Vibration of Circular Plates with Mixed Edge Conditions. Part II: Numerical Determination for Higher Frequencies

Yos Sompornjaroensuk

Department of Civil Engineering, Faculty of Engineering, Mahanakorn University of Technology, Thailand
E-mail: yos@mut.ac.th

ABSTRACT

The main goal of this research is to numerically deal with an accurate value for some higher natural frequencies represented in terms of frequency parameter of circular plates using finite element software with a dense meshed of element model. Two different cases of vibratory circular plate regarding mixed edge conditions are considered as (i) simply supported-free and (ii) clamped-free. The obtainable numerical results are given and presented as a dependent function of the angle over the circumferential plate sup-porting. The values for the first thirty frequency parameters are then carried out and numerically given in the tabular forms, which could be served for comparison with other methods. Some observations and limitations are also addressed.

Keywords: Circular plates, Finite element, Frequency parameter, Mixed edge conditions

1. INTRODUCTION

Plate members are formed to be a very important component of a structure, which have been usually applied as parts of fundamental components in the practical designs [1,2]. They are generally recog-nized and proved to be proper members in complicated structures.

As it is, however, well known that most structural engineering components must resist not only static loads [3] but also dynamic loads [4,5]. Thus, it is necessary to understand the behavior of these plate structures which are their deflections due to dynamic loads and their tendency to re-sonate.

Since the free vibration characteris-tics of the plate that are involved with frequen-cies and their corresponding mode shapes are of important and most interesting, free vibration results are then certainly useful in which knowledge of the natural frequen-cies can help the designer to avoid the peak resonances.

Although the subject of plates has a long history which goes back for more than two centuries [2,3], it remains true that most theoretical solutions to this class of problems are considered to be at best approximate in nature. This is due to the existing some troubles required to satisfy either the plate’s partial differential equa-tion or the prescribed boundary conditions exactly [6,7]. Thus, numerous numerical techniques have improved continually and used to overcome the situation. Emphasiz-ing the problems of plate bending, a valua-ble research was numerically treated to compare approximate methods [8,9].

Considering the analytical solution of plate vibration analysis with uniform thick-ness, it may suitably be used the polar coordinate system to formulate and obtain separable solutions to the partial differen-tial equation of the vibratory plate. Significant-ly, the mode shape of vibration can then be sought in the form of a series of products of Bessel and modified Bessel

Received 14-09-2020
Revised 19-11-2020
Accepted 25-12-2020

functions that have been shown in a Part I compa-nion paper of the present title [10] leading to an infinite number of solutions appearing in frequency characteristic equation. Com-putation is then difficult to carry out.

In order to theoretically analyze the circular plate vibration problems, analytical closed-form expressions can generally be found for the plates having common edge conditions [4,6,7,11], but very little is available in literature for the problem of plates with discontinuous or mixed conditions [12-15] in which numerical method is, however, required to this class of problem for determining the approximate solutions.

With the advent of high speed digital computer developments, a set of large number of simultaneous algebraic equations can be performed and numerically solved in a relatively short time [16-22]. Remarkably, it should not be disremembered about numerical methods that the obtained solutions are still approximate since their results do not satisfy all boundary conditions or the governing equation. An accurate approximate method has been reviewed and discussed by Yoowattana et al. [23] for problems of bending and vibration of plates having various shapes and involving stress singularities.

2. STATEMENT OF PROBLEM

In the field of engineering applications, numerical method is one of the useful approaches for determining the interested solutions in theoretical analysis and engineering design, especially for the complex problems. It can be noted that there are various methods developed and to be used in analyzing vibration of circular plates.

Olson and Lindberg [24] mathematically derived two finite plate bending elements in polar coordinates. The first element was in the form of a circular sector and had nine degrees of freedom, while the second was in the form of an annular sector and had twelve degrees of freedom. Static deflections of an annular and a com-plete circular plates and free vibrations of circular plates were analyzed and numerically compared to exact solutions. The good engineering accuracy had been ob-tained with the method.

