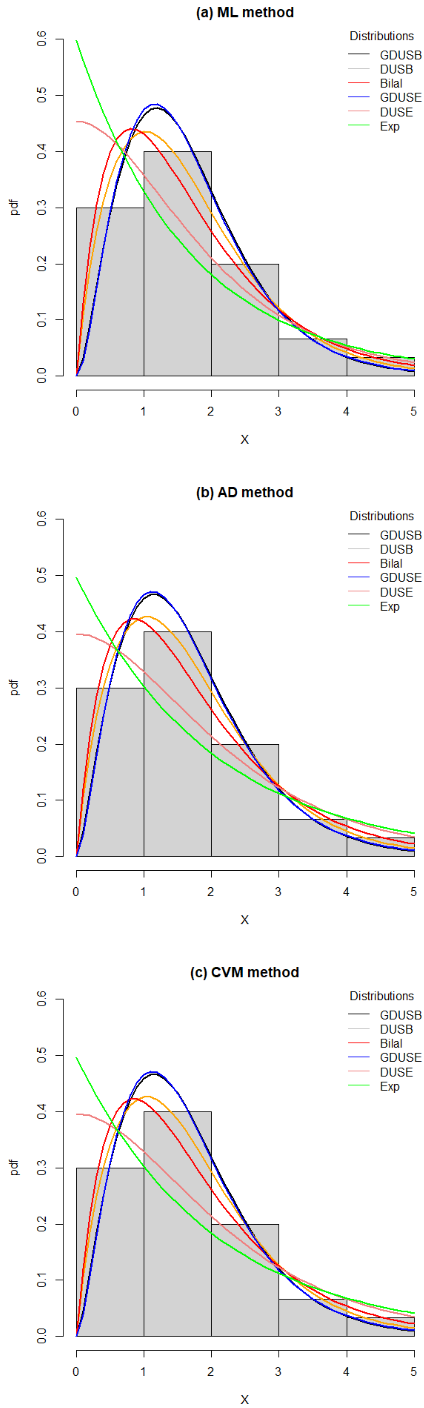

Figure 4 Empirical and fitting distributions of Data I with different estimation methods

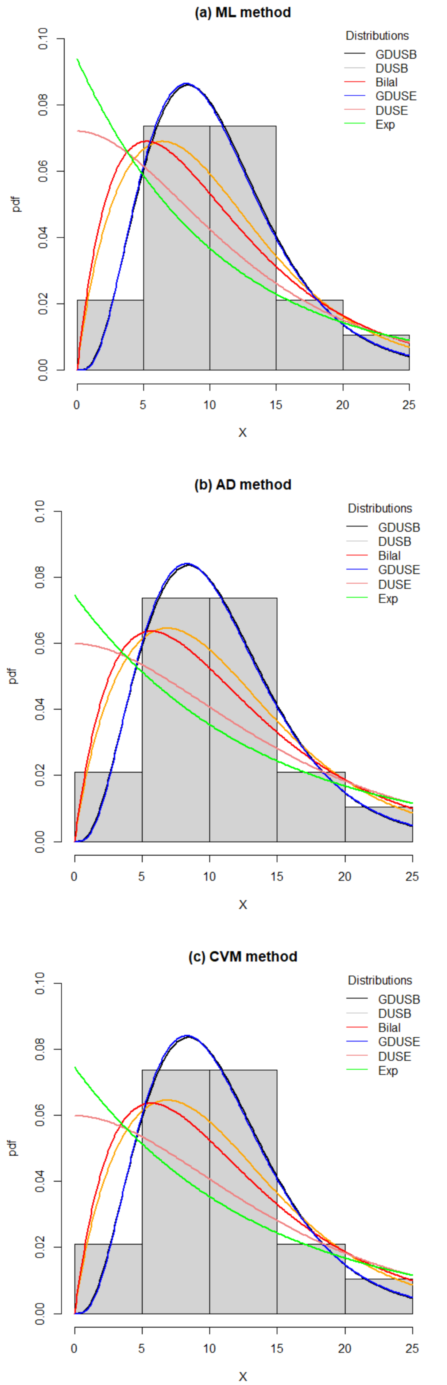

Figure 5 Empirical and fitting distributions of Data II with different estimation methods

