

Vibration of circular plates with mixed edge conditions. Part III: Localized frequency curve veering phenomena

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ABSTRACT

This paper aims to investigate and examine the phenomenon of mode localization and frequency curve veering for vibratory circular plates with mixed edge conditions. An important characteristic of curve veering is that when two loci of the curves approach each other as system parameters vary, they often exactly cross (intersections) or abruptly diverge away (avoided crossings). The latter case is termed as curve veering. Thus, in the present paper, the carried out results reveal a significant effect on the occurrence of curve veering possibilities due to support length variations that can be detected by plotting the variation of frequency parameter against the angle provided the circumferential plate support by means of finite element method with discretized refinement model.

Keywords: Circular plates, Curve veering, Finite element method, Mixed edge conditions, Vibratory plate

1. INTRODUCTION

Self-adjoint eigenvalue problems for ordinary or partial differential equations are very important in a variety of sciences and engineering disciplines. Eigenvalue loci veering and mode localization phenomena, which are encountered in some linear vibratory systems when a system parameter is varied, have received much attention in the published scientific or technical literature in which a considerable amount of work in this field deals with the computation of the eigenfrequencies of the systems.

Frequency curve veering is the surprising phenomenon that can be observed by a plot of the frequency variation with parameters of the system. It can also be noted that the characteristics of vibration modes are very sensitive to small parameter variations leading to the existence of vibratory mode localization possibilities since the eigenvalue of the system is placed as a function

of the characteristic parameters. This phenomenon is commonly found in the field of vibration problems.

In order to plot the frequency loci when the system parameter is altered, two adjacent loci of frequency may cross, or they may deviate immediately from one another. These interesting phenomena have been generally known and indicated to be frequency crossings and transitions, respectively. In the latter case, it may notably be stated as quasi-degeneracies [1] in which veering generally occurs when a system does not appear to have the symmetries to sustain equal frequencies with independent modal shapes where a crossing of frequency curves is referred to frequency degeneracy or equal frequencies. This exists when a system is something of symmetry [2].

However, before proceeding further to describe and explain the contexts of the present paper, the purpose of this title (Part I [3]) was to review most of the published

literature on vibration and bending problems of circular plates involving mixed edge conditions as background. Emphasis was given to the basic theory involved and analytical solution being applied for solving free vibration and static bending of circular plates with regular boundary conditions. Furthermore, both various analytical and numerical methods were explored including discussions of the existence of stress singularities in the problem considerations. In a companion Part II paper of this title [4], the aim was to numerically determine the first thirty accurate values for natural frequencies using an available well-known computer finite element code with a dense net of meshed generation representing the finite element model of circular plates. The extended investigation of obtainable results has led to the analysis of the present paper.

In this paper, the main objectives are to demonstrate the occurrence of frequency curve veering possibilities and indicate the mode localizations for free vibration problem of two different circular plates having discontinuity of edge conditions; the first is partially simply supported plate and the second is partially clamped plate, which have been investigated numerically in order to obtain some higher natural frequencies using finite element program with a dense net as previously presented in a companion Part II paper [4].

2. LITERATURE REVIEW

The word ‘frequency curve veering’ has long been referred to or characterized by the phenomenon observed, in this field, in plots of two adjacent vibrational frequencies versus a parameter such as the aspect ratio or the ratio of rigidities. Significantly, this phenomenon has usually been remarked in various approximate solution methods for system model discretization.

It is, in fact, not possible here to cite the existing literature or even a brief account of these because of the enormous size of the

information. Thus, in this section, some important published literature that related to the present work are reviewed.

One of the first studies involving the frequency curve veering found in the technical literature was made by Nair and Durvasula [1]. They attempted to analytically treat the free vibration problems of skew, rhombic, and rectangular plates by plotting the variations of frequencies with geometric parameters, namely, the aspect ratio and the angle of skew by making use of the method of perturbation techniques. Furthermore, they also described and defined the veering phenomena as quasi-degeneracies.

However, it should be emphasized that the occurrence of frequency curve veering was first carefully considered by Leissa [5] to be a phenomenon generated by an approximation of continuous systems. By observation from the obtainable results for natural free vibratory problems of clamped rectangular plate and rectangular plate having two oppositely clamped edges and free on the remained two edges [6], and rectangular plate having clamped on one edge [7] as a function of side ratio for the plate, and skew cantilever plates [8] as functions of two independent variables of side ratio and skew angle, the plots of frequency curves have shown that, the interesting phenomenon of curve veering, the curves appear to approach together and deviate away from each other. Furthermore, the mode shapes of vibration that are associated with the frequency loci before veering have an interchange of the modal patterns after veering away. It is also noted that Claassen and Thorne [6] call these regions as transition zones where the veering away occurs. Significantly, Leissa [5] demonstrated frequency veering possibilities due to the use of approximate solution of a rectangular membrane using the Galerkin method and has stated that the curve veering can arise from the application of an

approximate method in order to estimate the frequencies of vibration problems and also from the obtainable exact solution of a problem system defined by an improper mathematical model. In the latter, Kuttler and Sigillito [9] have checked the solution of the problem that was considered by Leissa [5], the fixed membrane vibration on rectangles. The results indicated that frequency curve veering can exist in the actual phenomenon for the accurate model and is not always necessarily caused by making use of approximation techniques.