It is obviously found that there exist many developable approximate numerical methods to be used to treat the plate problems in the last six decades. Significantly, these most numerical methods came into view as special cases in which their results have validated as a check with the results obtained from finite element method.

At the present time, finite element procedures [25-27] are an important and frequently indispensable part of engineering analysis and design. Finite element computer programs are also now widely used in practically all branches of engineering and science. The essence of a finite element solution of an engineering problem is that a set of governing algebraic equations is established and solved, and it was only through the use of the digital computer that this process could be rendered effective and given general applicability. Significantly, an important aspect of a finite element procedure is its reliability in which the present paper emphasizes this highlight throughout the study of free vibrations of circular plate with mixed edge conditions.

The objective of this paper is to deal with the accurate values for some higher natural frequencies represented in terms of frequency parameters of circular plates having mixed edge conditions using a well-known computer finite element code.

Two different cases of the plate are considered. The first is of partially simply supported circular plate and the second is partially clamped circular plate as depicted in Figures 1 and 2, respectively.

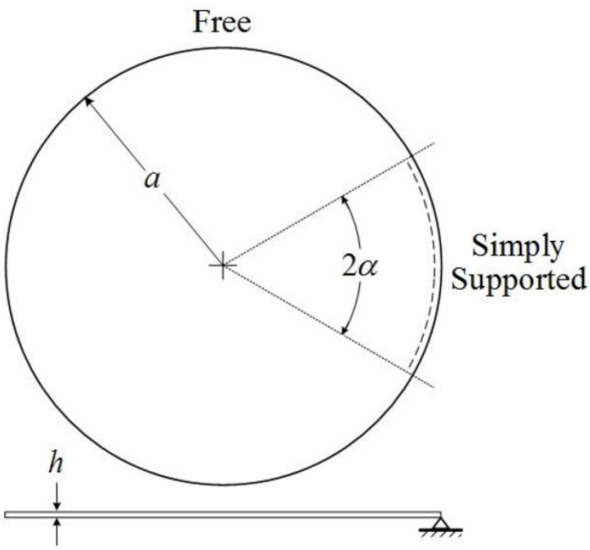

Figure 1. Configuration of partially simply supported circular plate.

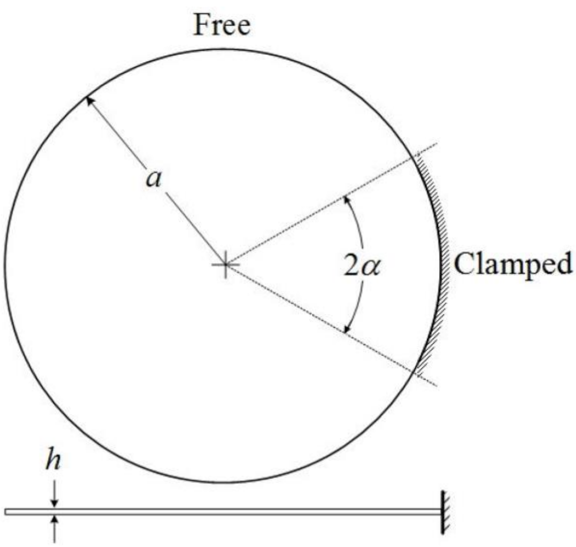

Figure 2. Configuration of partially clamped circular plate.

As shown in Figures 1 and 2 above, the plate has a uniform thickness of h with a radius of a in which the angle 2α is measured with respect to the plate center providing circumferential length of support.

In the present work, numerical analysis can be made with the use of finite element code through the ANSYS computer finite element program software package [28,29] that have analyzed free vibration problems of circular plate with three different regular boundary conditions by Malison and Sompornjaroensuk [30].

