Vibration of Circular Plates with Mixed EdgeConditions. Part I: Review of Research

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ABSTRACT

The vibration problems of elastic plates with uniformly constrained boundary edges have been several considered many times over the past years, but there are few scientific or technical literatures on vibrating plates under discontinuous edge conditions. Therefore, this paper attempts to review and summarize the extensive published technical literature relevant to the problems involving circular plates with mixed edge conditions. Of particular attentional interest is mainly addressed to plate vibration research. The static plate bending problems, however, are also explored optionally to complete the literature survey. The review is conducted with emphasis on the numerous methods being solved successfully previously for this class of plate problems.

Keywords: Circular plates, Mixed edge conditions, Plate bending, Plate vibration

1. INTRODUCTION

As generally known that plates have widely been used as part of fundamental structural components throughout various engineering design and application prob-lems [1,2], they are of interest to and have many applications in structural, aerospace or mechanical engineering such as in the design of aircraft, missiles, and ship struc-tures, as well as the construction of build-ings. Much attention of such problems was received to investigate the responses of static plate bending (deflections and stress resultants) and plate vibration (frequencies and corresponding vibration mode shapes).

Within the framework of thin elastic plate theory, many solutions concerning the bending of plates having regular or common boundary conditions can be found in numerous textbooks. A signi-ficant contribution and extensive study in the area of plate bending analysis based upon the analytical/numerical methods to-gether with its application have well been concisely collected and summarized in a fundamental monograph by Timoshenko and Woinowsky-Krieger [3] that gave a profound analysis of various plate bending problems.

It is, however, well known that the structural engineering components must resist not only static loads but also dyna-mic loads [4]. It is, therefore, necessary to understand the behaviors of the plates; namely, their deflections due to various static and dynamic loads, and their tenden-cy to resonate.

Hence, knowing the static response analysis of the plates alone is not sufficient to ensure proper and safe performance. Their design needs to include the effects of periodic or random time varying forces causing vibrations in order to determine the natural frequencies, mode shapes of

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vibration, and the dynamic responses [5,6]. The analysis of free vibration (eigenvalue) problems is of basic and applied interests in several fields of science and technology. An exhaustive summary of the published literature on the free vibrations of various shaped thin plates is available in Leissa [7] and Leissa's monograph [8].

Consequently, Leissa [9] considered and attempted to present comprehensive and accurate analytical results for the different twenty-one cases of free vibration of rectangular plates with various aspect ratios and Poisson's ratios. It is interesting and significant to note that exact characteristic equations involving frequency determinations were explicitly given only for the six cases of plate having two opposite simply supported edges. For the remaining fifteen cases, the Ritz method with 36 admissible terms containing the products of beam functions to determine their accurate natural frequencies was employed. The accurate (six significant figure) and extensive results for nondimensional frequency parameters of simply supported circular plates were carefully carried out and numerically examined by Leissa and Narita [10] with the use of asymptotic expansions of analy-tic Bessel functions. The first eighty frequency parameters for circular plates with three different boundary support conditions were numerically determined by Malison and Sompornjaroensuk [11] using ANSYS finite element program with a dense net.

In many cases of studied free vibration of thick plates which are beyond the scopes of the present work, Liew et al.[12] reviewed the existing literature on the vibration analysis of thick plates. Especially, emphasis was devoted to significant studies based on the Mindlin theory and the modified Mindlin plate theories for laminated plates. An extensive available numerical frequencies through the application of the Ritz method with the use of alge-braic polynomial as displacement functions in analyzing free vibrations of plates with various shapes according to the Mindlin theory was comprehensively presented by Liew et al.[13].

In order to determine the exact solutions of plate equations (both free vibration and static bending problems), it is recognized that the exact solutions are the most desirable, but not always easily attainable. This is because of the difficulties which have been encountered in trying to obtain the solutions satisfied all boundary conditions as well as the governing fourth-order partial differential equation exactly. Subsequently, research works on plate analyses were then conducted using a wide range of approximate mathematical techniques.

In this paper, the motivation and objective are to (a) review the methods that have successfully been used in the past for analyzing both the free vibration and static bending problems of circular plates focusing on the mixed edge boundary conditions and (b) discuss some significant behaviors on transition points of mixed edge conditions which may result to the solution accuracy obtained in each available solution technique for this class of problems.

Additionally, some higher natural frequencies of the plates and the localized frequency curve veering phenomena are then given and presented in the following Part II and Part III companion papers of this title, respectively.

2. CLASSICAL PLATE EQUATIONS

2.1 Free vibration equation

Mathematically, with limited to consideration of undamped free vibration analysis, the classical differential equation of motion for the vibratory transverse displacement (w) of a thin, homogeneous, isotropic elastic plate [3] can immediately be expressed in the following a governing 2-D partial differential equation represented in terms of spatial polar coordinates (r,θ) and time varying (t), which is [8]

$$D\nabla^4 w + \rho \partial^2 w / \partial t^2 = 0 \qquad (1)$$

where ρ is the mass density per unit area of plate surface and D is the flexural rigidity represented in terms of Young's modulus (E), Poisson's ratio (v), and plate thickness (h) by the expression as follow

$$D = Eh^3 / 12(1-v^2) \qquad (2)$$

and $\nabla^4 = \nabla^2\nabla^2$ is the biharmonic differential operator in which ∇^2 is the Laplacian operator expressed in polar coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \qquad (3)$$

On the assumption of assuming first a sinusoidal time response for free vibration, the transverse displacement (w) can be represented as

$$w(r,\theta,t) = W(r,\theta)\exp(i\omega t) \qquad (4)$$

in which

$$\exp(i\omega t) = \cos(\omega t) + i\sin(\omega t) \qquad (5)$$

where W is a transverse motion only of the position coordinates, ω is the circular frequency (expressed in radians per unit time) and $i = \sqrt{-1}$.

Substituting (4) into (1) and eliminating the time dependence yields

$$D\nabla^4 W - \rho\omega^2 W = 0 \qquad (6)$$

It is usually convenient to introduce a parameter k defined by

$$k^4 = \rho\omega^2 / D \qquad (7)$$

With the use of (7), equation (6) can be rewritten as

$$(\nabla^4 - k^4)W = 0 \qquad (8)$$

Equation (8) can also further be factored into

$$(\nabla^2 + k^2)(\nabla^2 - k^2)W = 0 \qquad (9)$$

It is immediately seen that the complete solution of (9) can be obtained by the sum of the two solutions W_1 and W_2 from the following equations

$$(\nabla^2 + k^2)W_1 = 0 \qquad (10)$$

$$[\nabla^2 + (ik)^2]W_2 = 0 \qquad (11)$$

Since the Fourier components in θ are assumed, the general solution of (9) for the transverse displacement W is then expressed in the form of infinite series as

$$W(r,\theta) = \sum_{m=0}^{\infty} W_m(r)\cos(m\theta) + \sum_{m=1}^{\infty} W_m^*(r)\sin(m\theta) \qquad (12)$$

where both W_m and W_m^* are the displacement functions only of r.