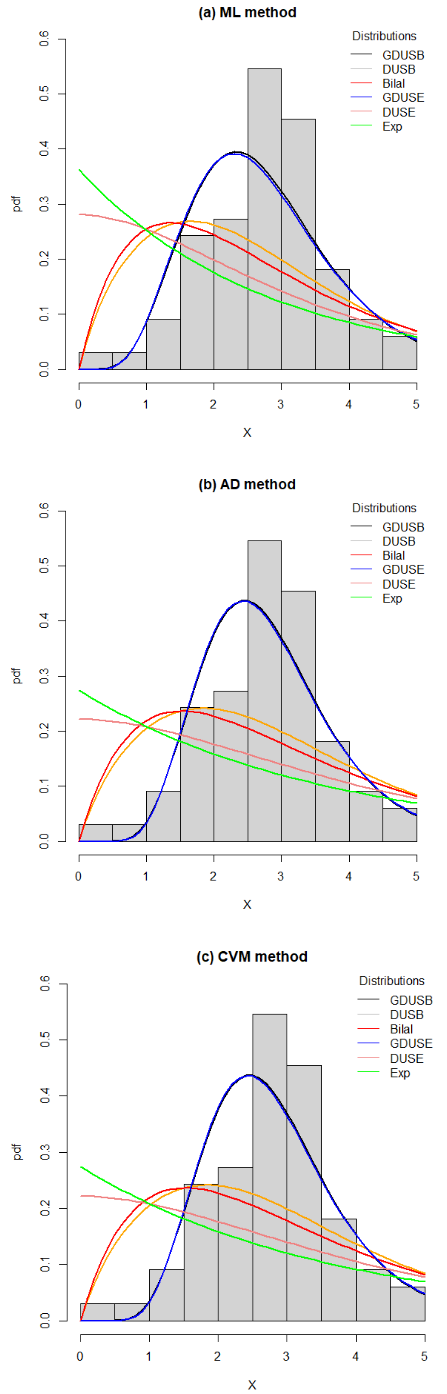

Figure 6 Empirical and fitting distributions of Data III with different estimation methods

4. Conclusions

This paper proposes a new distribution called the generalized DUS-Bilal distribution, which extends version of the Bilal distribution using the generalized DUS transformation. The function of survival and hazard, and order statistics are studied. The maximum likelihood, Anderson-Daring, and Cramer-Von Mises techniques are used to estimate parameters of distribution. The results show that the estimation methods have no different efficiency. The GDUSB gives lower KS values than the other distributions, indicating that the proposed distribution has a better fit for example datasets. The flexibility enables DUS-Bilal distributions to more accurately align with observed data that may not be sufficiently represented by simpler, existing distributions. This results in enhanced precision in statistical inference and predictions. However, when using this model, the model's suitability should be checked every time for accuracy in inference, and the parameters can be estimated using the various methods presented or other estimation methods such as Bayesian method.

Acknowledgements

Author is grateful to the Editor-In-Chief and the anonymous reviewer for fruitful comments on the paper which improved the presentation of the paper.

Declaration of Conflicting Interests

The author has declared no conflicts of interest related to the research, authorship, or publication of this article.

References

1. Abd-Elrahman AM. Utilizing ordered statistics in lifetime distributions production: a new lifetime distribution and applications. J Proba Stat Sci. 2013;11(2):153-64.
2. Abd-Elrahman AM. A new two-parameter lifetime distribution with decreasing, increasing or upside-down bathtub-shaped failure rate. Commun Stat-Theory Methods. 2017;46(18):8865-80.
3. Dinesh K, Umesh S, Sanjay Kumar S. A method of proposing new distribution and its application to Bladder cancer patients data. J Stat Appl Probab Lett. 2015;2:235-45.
4. Maurya SK, Kaushik A, Singh SK, Singh U. A new class of distribution having decreasing, increasing, and bathtub-shaped failure rate. Commun Stat-Theory Methods. 2017;46:10359-72.
5. AbuJarad MH, Khan AA, AbuJarad ESA. Bayesian survival analysis of generalized DUS exponential distribution. Austrian J Stat. 2020;49(5):80-95.
6. Irshad MR, Chesneau C, Nitin SL, Shibu DS, Maya R. The generalized DUS transformed log-normal distribution and its applications to cancer and heart transplant datasets. Mathematics. 2021;9(23):3113.
7. Akkanphudit T. Generalized DUS transformed Garima distribution: Properties, simulations and applications. Lobachevskii J Math. 2023;44(2):803-14.
8. R Core Team. R: A language and environment for statistical computing, R Foundation for Statistical Computing, Vienna, Austria, 2025.
9. do Espirito Santo APJ, Mazucheli J. Comparison of estimation methods for the Marshall Olkin extended Lindley distribution. J Stat Comput Simul. 2014;85:3437-50.
10. Anderson TW, Darling DA. A test of goodness-of-fit. J Am Stat Assoc. 1954;49(268):765-9.
11. Anderson TW. On the distribution of the two-sample Cramer-von Mises criterion. Ann Math Stat. 1962;33(3):1148-59.
12. Hinkley D. On quick choice of power transformations. Appl Stat. 1977;26:67-9.
13. Korkmaz MC, Yousof HM, Rasekhi M, Hamedani GG. The odd Lindley Burr XII model: Bayesian analysis, classical inference and characterizations. Data Sci J. 2018;16: 327-54.
14. Kibria BG, Shakil M. A new five-parameter Burr system of distributions based on generalized Pearson differential equation. JSM Proceedings, Section on Physical and Engineering Sciences, 2011: 866-80.
15. Nichols MD, Padgett WJ. A bootstrap control chart for Weibull percentiles. Qual Reliab Eng. 2006;22:141-51.