3. FINITE ELEMENT MODEL

To numerically determine the vibration behaviors of the plate, the ANSYS finite element program is implemented to model the problems by making use of quadrilateral shape of SHELL181 element type. This mentioned element consists of a 4 nodes with 6 degrees of freedom specified at each node (3 translations in the direction of *x*, *y*, and *z*, and 3 rotations about the *x*, *y*, and *z*-axis). It is suitable for analyzing the problems of thin to moderately-thick plates. In addition, SHELL181 element has been well-suited for the analysis of both linear and nonlinear problems. Moreover, it has also provided for the full and reduced integration within the domain of element in which the logarithmic strain and true stress measures are accounted for element formulation. Further detailed information about this element type, the ANSYS Theory Reference [28] is preferably consulted. Figure 3 presents the mesh of finite element dense net. The number of 12720 elements with 73482 degrees of freedom is used to model the circular plate for two different cases of the plate under consideration.

With the use of plate discretization that shown in Figure 3, it is significant to suppose the proper boundary conditions at the transition points (the points having two different types of boundary support) due to the plate having mixed boundary support conditions. The basis can be described that higher constraint support condition than another is to be selected to identify the support condition at those transition points.

To clearly understand, consider the circular plate as demonstrated in Figure 1, the tips of partial simple support are the points of discontinuity where the simple support condition is immediately changed to be unsupported edge. Thus, the simply supported condition is used to suppose the boundary constraint at those points for the representative support model of circular plate. Similarly, the clamped condition is then used for the case of the plate that illustrated in Figure 2.

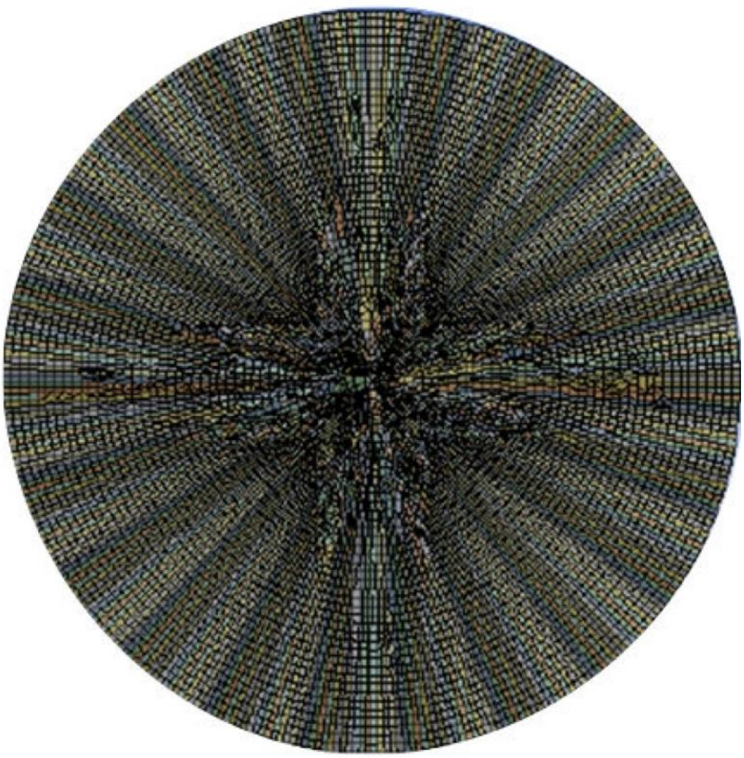

Figure 3. Mesh of finite element net used in circular plate.

4. NUMERICAL RESULTS

After performing the computer program, the required frequencies of the plates are then immediately carried out. However, it can be noted that numerical results determined from the finite element code are generally expressed in terms of natural frequencies, *f*, within the Hertz unit (Hz), which can be related to the circular frequency (ω) that defined by

$$\omega = 2\pi f \quad (1)$$

It is convenient to express the natural frequency in the dimensionless form of frequency parameter as [6,10],

$$\lambda^2 = \omega a^2 \sqrt{\rho/D} \quad (2)$$

$$D = Eh^3/12(1-\nu^2) \quad (3)$$

where a is the radius of circular plate, ρ is the mass density per unit area of plate surface, D is the plate's flexural rigidity, h is the plate thickness, E and ν are the Young's modulus and Poisson's ratio, respectively.