Back substituting (12) into (10) and (11) yields the following ordinary differential equations below

$$\frac{d^2 W_{m1}}{dr^2} + \frac{1}{r}\frac{dW_{m1}}{dr} - \left(\frac{m^2}{r^2} - k^2\right)W_{m1} = 0 \quad (13)$$

$$\frac{d^2 W_{m2}}{dr^2} + \frac{1}{r}\frac{dW_{m2}}{dr} - \left(\frac{m^2}{r^2} + k^2\right)W_{m2} = 0 \quad (14)$$

$$\frac{d^2 W_{m1}^*}{dr^2} + \frac{1}{r}\frac{dW_{m1}^*}{dr} - \left(\frac{m^2}{r^2} - k^2\right)W_{m1}^* = 0 \quad (15)$$

$$\frac{d^2W_{m2}^*}{dr^2}+\frac{1}{r}\frac{dW_{m2}^*}{dr}-\left(\frac{m^2}{r^2}+k^2\right)W_{m2}^*=0 \quad (16)$$

Equations (13) through (16) are found to be of the forms of Bessel's equation. Thus, the general solutions of (13) and (14) can be taken as

$$W_{m1}=A_mJ_m(kr)+B_mY_m(kr) \quad (17)$$

$$W_{m2}=C_mI_m(kr)+D_mK_m(kr) \quad (18)$$

where J_m and Y_m are the Bessel functions of the first and second kinds, respectively, and I_m and K_m are the modified Bessel functions of the first and second kinds, res-pectively. For the coefficients $A_m,\ldots, D_m$, they are solved from the prescribed boun-dary conditions of the plate.

The remaining equations of (15) and (16) have the similar form of solutions to those of (17) and (18) as follow

$$W_{m1}^*=A_m^*J_m(kr)+B_m^*Y_m(kr) \quad (19)$$

$$W_{m2}^*=C_m^*I_m(kr)+D_m^*K_m(kr) \quad (20)$$

Utilizing (12), (17) and (19), the general solution to (10) is

$$W_1=\sum_{m=0}^{\infty}[A_mJ_m(kr)+B_mY_m(kr)]\cos(m\theta)$$
$$+\sum_{m=1}^{\infty}[A_m^*J_m(kr)+B_m^*Y_m(kr)]\sin(m\theta) \quad (21)$$

and, in the same manner of (21), general solution of (11) can take in the form

$$W_2=\sum_{m=0}^{\infty}[C_mI_m(kr)+D_mK_m(kr)]\cos(m\theta)$$
$$+\sum_{m=1}^{\infty}[C_m^*I_m(kr)+D_m^*K_m(kr)]\sin(m\theta) \quad (22)$$

Superimposing the solutions of (21) and (22), the complete solution to (9) in polar coordinates is

$$W(r,\theta)=\sum_{m=0}^{\infty}[A_mJ_m(kr)+B_mY_m(kr)$$
$$+C_mI_m(kr)+D_mK_m(kr)]\cos(m\theta)$$
$$+\sum_{m=1}^{\infty}[A_m^*J_m(kr)+B_m^*Y_m(kr)$$
$$+C_m^*I_m(kr)+D_m^*K_m(kr)]\sin(m\theta) \quad (23)$$

It is remarkable that only for the circular plate having the origin of a polar coordinate system placed at the center of the plate, equation (23) must be reduced to

$$W(r,\theta)=\sum_{m=0}^{\infty}[A_mJ_m(kr)+C_mI_m(kr)]$$
$$\times\cos(m\theta)+\sum_{m=1}^{\infty}[A_m^*J_m(kr)$$
$$+C_m^*I_m(kr)]\sin(m\theta) \quad (24)$$

This is to avoid infinite deflections and stresses at the origin $r=0$ due to infinite values of functions Y_m and K_m at $r=0$. Equation (24) presents that at every interior point of the circular plate the solution must be finite. Additionally, the subscript m corresponds to the number of nodal diameters of plate vibrating. Since the plate supported by a uniform boundary condi-tion leads to have symmetry of vibratory displacements with respect to one or more diameters of the circle, the terms involving $\sin(m\theta)$ are not needed, and equation (24) can further be reduced to

$$W(r,\theta)=\sum_{m=0}^{\infty}[A_mJ_m(kr)+C_mI_m(kr)]$$
$$\times\cos(m\theta) \quad (25)$$

It can be concluded that for a circular plate with regular boundary conditions, only two condition equations are required to determine two unknown coefficients of the problem.

However, it is useful to express the circular frequency (ω) in terms of dimensionless form of frequency parameter (λ^2) by introducing $\lambda = ka$ where a is a radius of circular plate. With the use of (7), the frequency parameter is found to be

$$\lambda^2 = \omega a^2 \sqrt{\rho/D} \tag{26}$$

2.2 Static bending equation

Supposing the origin of a polar coordinate system taken to coincide with the center of circular plate, the governing differential equation for deflection (w) of a laterally loaded circular isotropic plate that expressed in polar coordinates can be derived in the following form as [3]

$$D\nabla^4 w = q \tag{27}$$

in which $q = q(r,\theta)$ is the static load distributed over the upper surface of the plate, and plate deflection $w = w(r,\theta)$ is a function of the position coordinates.

It is interesting to note that when comparing between (27) and (1), the second term as presented in the left-hand side of (1) is the inertia term which acted in the opposite direction to the plate during vibration.

The general solution of (27) can be taken in the form of a sum

$$w(r,\theta) = w_c(r,\theta) + w_p(r,\theta) \tag{28}$$

that satisfied the following equations

$$\nabla^4 w_c(r,\theta) = 0 \tag{29}$$

$$D\nabla^4 w_p(r,\theta) = q \tag{30}$$

where $w_c(r,\theta)$ is the complementary solution involved with support conditions and $w_p(r,\theta)$ is the particular solution depending on a given loading function (q).

By an identical procedure to represent the function $W(r,\theta)$ that given in (12), the solution of (29) can be expressed in the form of infinite series as

$$w_c(r,\theta) = \sum_{m=0}^{\infty} R_m(r)\cos(m\theta) + \sum_{m=0}^{\infty} R_m^*(r)\sin(m\theta) \tag{31}$$

in which R_m and R_m^* are functions of the radial distance r only.

Taking $m = 0$ in (31) yields

$$w_c(r,\theta) = R_0(r) + \sum_{m=1}^{\infty} R_m(r)\cos(m\theta) + \sum_{m=1}^{\infty} R_m^*(r)\sin(m\theta) \tag{32}$$

This is shown that the first term, $R_0(r)$, on the right-hand side of (32) represents the solution for symmetrical bending of circular plate which is independent of θ.