Substitution of (1) into (2) yields

$$\lambda^2 = 2\pi f a^2 \sqrt{\rho/D} \quad (4)$$

As it is generally known that for the simply supported, clamped, and completely free circular plates, the frequency equations for each case of the plates can be expressed in the following transcendental characteristic equations, respectively, [6]

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)} + \frac{I_{n+1}(\lambda)}{I_n(\lambda)} = \frac{2\lambda}{1-\nu} \quad (5)$$

$$J_n(\lambda) I_{n+1}(\lambda) + I_n(\lambda) J_{n+1}(\lambda) = 0 \quad (6)$$

$$\frac{\lambda^2 J_n(\lambda) + (1-\nu)[\lambda J_n'(\lambda) - n^2 J_n(\lambda)]}{\lambda^2 I_n(\lambda) - (1-\nu)[\lambda I_n'(\lambda) - n^2 I_n(\lambda)]} = \frac{\lambda^2 J_n'(\lambda) + (1-\nu)n^2[\lambda J_n'(\lambda) - J_n(\lambda)]}{\lambda^2 I_n'(\lambda) - (1-\nu)n^2[\lambda I_n'(\lambda) - I_n(\lambda)]} \quad (7)$$

where J_n and I_n are the Bessel function and modified Bessel function of the first kind and order n, respectively, the superscript primes denote the differentiation with respect to the argument, in this case $\lambda = kr$, r defines to be the radial coordinate in polar coordinates system, and k is defined by

$$k^4 = \rho\omega^2/D \quad (8)$$

It can be seen that equations (5) and (7) have the term of Poisson's ratio in their expressions for the cases of simply supported plate and completely free plate, respectively. Therefore, all numerical results presented in this paper are provided for the Poisson's ratio value of 0.3.

Numerical values for the first thirty frequency parameters (λ^2) of circular plates having partially simply supported edge are prepared and listed in Table 1 with the increasing of angle $\alpha = \pi/6$. For the case of partially clamped circular plates, the numerical results are given in Table 2.

5. DISCUSSION

It can be observed in Tables 1 and 2 for three limiting cases that when $\alpha = 0$, both circular plates as presented in Figures 1 and 2 become a problem of completely free plate. Since α is taken to be π, the problems are to be simply supported and clamped circular plates, respectively.

The accurate values for frequency parameter corresponding to those of three cases of the plate can numerically be evaluated using (5) to (7), which is, however, a tedious work due to computation of the analytic functions of familiar Bessel functions of order n. However, some numerical values of frequency parameters were given and collected in Leissa's monograph [6].

It is interesting to note that the first six values of frequency parameter as presented in Tables 1 and 2 are all zeros for the case of α = 0 (a case of completely free plate). These are in accordance with the rigid body movements due to the SHELL 181 element type used of which has 6 degrees of freedom at the node of the quadrilateral element as described previously in section 3.

An another observation can also be seen from the analysis as shown in Tables 1 and 2 that there are many repeated values of frequency parameters for the cases of the plate having regular boundary conditions namely, a completely free circular plate (α = 0), a fully simply supported circular plate ($\alpha = \pi$ as shown in Table 1), and a fully clamped plate ($\alpha = \pi$, Table 2), but not for the plates having partial supports. These behaviors are due to the degeneracy of frequencies for plate having symmetric vibrations. These pheno-mena are discussed later in a more details that given in a following companion paper of this title.

Table 1. Frequency parameters (λ^2) for partially simply supported circular plates.

Mode	α/π						
	0	**1/6**	**1/3**	**1/2**	**2/3**	**5/6**	**1**
1	0.000	0.362	0.867	1.787	3.663	4.883	4.936
2	0.000	2.188	3.689	6.306	6.748	12.692	13.906
3	0.000	4.619	6.082	6.577	12.473	13.897	13.907
4	0.000	7.983	8.973	13.988	16.478	19.804	25.634
5	0.000	8.694	12.808	15.495	16.614	25.478	25.635
6	0.000	13.444	16.992	17.341	26.655	27.808	29.763
7	5.359	17.697	18.748	24.074	28.576	32.054	39.997
8	5.359	17.746	20.467	27.700	29.463	38.885	39.997
9	9.005	21.059	28.418	30.876	34.064	42.257	48.599
10	12.441	25.547	31.370	33.200	43.358	48.258	48.605
11	12.441	30.017	31.567	36.942	43.538	51.178	56.908
12	20.489	31.724	34.034	42.690	47.179	52.059	56.908
13	20.490	35.742	39.980	47.158	53.663	59.400	70.329
14	21.840	37.147	47.074	50.257	54.478	63.900	70.334
15	21.841	40.505	47.212	54.119	61.060	71.277	74.476
16	33.511	46.111	49.253	54.140	61.204	71.893	76.311
17	33.511	50.285	54.068	60.435	71.216	74.702	76.317
18	35.293	52.371	58.118	66.013	75.589	80.115	94.864
19	35.294	55.129	61.706	68.171	77.311	82.145	94.864
20	38.505	59.184	65.684	76.050	80.208	94.905	98.170
21	47.414	59.369	67.187	76.870	81.310	95.602	98.170
22	47.415	66.971	75.004	80.638	83.494	97.555	103.369
23	53.063	70.903	79.141	81.264	99.226	102.738	103.394
24	53.063	71.118	79.336	87.323	99.749	105.147	122.126
25	59.974	77.680	82.678	88.276	102.154	107.254	122.152
26	59.979	81.294	86.462	101.062	104.466	114.160	122.444
27	63.526	82.140	88.497	101.287	105.322	121.957	122.449
28	63.526	87.123	93.987	106.009	110.992	124.002	135.214
29	73.636	91.988	103.297	109.479	114.144	129.175	135.239
30	73.636	92.444	104.466	110.940	126.519	131.892	139.505

Table 2. Frequency parameters (λ^2) for partially clamped circular plates.

Mode	α/π						
	0	**1/6**	**1/3**	**1/2**	**2/3**	**5/6**	**1**
1	0.000	0.939	1.426	2.461	5.178	9.631	10.217
2	0.000	2.388	4.021	7.211	9.962	17.163	21.279
3	0.000	5.253	7.845	9.325	15.232	21.189	21.280
4	0.000	9.063	9.859	14.890	20.772	23.635	34.917
5	0.000	9.078	13.803	19.877	21.545	34.281	34.918
6	0.000	13.869	20.335	20.958	31.140	35.641	39.853
7	5.359	18.554	20.671	25.449	34.796	39.893	51.099
8	5.359	20.454	21.596	32.838	35.094	48.036	51.100
9	9.005	21.148	29.463	35.240	39.832	51.654	61.025
10	12.441	25.939	34.634	37.562	50.681	58.087	61.035
11	12.441	31.561	35.221	40.257	51.479	60.635	69.775
12	20.489	35.099	37.896	47.923	53.222	61.645	69.780
13	20.490	36.298	40.664	52.269	60.435	70.201	84.914
14	21.840	38.089	49.312	54.089	61.568	72.918	84.924
15	21.841	41.363	51.771	60.502	69.862	84.375	89.579
16	33.511	48.228	52.838	60.958	70.688	85.149	90.906
17	33.511	52.868	60.092	65.156	79.526	88.569	90.911
18	35.293	53.684	60.353	71.908	84.580	90.947	111.494
19	35.294	57.600	63.075	73.092	85.021	93.116	111.494
20	38.505	59.999	70.262	84.165	89.686	107.500	114.467
21	47.414	61.353	72.144	84.734	91.614	111.371	114.467
22	47.415	68.827	77.434	85.272	92.736	111.904	120.921
23	53.063	72.933	84.129	88.769	109.074	116.497	120.952
24	53.063	73.277	84.944	94.059	111.494	118.866	140.422
25	59.974	80.213	87.328	94.731	112.058	120.875	140.427
26	59.979	84.273	90.158	106.388	116.051	124.771	140.730
27	63.526	84.760	93.987	111.397	118.030	139.653	140.766
28	63.526	88.605	96.797	112.119	120.311	140.899	155.073
29	73.636	92.962	105.660	118.609	122.059	142.847	155.109
30	73.636	95.269	111.176	118.948	138.126	144.846	159.774

It is remarkable that the carried out results as shown in Tables 1 and 2 have no considering the correct singularity order at the transitions of discontinuous support in the present finite element models. In order to account for the nature of singularity in finite element analysis of this class of problems, the following works can then be extended and adopted to analyze the present problems.