Substituting (32) into (29) and separating out the terms that involved with the angle θ leads to the following ordinary differential equations

$$\left[d^2/dr^2 + (1/r)(d/dr) - (m/r)^2\right] \times \left[d^2R_m/dr^2 + (1/r)(dR_m/dr) - (m/r)^2 R_m\right] = 0\,;\ m \ge 0 \tag{33}$$

$$\left[d^2/dr^2 + (1/r)(d/dr) - (m/r)^2\right] \times \left[d^2R_m^*/dr^2 + (1/r)(dR_m^*/dr) - (m/r)^2 R_m^*\right] = 0\,;\ m \ge 1 \tag{34}$$

The general solution to (33) for $m = 0$ is given by

$$R_0 = a_0 + b_0 r^2 + c_0 \log r + d_0 r^2 \log r \qquad (35)$$

For $m = 1$, the solutions of (33) and (34) are found to be

$$R_1 = a_1 r + b_1 r^3 + c_1 r^{-1} + d_1 r \log r \qquad (36)$$

$$R_1^* = a_1^* r + b_1^* r^3 + c_1^* r^{-1} + d_1^* r \log r \qquad (37)$$

and for $m > 1$,

$$R_m = a_m r^m + b_m r^{-m} + c_m r^{m+2} + d_m r^{-m+2} \qquad (38)$$

$$R_m^* = a_m^* r^m + b_m^* r^{-m} + c_m^* r^{m+2} + d_m^* r^{-m+2} \qquad (39)$$

where the coefficients $a_m, b_m, \ldots, c_m^*$, and d_m^* for any arbitrary integers m can be determined from the specified boundary conditions of circular plate.

It is also noted that only two coefficients in each integer m are required to be determined from the specific problem of circular plates having regular boundary conditions, and to eliminate all possible singularities existing in the plate as explained previously in subsection 2.1.

For more detailed information to the solutions of (9) and (29), the interested reader is suggested to consult the monographs of Leissa [8] and Timoshenko and Woinowsky-Krieger [3], respectively.

3. BOUNDARY CONDITIONS

For a circular plate with radius of a and having no interior supports, there are three different support conditions along its circular edge namely, clamped, simple, and free supports in which the condition equations for each type of supports depend on their characteristics.

Considering first the circular plate with clamped edge at $r = a$, two boundary conditions are given by

$$W(a,\theta) = 0 \qquad (40)$$

$$\frac{\partial W(a,\theta)}{\partial r} = 0 \qquad (41)$$

for the problems of plate vibration. In the case of static bending problems, equation (40) and (41) are changed to be

$$w(a,\theta) = 0 \qquad (42)$$

$$\frac{\partial w(a,\theta)}{\partial r} = 0 \qquad (43)$$

in which W and w are previously presented in (25) and (28), respectively.

If the circular plate is simply supported at $r = a$, then the boundary conditions are, for the plate vibration,

$$W(a,\theta) = 0 \qquad (44)$$

$$M_r(a,\theta) = 0 \qquad (45)$$

and for the plate bending,

$$w(a,\theta) = 0 \qquad (46)$$

$$M_r(a,\theta) = 0 \qquad (47)$$

whereas M_r is the radial bending moment acted along circumferential sections of the plate defined by

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \qquad (48)$$

The function w as shown in (48) is replaced by function W for plate vibration problems.

The boundary conditions of a completely free circular plate with radius of a can be written as

$$M_r(a,\theta) \qquad (49)$$

$$V_r(a,\theta)=0 \qquad (50)$$

in which V_r is the supplemented shearing force (the Kelvin-Kirchhoff edge reaction) as given by

$$V_r=-D\left[\frac{\partial\nabla^2 w}{\partial r}+\frac{(1-\nu)}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial\theta^2}\right)\right] \qquad (51)$$

and function w is also changed to be W for problem of plate vibrations.

4. VIBRATION PROBLEM

In the mid-1950's, a more general problem of forced vibration due to a periodic load, perpendicular to the circular plate with mixed boundary conditions between simply supported and clamped edges and also loaded by a uniform compressive load (N) acting in the middle surface of the plate was studied by Nowacki and Olesiak [14]. The governing differential equation of the problem can be drawn as

$$D\nabla^4 w(r,\theta,t)+\rho\frac{\partial^2 w(r,\theta,t)}{\partial t^2}+N\nabla^2 w(r,\theta,t)=p(r,\theta)\exp(i\omega t) \qquad (52)$$

where the right-hand side term of (52) is a periodic load. The transverse displacement (deflection) of the plate with mixed edge conditions, due to the external loads, can be obtained by superposition of the deflec-tion of the same plate simply supported all around (w_1) and the deflection of that plate simply supported and subjected to an unknown function of clamping moment alone that treated as external load (w_2),

$$w(r,\theta,t)=w_1(r,\theta,t)+w_2(r,\theta,t) \qquad (53)$$

The solution of problem was thus reduced to determine the solution of Fredholm integral equation of the first kind in terms of an unknown function of clamping moment. Later, Bartlett [15] considered the free vibration and buckling problems of circular plate having mixed boundary conditions in the same manner of the previous work [14]. The governing equation of (52) is reduced to

$$D\nabla^4 w(r,\theta,t)+\rho\frac{\partial^2 w(r,\theta,t)}{\partial t^2}+N\nabla^2 w(r,\theta,t)=0 \qquad (54)$$

Two variational principles were derived for the lowest eigenvalues showing that the governing expressions can be determined by separation of variables. The upper and lower bounds for the lowest eigenvalue (fundamental frequency parameter) were shown to be close together.

Noble [16] reexamined the problem that treated by Bartlett [15]. The differential equation of the plate is the same as given in (54). Considering simple harmonic vibrations, the function $w(r,\theta,t)$ that presented by (4) can be used. Applying the mixed boundary conditions have led to the dual series equations and was reducible to an integral equation. The frequency or buckling load was obtained from an approximate solution of the integral equation by analytical methods which yield very good agreement with Bartlett's results [14].

The problem of large amplitude vibrations of circular plates have studied by Ramachandran [17]. The edge of the plate is a mixing between simple and clamped supports. The energy method and method of elastically restrained boundary condi-tions that generalized for simply supported and clamped supports were applied. The total potential energy (U) of the system can be expressed as

$$U = U_B + U_M + U_r + U_m \quad (55)$$

where U_B, U_M, U_r, and U_m are strain energy due to bending, stretching of the middle plane, edgewise reaction (V_n), and edgewise moment (M_n), respectively.

The expressions for V_n and M_n are given by

$$V_n = K_v w \quad (56)$$

$$M_n = -K_m \frac{\partial w}{\partial n} \quad (57)$$

where K_v and K_m are coefficients to fixity.