Pramod et al.[31] analyzed the dynamic behavior of annular plates with circumferential cracks using finite element method in which singular elements are employed to model around the crack tips and regular elements in the rest of the domain. Wang [32] constructed an analytical singular element with arbitrary high-order precision to analyze the problems for bi-material V-shaped notches in Kirchhoff plate bending. This singular element can be used to describe the rapid change from finite values to infinities of the stress field around the tip of a V-shaped notch formed of two dissimilar materials. Ayatollahi et al.[33] applied the finite element over-deterministic method to determine the coefficients of the crack tip asymptotic fields in an anisotropic plane giving the exact solution of the crack tip parameters. Moreover, numerical examples of cracks in finite and infinite plates were also investigated which shows good agreement with exact or alternative numerical solutions. An efficient numerical method was made by Sun and Wei [34] who develops the singular boundary method for the analysis of thin elastic plates subjected to dynamic loads. Based on the analytical approaches, Huang and Chan [35] adopted the basis functions comprising polynomials and crack functions to generate the admissible functions by the moving least-squares approach in the Ritz method for analyzing the vibration of cracked plates. The crack functions were accounted for the singular behaviors of stress resultants at crack tips, which are discontinuous in displacement and slope across the crack.

6. CONCLUDING REMARKS

To the best knowledge of the author, the higher frequency parameters especially for the case of circular plates with mixed edge conditions that corresponded to the plates considered here have never been presented and reported in the scattering scientific or technical literatures.

Although the given results seem to be an approximate solution, the accurate numerical values are then carried out using a finite element dense net of reliable well-known computer finite element code. The details of a representative finite element model that emphasized on the boundary conditions at the transition point between two different supporting conditions, have proposed. Importantly, the obtained values are presented in tabular form for easy reference by other alternative analytical and numerical methods, which could be served as a benchmark for comparison.

7. REFERENCES

[1] Ventsel E, Krauthammer T. Thin Plates and Shells: Theory, Analysis, and Applications. New York: Marcel Dekker, Inc.; 2001.

[2] Szilard R. Theories and Applications of Plates Analysis: Classical, Numerical and Engineering Method. New Jersey: John Wiley & Sons, Inc; 2004.

[3] Timoshenko SP, Woinowsky-Krieger S. Theory of Plates and Shells. 2nd ed. Singapore: McGraw-Hill; 1959.

[4] Nowacki W. Dynamics of Elastic Systems. London: Chapman & Hall Ltd.; 1963.

[5] Rao JS. Dynamics of Plates. New Delhi: Narosa Publishing House; 1999.

[6] Leissa AW. Vibration of Plates. Reprinted ed. Washington, DC: Acoustical Society of America; 1993.

[7] Meirovitch L. Analytical Methods in Vibrations. New York: Macmillan Publishing Co., Inc.; 1967

[8] Chen SS-H, Pickett G. Bending of plates of any shape and with any variation in boundary conditions. J Appl Mech. 1967; 34:217-8

[9] Leissa AW, Clausen WE, Hulbert LE, Hopper AT. A comparison of approximate methods for the solution of plate bending problems. AIAA Journal. 1969; 7:920-8.

[10] Sompornjaroensuk Y, Chantarawichit P. Vibration of circular plates with mixed edge conditions. Part I: Review of research. UTK Research Journal. 2020. (Accepted for publication)

[11] Leissa AW, Narita Y. Natural frequencies of simply supported circular plates. J Sound Vib. 1980; 70:221-9

[12] Nowacki W, Olesiak Z. Vibration, buckling and bending of a circular plate clamped along part of its periphery and simply supported on the remaining part. Bulletin de L'Academie Polonaise des Sciences. 1956; 4: 247-58.