The usual boundary conditions can then be obtained by choosing K_v and K_m properly as

$$K_v = \infty,\ K_m = 0 \rightarrow \text{simple support} \quad (58)$$

$$K_v = \infty,\ K_m = \infty \rightarrow \text{clamped} \quad (59)$$

K_v or K_m or both finite → elastic support

(60)

Hirano and Okazaki [18] dealt with the vibration problem of circular plates with three different mixed boundary condi-tions of clamped-free, simply supported-free, and clamped-simply supported by means of the weighted residual method. The frequency equation was derived by finding stationary conditions and dynamic boundary conditions with the use of the Lagrange method of multipliers. For free vibrations, the displacement of vibrating plate was assumed to be

$$w(r,\theta,t) = W(r,\theta)\sin\omega t \quad (61)$$

and function $W(r,\theta)$ was expanded into a Fourier series as

$$W(r,\theta) = \frac{1}{2}W_0(r) + \sum_{m=1}^{\infty}[W_m(r)\cos(m\theta) + W_m^*(r)\sin(m\theta)] \quad (62)$$

Narita and Leissa [19] extended the method as previously developed by Leissa et al.[20] which analyzed the free vibration problem of circular plates having nonuniform edge constraints by representing translational and rotational springs for elastic support, to analyze free vibration of simply supported circular plates with rotational springs along part of the circumference of the edge. The boundary conditions for this edge are given as follow

$$W(a,\theta) = 0 \quad (63)$$

$$M_r(a,\theta) = K_\psi \frac{\partial W(a,\theta)}{\partial r} \quad (64)$$

where a is the radius of circular plate, W is defined in the same manner of (24), and K_ψ is the spring stiffness coefficient which vary circumferentially around the boundary, and can be expanded in a Fourier series,

$$K_\psi(\theta) = \sum_{n=0}^{\infty} L_n\cos(n\theta) + \sum_{n=1}^{\infty} L_n^*\sin(n\theta)$$

(65)

where L_n and L_n^* are Fourier coefficients determined in the usual manner.

Since K_ψ is set to be zero, the plate becomes a simply supported circular plate. If K_ψ goes to infinity, then that part of the plate is to be clamped edge. Numerical results were demonstrated that if a non-dimensional spring constant $aK_\psi / D = 10^4$ is taken, the rotationally constrained parts of the edge can be considered as clamped.

Some results as special cases were compared and shown to be close to Bartlett [15] and Hirano and Okazaki [18].

Narita and Leissa [21] also further applied the method [20] to the problem of free vibrations of a circular plate elastically constrained along parts of its edge and free on the remainder. The elastic constraints are consisted of both translational and/or rotational springs, in which translational spring stiffness coefficient (K_w) can be expanded into Fourier components around the circumference of the plate similar to that of (65),

$$K_w(\theta) = \sum_{n=0}^{\infty} K_n \cos(n\theta) + \sum_{n=1}^{\infty} K_n^* \sin(n\theta) \tag{66}$$

where K_n and K_n^* are Fourier coefficients.

The boundary condition for parts of the edge attached to translational spring is given by

$$V_r(a,\theta) = -K_w W(a,\theta) \tag{67}$$

As a special case for a circular plate having uniformly constrained by a translational spring, the obtained results were shown that if a non-dimensional spring constant $a^3K_w/D = 10^6$ is taken, this stiffness value can be used to treat the spring system as rigid, and the plate is considered to be a simply supported plate. With utilizing the presented series-type method, numerical results were also given for the problem of the plate which is partially free and partially simply supported, as well as for the clamped-free boundary.

A more generally complicated problem wherein additional mass also appears along a segment of the plate edge with elastic constraints was analytically treated by Leissa and Narita [22]. A series-type solution was derived in the same manner of previous works [19,21] with an exact solution to the governing partial differential equation of (6) taken as (12).

For a free circular plate elastically constrained along parts of the edge by the attachment of translational and rotational springs (such springs having stiffnesses K_w and K_ψ, respectively) and also having a uniformly distributed, additional strip of mass m and rotary inertia I_G (per unit of length) acted upon a segment of the boundary, the following boundary conditions are then required along typical portions of the edge,

$$V_r(a,\theta) = -(K_w - m\omega^2)W(a,\theta) \tag{68}$$

$$M_r(a,\theta) = (K_\psi - I_G\omega^2)\frac{\partial W(a,\theta)}{\partial r} \tag{69}$$

where K_w and K_ψ have defined in (66) and (65), respectively. The coefficients m and I_G can be expanded into Fourier series as

$$m(\theta) = \sum_{n=0}^{\infty} m_n \cos(n\theta) + \sum_{n=1}^{\infty} m_n^* \sin(n\theta) \tag{70}$$

$$I_G(\theta) = \sum_{n=0}^{\infty} I_{G_n} \cos(n\theta) + \sum_{n=1}^{\infty} I_{G_n}^* \sin(n\theta) \tag{71}$$

in which m_n, m_n^*, I_{G_n}, $I_{G_n}^*$ are Fourier coefficients.

For the numerical determination of natural frequencies of circular plate hav-ing partially free and partially clamped edges, Eastep and Hemmig [23] used a high precision conforming triangular and quadrilateral plate bending elements that provided by the NASTRAN finite element program to represent the circular plate with mixed edge conditions. The circular plate was modelled as seven rings of elements by the use of 144 quadrilateral (for the outer rings) and 24 triangular plate ele-ments (for the inner ring joined at the plate center).

Based on the finite element method together with the use of consistent and conforming plate bending elements gave upper bounds to the vibration frequencies. The governing equation as presented in (8) can be written in a form of matrix equation to evaluate the natural frequencies (ω) and the corresponding mode shapes $\{W\}$ of the plate as

$$\{[K]-\omega^2[M]\}\{W\}=\{0\} \qquad (72)$$

where $[K]$ is the assembly of the stiffnesses of the individual plate bending elements and $[M]$ is a consistent mass matrix.

Referred to the obtained numerical results, the analysis has led to observe that the mixed boundary conditions cause some of the higher modes of vibration to split into two branches of vibration mode shape existing at approximately the same frequency with increasing arc length of the free boundary.

Liew [24] proposed an approximate model with combinations of the Rayleigh-Ritz and the Lagrangian multiplier methods to study the free vibration of circular plates having point supports, partial internal curved supports, and mixed edge boundary conditions. The asmissible *pb*-2 Ritz function consisting of the product of a two-dimensional polynomial and a basic function was used to analyze the problems. For the vibration problem of circular plates having mixed edge conditions, numerical results were compared with the results obtained by Narita and Leissa [19,21] which are in close agreement together.

Yuan and Dickinson [25] applied the Ritz method in which the plate was treated as composed of sectorial plate elements joined together by means of very stiff, artificial springs to study of the free vibration of annular, circular and sectorial plates. The plate may be supported only on portions of the boundary or may involve cut-outs. The trial functions (admissible functions) for the Ritz solution are combinations of simple and orthogonal polynomials in cylindrical polar coordinates. The appropriate stiffness and mass coefficients for a generic element were formulated. The Rayleigh quotient was obtained by summing the strain and kinetic energies over the system, and the Ritz minimization procedure was then carried out to yield the eigenvalue problem. Numerical results for a specific case of circular plate having partially clamped edge were carried out and compared with the results as given by Narita and Leissa [21] who used a Fourier series solution, which yield a reasonably good agreement.

The accurate values of frequency parameter for a thin, elastic circular plate with various mixed boundary conditions, namely (a) simply supported-clamped, (b) simply supported-free and (c) clamped-free that obtained by using a well-known computer finite element code (ALGOR) with a dense net of 20232 elements were numerically carried out by Rossi and Laura [26]. Very good agreement was clearly observed when examining the results were compared with the values as given in Bartlett [15] and Noble [16]. The first nine frequency parameters were presented for three different mixed boundary conditions of the plate with the Poisson's ratio taken as 0.3.

The Rayleigh-Ritz method was successfully applied to study the problem of transverse vibrations of elliptical plate with half of the boundary clamped and the rest simply supported by Hassan and Makary [27]. Additionally, in this work, the thickness of the plate is varying linearly in the space coordinates. Comparison have made with the known results given by Bartlett [15] and Narita and Leissa [19] for circular plates. Based on the Rayleigh-Ritz method, the method consists of minimizing the Rayleigh quotient as

$$\omega^2 = (\iint_{R'} D[(\nabla^2 W)^2 + 2(1-\nu)\{(W_{XY})^2$$

$$-W_{XX}W_{YY}\}]dXdY) \Big/ \rho h \iint_{R'} hW^2 dXdY \tag{73}$$

where D is the flexural rigidity as given in (2), the subscripts denote differentiation.

For the free harmonic motions of plate vibration, the transverse displacement $w(X,Y,t)$ of the plate at the point (X,Y) at time t can be assumed to be of the form similar to that of (61) as

$$w(X,Y,t) = W(X,Y)\sin\omega t \tag{74}$$

where $W(X,Y)$ is the maximum displacement at a certain point (X,Y) of the plate, and the domain R' defined by

$$R' = \left\{(X,Y) : \frac{X^2}{a^2} + \frac{Y^2}{b^2} \le 1\right\} \tag{75}$$

where a is the semi-major axis and b is the semi minor axis of the elliptical plate.

The function $W(X,Y)$ was approximated with N-term of the form

$$W = (y^3 - rzy^2 - r^2z^2y + r^3z^3)\sum_{j=1}^{N} c_j x^{m_j} y^{n_j} \tag{76}$$

$$x = X/a,\ y = Y/a,\ z = \sqrt{1-x^2} \tag{77}$$

in which c_j are constants, m_j and n_j are polynomials of various degrees, and x, y, z are non-dimensional variables.

Later, Hassan [28] also applied the previous method to consider the problem of elliptical plate having mixed boundary conditions between simple support and free edge by changing the function W to be

$$W = z^2(y + rz)\sum_{j=1}^{N} c_j x^{m_j} y^{n_j} \tag{78}$$

Furthermore, free vibration problem of elliptical plates of variable thickness with mixed boundary conditions between clamped and free edge was investigated and presented by Hassan [29]. In this case of the plate, the representation of function W can be approximated and taken in the following form

$$W = (y - rz)^2 z\sum_{j=1}^{N} c_j x^{m_j} y^{n_j} \tag{79}$$

The method of perturbation of boundary conditions was applied to solve plate vibration problems with mixed boundary conditions by Febbo et al.[30] in which the boundary condition of the plate is mixed between simple and clamped supports.

Assuming simple harmonic motion at frequency ω, the equation of motion that satisfies the displacement amplitude W as given by (8), the frequency of the plate can then be obtained from the following equation as

$$k^4 - (k^0)^4 = \oint_C \left(\nabla^2 W^0 \frac{\partial W}{\partial n} - \nabla^2 W \frac{\partial W^0}{\partial n}\right.$$

$$\left. + W^0 \frac{\partial \nabla^2 W}{\partial n} - W \frac{\partial \nabla^2 W^0}{\partial n} \right) dl \Big/ \int_S (W^0)^2 dA \tag{80}$$

and

$$C = C_1 \cup C_2 \tag{81}$$

where k^4 is defined by (7), $(k^0)^4$ is for the unperturbed value, W^0 and W are the unperturbed and perturbed solutions, respectively, C_1 and C_2 are the boundary contours for the clamped and simply supported edges, respectively, in which C is the total boundary, and S is the boundary domain (surface) of the plate.

A modified Galerkin approach with the use of very simple polynomial coordinate functions was also additionally developed to determine the fundamental frequency which yields good engineering accuracy. However, the proposed method has been mentioned that it can be extended to analyze higher modes of vibration.

The lower natural frequencies of circular plate having various partially mixed boundary conditions were numerically evaluated by Bauer and Eidel [31]. Based on a semi-analytical method, the transverse displacement of (4) was first assumed and the displacement amplitude as given in (24) has used to satisfy the mixed boundary conditions for a finite number of boundary points. The boundary conditions at $r = a$ may be either,

$$W = 0\,;\ \frac{\partial W}{\partial r} = 0 \qquad (82)$$

for clamped support,

$$W = 0\,;\ M_r = 0 \qquad (83)$$

for simple support,

$$M_r = 0\,;\ V_r = 0 \qquad (84)$$

for free edge,

$$\frac{\partial W}{\partial r} = 0\,;\ V_r = 0 \qquad (85)$$

for guided support, and for elastic support

$$M_r - K\frac{\partial W}{\partial r} = 0;\ V_r + kW = 0 \qquad (86)$$

where M_r and V_r are, respectively defined by (48) and (51) with instead of w by W, and the terms K and k as presented in (86) are rotational spring distributed stiffness (moment per unit length) opposing the edge rotation and translational spring dis-tributed stiffness (force per unit length) opposing the translational motion in the direction of W, respectively.

Hassan [32] used the boundary characteristic orthogonal polynomials in two variables of *xy*-coordinates together with the Rayleigh-Ritz method for solving the vibration problem of a clamped-free elliptical plate which makes the obtained solutions faster in convergence. Comparisons have been made with known results as given previously by Hassan [29].

5. STATIC BENDING PROBLEM

There are few articles in the technical literature on the subject of bending of circular plates with mixed edge conditions. The first investigation is that of Nowacki and Olesiak [14] who considered the vibration, buckling, and bending of such plates.