[13] Bartlett CC. The vibration and buckling of a circular plate clamped on part of its boundary and simply supported on the remainder. Quarterly Journal of Mechanics and Applied Mathematics. 1963; 16:431-40.
[14] Noble B. The vibration and buckling of a circular plate clamped on part of its boundary and simply supported on the remainder. Proceedings of the 9th Midwestern Mechanics Conference, 1965 April 16-18, Madison, Wisconsin. 1965.
[15] Hirano Y, Okazaki K. Vibrations of a circular plate having partly clamped or partly simply supported boundary. Bulletin of the JSME. 1976; 19:610-8.
[16] Eastep FE, Hemmig FG. Natural frequencies of circular plates with partially free, partially clamped edges. J Sound Vib. 1982; 84:359-70.
[17] Yuan J, Dickinson SM. On the vibration of annular, circular and sectorial plates with cut-outs or on partial supports. Computers and Structures. 1996; 58:1261-4.
[18] Rossi RE, Laura PAA. Transverse vibrations of a thin, elastic circular plate with mixed boundary conditions. J Sound Vib. 2002; 255:983-6.
[19] Bauer H, Eidel W. Determination of the lower natural frequencies of circular plates with mixed boundary conditions. J Sound Vib. 2006; 292:742-64.
[20] Hassan SM. Numerical solution for frequencies and mode shapes of elliptical plate half of whose boundary is simply supported and the rest free. Int J Mech Sci. 2004; 46:1747-61.
[21] Hassan SM. Free transverse vibration of elliptical plates of variable thickness with half of the boundary clamped and the rest free. Int J Mech Sci. 2004; 46:1861-82.
[22] Hassan SM. Numerical computation of BCOPs in two variables for solv-ing the vibration problem of a CF-elliptical plate. Journal of King Saud University (Science). 2010; 22:195-204.
[23] Yoowattana T, Chantarawichit P, Vibooljak J, Sompornjaroensuk Y. Accurate approximate and analytical methods for vibration and bending problems of plates: A literature survey. Applied Mathematical Sciences. 2015; 9:1697-719.
[24] Olson MD, Lindberg GM. Annular and circular sector finite elements for plate bending. Int J Mech Sci. 1970; 12:17-33.
[25] Zienkiewicz OC, Taylor RL. The Finite Element Method: Volume 2-Solid and Fluid Mechanics, Dynamics and Non-Linearity. 4th ed. Singapore: McGraw-Hill Book Company; 1991.
[26] Reddy JN. An Introduction to the Finite Element Method. 2nd ed. Singapore: McGraw-Hill Inc.; 1993.
[27] Bathe KJ. Finite Element Procedures. New Jersey: Prentice-Hall, Inc.; 1996.
[28] ANSYS Inc. ANSYS Mechanical APDL Theory Reference, Release 14.5; 2012.
[29] ANSYS Inc. ANSYS Mechanical APDL Element Reference, Release 14.5; 2012.
[30] Malison R, Sompornjaroensuk Y. Numerical finite element determination on free, transverse vibrations of circular plates. Engineering Transactions. 2015; 18:61-9.
[31] Pramod R, Shashi Kumar ME, Mohan Kumar S. Modal analysis of annular plate with crack and its effect on natural frequency. Appl Mech Mater. 2015; 813-814:910-4.
[32] Wang S. An analytical singular element for interface V-shaped notches in bi-material Kirchhoff plate bending. Eng Fract Mech. 2017; 180:282-95.
[33] Ayatollahi MR, Nejati M, Ghouli S. The finite element over-deterministic method to calculate the coefficients of crack tip asymptotic fields in anisotropic planes. Eng Fract Mech. 2020; 231:106982.

[34] Sun L, Wei X. A frequency domain formulation of the singular boundary method for dynamic analysis of thin elastic plate. Eng Anal Bound Elem. 2019; 98:77-87.

[35] Huang CS, Chan CW. Vibration analysis of cracked plates by the Ritz method with moving least-squares interpolation functions. International Int J Struct Stabil Dynam. 2014; 14: 1350060.