Conway and Farnham [33] used a direct point-matching approach by choosing a solution of (29) which satisfies the various boundary conditions at a prescribed number of boundary points to analyze the bending of a uniformly loaded circular plate in which there are simply supported-clamped, simply supported-free, or clamped-free combinations of edge conditions. An appropriate solution of (29) was taken as

$$w_c = a_0 + b_0 r^2 + \sum_{m=1}^{\infty} (a_m r^m + c_m r^{m+2}) \cos(m\theta) \qquad (87)$$

which is similar form to that of (32) where a_0, b_0, a_m, and c_m are constants to be determined from prescribed boundary conditions. For the problem of simply supported circular plate having an unknown of distributed edge moment (M) applied along the arc portion of the edge at $r=a$, the moment can be expressed in the Fourier series form as

$$(M_r)_{r=a} = \frac{M\phi}{\pi}\left[1+2\sum_{m=1}^{\infty}\frac{\sin(m\phi)}{m\phi}\cos(m\theta)\right] \tag{88}$$

where ϕ is the angle over the segment that M is applied.

Applying the boundary conditions for simply supported edge as presented in (44) and (45) to (87) yields

$$w_c = -\frac{M(a^2-r^2)\phi}{2\pi D}\left[\frac{1}{1+\nu} + 2\sum_{m=1}^{\infty}\frac{r^m}{a^m(1+\nu+2m)}\frac{\sin(m\phi)}{m\phi}\cos(m\theta)\right] \tag{89}$$

The particular solution for the uniformly loaded (q_0), simply supported plate satisfying (30) is generally founded to be

$$w_p = \frac{q_0(a^2-r^2)}{64D}\left[\left(\frac{5+\nu}{1+\nu}\right)a^2 - r^2\right] \tag{90}$$

Superimposing (89) and (90) with enforcing zero edge slope condition along the portion that edge moment is applied, the unknown edge moment can be deter-mined. Therefore, problem of circular plate with simply supported-clamped boundary conditions is solved.

The solution to a uniformly loaded circular plate having simply supported-free boundary conditions can be obtained by solving the problem of a plate, which has free edge and is supported by uniform shear force along the arc segment. The shearing force distribution around the edge is then expanded in the Fourier series form. By balancing the shearing force to the load q with applying the boundary conditions for free edge, the unknown shearing force can be determined.

For the case of circular plate with clamped-free boundary conditions, by superposing appropriate solutions for (a) a uniformly loaded, free edge plate supported by uniform edge shear, and (b) a free edge plate bent by distributed edge moment, this plate problem can be solved by techniques similar to those previously used.

Leissa and Clausen [34] also applied the point matching method to numerically solve the problem of uniformly loaded (q_0) circular plate having two portions of its boundary clamped and the remainder simply supported. Due to two-fold symmetry of the problem, an exact solution to (27) was taken as

$$w = \frac{q_0 r^4}{64D} + \sum_{m=0,2,4}^{\infty}(a_m r^m + c_m r^{m+2})\cos(m\theta) \tag{91}$$

Das Gupta [35] analytically treated a uniformly loaded circular plate with partly clamped and partly simply supported edge by conformal transformation of the circular area of the plate to a semi-infinite half space. The mixed boundary conditions of the problem can be written as the dual series equations and further reduced to a pair of integral equations which was solved by the Mellin transform method.

Stahl and Keer [36] have considered the bending of a uniformly loaded circular plate with mixed conditions on the boundary. Two cases of mixed boundary conditions are (a) clamped-simply supported and (b) simply supported-free. The general solution which exhibits the two-fold symmetry of the problem can be taken as (91). The mixed boundary conditions were formulated as the dual series equations and reduced to a Fredholm integral equation of the second kind using the finite Hankel integral transform.

Two problems of torsion of a stamp on a circular plate were investigated by Alexsandrov and Chebakov [37]. Mixed boundary conditions of the problem were reducible to dual series in terms of Bessel functions and solved by reducing the series to an infinite algebraic system of the first kind with a singular matrix of coefficients.

Hamada et al.[38] presented a new approach which is an iterative method, to solve the bending problems of circular plates with uniform or a concentrated load under mixed boundary conditions between simple and clamped supports. The general complementary solution that satisfied (29) was expressed as the same form of series term of (91) due to symmetry of problem.

Hamada et al.[39] adopted the iterative method [38] to analyze the bending of a simply supported circular plate subjected to elastic constraint. By this method, several kinds of the distribution of the elastic constraint varying continuously or stepwise were easily treated and capable of dealing with nonlinear constraints.

Kiattikomol and Sriswasdi [40] dealt with the analytical solution of a uniformly loaded (q_0) annular plate in which the inner edge is free and the outer edge has mixed conditions between simple and clamped supports. The method of finite Hankel integral transform was used similar to that of Stahl and Keer [36]. The solution was set up by using the deflection equation satis-fying the differential equation of (27) and two-fold symmetry of the plate as

$$w = \frac{q_0 r^4}{64D} + a_0 + b_0 r^2 + c_0 \log r + d_0 r^2 \log r + \sum_{m=0,2,4}^{\infty} (a_m r^m + b_m r^{-m} + c_m r^{m+2} + d_m r^{-m+2}) \cos(m\theta) \quad (92)$$

in which the undetermined constants (a_0, b_0 ,…, c_m and d_m) have previously been defined in section 2.

Mifune et al.[41] have proposed an effective coefficient comparison method to numerically solve the bending problem of axi-symmetrically loaded circular plates having mixed boundary conditions in three cases of (a) clamped-free, (b) simply supported-free, and (c) simply supported-elastically constrained for rotation boundary condition. With the use of a Fourier series expansion to fit the mixed boundary conditions has led to the result of many coupled equations for finding the Fourier coefficients. Numerical results were shown to be in excellent agreement with those obtained by the iterative method [39].

Zheng et al.[42] proposed an analytical method using the concept of mixed boundary functions to analyze the problem of circular plates subjected to arbitrary lateral loads. Based upon this method, the trial functions were constructed by using the series of particular solutions of the biharmonic equations. Results were carried out to show the stability and high convergence rate of the proposed method.

Strozzi and Vaccari [43] analyzed the bending of circular plate partially supported along an edge arc and deflected by a central transverse force. The problem was first formulated in terms of a Fredholm integral equation of the first kind and then modified into a new Fredholm integral equation of the first kind. An approximate solution has introduced and numerically computed with the use of collocation method to determine the integral equation for obtaining the interested problem solutions.

Monegato and Strozzi [44] have analytically considered the problems of circular plate simply supported along two antipodal edge arcs and subjected to a centrally concentrated force. Two kinds of contact reactions were analyzed which are the cases of distributed reaction force alone and a distributed force joined to a distributed couple. Both of these problems were formulated in terms of an integral equation of the Prandtl type with Hilbert and Volterra operators that associated with two constraint conditions. The edge deflections along the free edge of the plate with radius of *a* can be expressed as

$$w(\theta)=\frac{Pa^2}{\pi D(1-\nu)(3+\nu)}\left\{\frac{1}{2}[(1+\cos\theta)\times\ln(1+\cos\theta)+(1-\cos\theta)\ln(1-\cos\theta)]+\frac{(1+\nu)}{4}\left[\pi|\theta-\sin\theta|-\theta^2\right]-\pi^2\frac{(1+\nu)}{24}+\frac{(1-\nu)^3}{16(1+\nu)}\ln 2\right\};\ -\pi\le\theta\le\pi \tag{93}$$

for the first kind of problem, and for the second kind,

$$w(\theta)=\frac{Ma}{\pi D(1-\nu)(3+\nu)}(1+\nu)\times\left[\theta-\frac{\pi}{2}(1-\cos\theta)\right]-2\sin\theta\ln\left[\tan\left(\frac{\theta}{2}\right)\right];\quad 0\le\theta\le\pi \tag{94}$$

where *P* is a concentrated force applied at the plate center, and *M* is the self-equilibrated boundary couple applied at each opposite free edge.

Monegato and Strozzi [45] further analyzed the problem as previously investigated in [43]. The analysis was, however, emphasized on the contact reaction in a circular plate in which the contact reaction has first assumed to be formed by a distributed reaction force that accompanied by a distributed a distributed moment with radial axis. The problem was then formulated in terms of an integral equation of the Prandtl type together with supplemented by a vertical and a rotational equilibrium condition. Furthermore, a design chart has been derived for the normalized deflection of the plate center versus the angular ex-tent of the supports.

Zheng et al.[46] have developed the boundary collocation method based on the least-square technique and a corresponding adaptive computation process to analyze the bending problem of circular plates with mixed edge conditions. A series of the biharmonic polynomials function was used as the trial functions and the local error indicators were determined by considering the residuals of the energy density on the boundary. Numerical results showed good agreement with other available results as presented in the literature.

Strozzi and Monegato [47] extended the previous study [44] to analytically investigate the bending problem of circular plate partially clamped along two antipodal arcs, the remaining part of the border being free, and loaded by a concentrated transverse central force. This problem has formulated in terms of the complex-valued Hilbert singular integral equation of the second kind in the complex domain, and the solution to integral equation was de-rived and obtained in analytical, integral form.

Ostryk and Ulitko [48] have considered the bending of a partially simply supported circular plate under the action of concentrated normal force at its center similar to that studied by Monegato and Strozzi [44]. The formulation of problem was reduced to the integral equation of a Prandtl-type in which the solution of integral equation can be solved numerically by three different methods, namely (i) the method of mechanical quadrature with the use of the interpolation Lagrange polynomial, (ii) the method of orthogonal polynomials by reduction to an infinite system of algebraic equations for the coefficients of expansion in a series with respect to the Chebyshev polynomials of the first kind, and (iii) the method of reduction to the Fredholm integral equation of the second kind using the inversion of a singular integral that can be solved numerically in the sense of quadrature method.

6. DISCUSSION

As it is known that, for a plate with mixed edge conditions, singularities in the bending fields are to be expected at the transition points of discontinuous boundary which is proportional to the inverse square root of the distance measured from the transition point. The nature of these singularities has first been studied using the Fadle eigenfunction expansion techniques by Williams [49]. Since the moment singularities are of an inverse square root type, it turns out that the distribution of supplemented or Kirchhoff shearing force becomes nonintegrable. However, the total reaction force exerted by the supports can still be determined from the equilibrium condition of the plate.

Williams [49] derived and obtained the six characteristic equations associated the possible combinations of three different boundary conditions namely, simply supported, clamped, and free edges along the radial boundaries of isotropic sectorial plates having the vertex angle of α, the governing equation of the plate bending problem is still to be of (27). Since the boundary condition of the plate is first mainly interested, only the complementary solution to (29) is considered which can be written as

$$w_c(r,\theta)=w_c^c(r,\theta)+w_c^r(r,\theta) \qquad (95)$$

in which $w_c^c(r,\theta)$ and $w_c^r(r,\theta)$ are the solutions that satisfied the circumferential and radial edge boundary conditions, respectively.

The solution of $w_c^c(r,\theta)$ can be expressed in the similar form as (32), which is [3]

$$w_c^c(r,\theta)=\sum_{m=0}^{\infty}R_m(r)\cos\left(\frac{m\pi\theta}{\alpha}\right)+\sum_{m=0}^{\infty}R_m^*(r)\sin\left(\frac{m\pi\theta}{\alpha}\right) \qquad (96)$$

whereas $R_m(r)$ and $R_m^*(r)$ have previously been defined in subsection 2.2.

For the remaining complementary solution $w_c^r(r,\theta)$, Williams [49] has used the method of separation of variables to represent $w_c^r(r,\theta)$ as

$$w_c^r(r,\theta)=G(r)F(\theta) \qquad (97)$$

and assuming $G(r)$ is a function of power series in r,

$$G(r)=\sum_{n=1}^{\infty}c_n r^{\lambda_n+1} \qquad (98)$$

where c_n is the unknown constant and λ_n is the eigenvalue parameter to be determined as part of the solution later in which the values of λ_n need not be an integer and can mathematically be complex number in general.

Application of the Laplacian operator as given in (3) to (97), yields

$$\nabla^2 w_c^r(r,\theta)=\sum_{n=1}^{\infty}c_n r^{\lambda_n-1}\times\left[\frac{d^2}{d\theta^2}+(\lambda_n+1)^2\right]F(\theta) \qquad (99)$$

The homogeneous biharmonic equation of $w_c^r(r,\theta)$ can be taken as

$$\nabla^4 w_c^r(r,\theta) = \nabla^2\nabla^2 w_c^r(r,\theta) = 0 \qquad (100)$$

and

$$\sum_{n=1}^{\infty} c_n r^{\lambda_n - 3}\left[\frac{d^2}{d\theta^2} + (\lambda_n+1)^2\right]$$

$$\cdot\left[\frac{d^2}{d\theta^2} + (\lambda_n-1)^2\right]F(\theta) = 0 \qquad (101)$$

Because of the linear independence of the terms $r^{\lambda_n - 3}$ for different values of λ_n, one can immediately be obtained the ordinary differential equation as

$$\left[\frac{d^2}{d\theta^2} + (\lambda_n+1)^2\right]\left[\frac{d^2}{d\theta^2} + (\lambda_n-1)^2\right]F(\theta)$$

$$= 0 \qquad (102)$$

The general solution of (102) can be written in the form as

$$F(\theta) = \tilde{C}_1\cos(\lambda_n - 1)\theta + \tilde{C}_2\sin(\lambda_n - 1)\theta$$

$$+\tilde{C}_3\cos(\lambda_n + 1)\theta + \tilde{C}_4\sin(\lambda_n + 1)\theta \qquad (103)$$

where $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3$, and $\tilde{C}_4$ are the arbitrary constants.

With the use of (98) and (103), the expression of $w_c^r(r,\theta)$ that given by (97) becomes

$$w_c^r(r,\theta) = \sum_{n=1}^{\infty} W_n(r,\theta) \qquad (104)$$

$$W_n(r,\theta) = r^{\lambda_n + 1}[A_n\sin(\lambda_n + 1)\theta$$

$$+B_n\cos(\lambda_n + 1)\theta + C_n\sin(\lambda_n - 1)\theta$$

$$+D_n\cos(\lambda_n - 1)\theta] \qquad (105)$$

in which the unknown constants A_n, B_n, C_n, and D_n can be determined from the four boundary conditions, two conditions along each radial edge at $\theta = 0$ and $\theta = \alpha$ resulting a set of four homogeneous algebraic equations in terms of these constants.

In order to obtain a non-trivial solution, the determinant of the coefficients of unknown constants must be vanished leading to the final result of eigenequation (characteristic equation) to the determination of λ_n. The terms $W_n(r,\theta)$ have been recognized and called the corner functions [50-52] characterizing the local behavior, which can be singular at the vertex of angular corner of the plate. It is interesting to note that there are an infinite number of eigenfunctions λ_n for each eigenfunctions, there will be an infinite number of corner functions that correspond to the boundary conditions along two radial edges.

Since λ_n is the complex number, only the real part Re(λ_n) of any values λ_n is then considered and has to be positive, Re(λ_n) ≥ 0 to meet the requirement of regu-larity conditions at the vertex, namely the deflection and slope must be finite at the origin ($r = 0$) in the physical sense and also the strain energy density U_0 is not integrable over any area near the vertex of the sector domain if Re(λ_n) < 0. With these conditions, the minimum values of Re(λ_n) are desired to be determined. For the ver-tex angle (α) varied from 0 to π, the values of minimum Re(λ_n) were first provided by Williams [49] and for the angles exceeding π radian ($\alpha > \pi$) which are re-entrant corners, the values of minimum Re(λ_n) were presented by Leissa, et al.[50].

In order to demonstrate the moment singularities existing at the vertex of sectorial plate with vertex angle of α, one introduces

$$\mathrm{w} = w_c^r(r,\alpha) \qquad (106)$$

and the circumferential bending moment (M_θ) acted along radial sections of the plate is defined by

$$M_\theta = -D\left[\frac{1}{r}\frac{\partial \mathrm{w}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \mathrm{w}}{\partial \theta^2} + \nu\frac{\partial^2 \mathrm{w}}{\partial r^2}\right] \quad (107)$$

Substituting (106) into (107) for w yields

$$M_\theta(r,\alpha) = -D\sum_{n=1}^{\infty} r^{\lambda_n - 1}[(\lambda_n + 1)(1+\nu\lambda_n) \times F(\alpha;\lambda_n) + F''(\alpha;\lambda_n)] \quad (108)$$

where superscript primes denote the derivative with respect to θ.

Considering a sectorial plate that has a combination of three sets of boundary condition along two radial edges namely, simply supported-free, simply supported-clamped, and clamped-free, the minimum $\mathrm{Re}(\lambda_n)$ for $\alpha = \pi$ corresponded to a semi-circular plate is found to be 0.5 [49,50]. Thus, substituting $\lambda_n = 0.5$ for $\alpha = \pi$ into (108) and taking $r \rightarrow 0$, it can be shown that the bending moment is square-root-singular at the transition point of discon-tinuous support at the origin ($r = 0$) of the plate coordinates.

Due to the stress singularities arising in the problems of plate with mixed edge conditions, the necessity of considering the singularity in solutions was confirmed by Chen and Pickett [53] and Leissa et al.[54]. Both of these numerical works concluded that to get sufficient accuracy in the solutions, it requires the use of appropriate singularity functions to represent the singular behavior at that point of singularity.

Leissa [55] also suggested that, for the problems of plate having singularities due to the abrupt change in geometry or mixed boundary conditions, the singularities are not negligible and have to be incorporated in the analysis. If these singularities are not considered properly, then highly inaccurate or even meaningless results can arise. Therefore, the infinite quantities may have strong effects upon the global behavior of the configuration such as static or dynamic deflections, free vibration frequencies and buckling loads.

In order to derive and obtain the solution to (100), the reader is suggested to consult the classic paper of Williams [49] and some discussions have been given by Kongtong and Sompornjaroensuk [56].

An exact analytical method based on the finite Hankel integral transform techniques for solving the bending and free vibration problems of plate having mixed boundary conditions which can cope with the moment singularities in the solution was explained in Sompornjaroensuk and Kiattikomol [57,58] previously.

Recently, Huang et al.[59,60] proposed new sets of enriched basis functions that can yield admissible functions for the Ritz method using the moving least-squares (MLS) approach for analyzing the vibration and buckling of cracked plates, which give the correct singularity order for the stress resultants at the crack tips. Another method was made by Huang et al.[61] who employs a domain decomposi-tion technique in combination with a set of admissible functions in the Rayleigh-Ritz method to determine the generalized coor-dinates and the corresponding modal fre-quencies and shapes of vibratory cracked plates. Xue et al.[62,63] considered and analyzed the free vibration and buckling problems of cracked plates based on the Ritz method. The correct singularity order in stress near the tip of the crack as well as the discontinuities in both displacement and bending rotation across the crack can mathematically be described by a series of corner functions that introduced and incor-porated into the admissible functions of the displacement which consist of the modi-fied characteristic functions. The accuracy of the obtainable solutions was verified through a convergence test.

However, it is observable from the mentioned works [59-63] that they are involved with the crack problem of rectangular plates. The problem of circular plates having either mixed edge boundary conditions or cracks can then be treated successfully based on their proposed methods.

7. SUMMARY

Because the vibration problem of circular plates with mixed edge conditions has become a challenging problem for scientists and engineers, this paper is then aimed to review and summarize the various methods, both analytical and numerical methods, which have used successfully and found in the published academic or technical literature. Furthermore, a review of bending problems for this class of the plate is also given for completing the literature survey.

The basic information of fundamental equations including their general solutions for classical plate theory is presented in details. Significantly, some behaviors on stress singularities existing at the point of transitions between two different boundary conditions are mathematically discussed and demonstrated in the present paper.

Since the vibration problem of circular plates with mixed edge conditions is of academic, technical, and technological importance, the need of knowledge of some higher natural frequencies is numerically determined and given in a Part II companion paper of this title. Moreover, the interesting phenomena of frequency curve veering and vibratory mode localization of the plates are also presented in a Part III companion paper.

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