Since the influence of discretized model causing the curve veering has raised questioning on the exactitude of various approximate solutions, Perkins and Mote [10] validated the occurrence of frequency veering phenomena in continuous systems with utilization of an exact eigensolution in the simple eigenvalue problem and have derived conditions for detecting veering and crossing possibilities in either continuous or discrete models.

Pierre [11] analytically investigated the results of disorder on vibratory modes for nearly periodic systems of structures using the modified perturbation methods in solving the eigenvalue problems. The obtained results were illustrated that, in the systems having close eigenvalues, some kind of small structural inconsistencies or disproportion can be led to both robust localization of the vibratory shapes and sudden veering away of the eigenvalue loci when there is a plot of the loci against a representative parameter of disorder in the system. Subsequently, the phenomena of load loci veering and buckling pattern localization in accordance with the case of structural stability problem were analytically examined by Pierre and Plaut [12].

Chen and Ginsberg [13] considered and studied the behavior of vibratory problems in which a changeable system parameter can produce two natural frequencies to become close using a perturbation series

solution. They have found that the vibratory behavior of rectangular clamped membrane in the veering region can be described in terms of the eigensolutions. In a particular to the small veering region, it was also noted that the modes of vibration cannot be determined correctly, although reducing the error due to the approximation is made to decrease the extent of the veering zone until it cannot be discovered. At the same time, Chen and Ginsberg [14] combined the assumed modes and the energy functionals method that based on the theory of classical linear bending, to investigate the properties of mode shape and frequency veering phenomena in the free vibration problem of axisymmetric thin spheroidal shells with arbitrary slender shape. The analysis demonstrated that at certain parameters of system, the different loci of eigensolution come close and after that then veer away with no intersection. The number of nodes in the mode associated with an increasable root number for the frequency is found to change irregularly when veering occurs. Importantly, further investigation showed that two loci of eigenvalue may veer as the result of approximation error.

Rim and Lee [15] employed the perturbation and Galerkin's methods to confirm the occurrence of frequency curve veering phenomenon in the problem of annular plates having clamped on the outer edge and subjected to in-plane force. From the obtained results, it confirmed that the curve veering phenomenon can arise as changing the magnitude of nonuniform in-plane force.

For the free in-plane vibration problem, Shin et al. [16] analytically studied the vibration of an axially moving membrane in which the effects of both the translating speed and aspect ratio of the membrane are considered. The coupled longitudinal and lateral equations of motion have derived with the use of Hamilton principle. These

equations were then further discretized based upon the method of Galerkin. The obtained results showed that the translating speed, aspect ratio, and boundary conditions have significant effects on the in-plane vibrations of the moving membrane and additionally, the curve veering phenomena between loci of frequency are existed among the in-plane modes of vibration.

Saito et al.[17,18] investigated veering phenomena in the linear and nonlinear vibration frequencies of a cantilevered cracked plate as the crack location or crack length is varied. Utilization of a finite element model, it was shown that the veerings of frequency curve due to variation of the crack length involve the interchanging of vibration mode shapes. By making use of a method of hybrid frequency/time domain, the analysis for nonlinear forced vibration responses was determined. Furthermore, the nonlinear vibration response near both loci veerings and crossings due to the changes of crack length parameter was also numerically investigated.

In order to physically describe the modal interactions in eigenvalue curve veering, du Bois et al.[19] have successfully derived three normalized indices. Two of these, namely the quotient of cross-sensitivity and the factor of modal dependence, are usable as quantification of the physical conditions while the third, the veering index, is provided as measurement of the global intensity of the effect.

Recently, the free vibration problem of cracked plates was analytically treated by Huang et al.[20]. The domain decomposition technique was employed to separate the domain of the plate into subdomains of rectangular and triangular configurations. The method of Rayleigh-Ritz was applied to determine the frequencies and mode shapes of vibration. With introducing and utilizing a special set of admissible func-

tions, the involved integrals found in the energy equations can be analytically evaluated. The effects of varying parameters of the crack influenced on frequency curve veering, mode separating, and mode localization were investigated and discussed for the case of square plates having all edges simply supported and with various crack configurations.

As is reviewed and discussed in the aforementioned literature, it is interesting and significant to observe that there is no study focusing on the frequency curve veering for the free vibration problems of circular plates with mixed edge conditions found in the past.

However, it should be mentioned that there are only two research works by Bauer and Eidel [21,22] seemed likely to be involved in this class of problem field, which numerically evaluated the lower natural frequencies of circular plates. In their studies, the edge of the plate can be supposed to be various cases of partially mixed boundary conditions for the combinations of clamped, simply supported, free or guided conditions. The analysis was based on a semi-analytical method. Numerical results as given in terms of natural frequency parameters were shown depending on the variation of angle provided the circumferential plate support. Some phenomena were detected in their works that the variation of the natural frequency for the asymmetric modes illustrates two branches of frequency curves depending on the magnitude of the angles of mixed conditions varies. For the cases of axisymmetric vibration modes, these phenomena cannot be found, such as the plate boundary being either completely clamped or completely simply supported edges.

However, the phenomenon of frequency curve veering and mode localization possibilities can also be found and continually investigated in other systems. Brouet et al.[23] considered the longitudi-

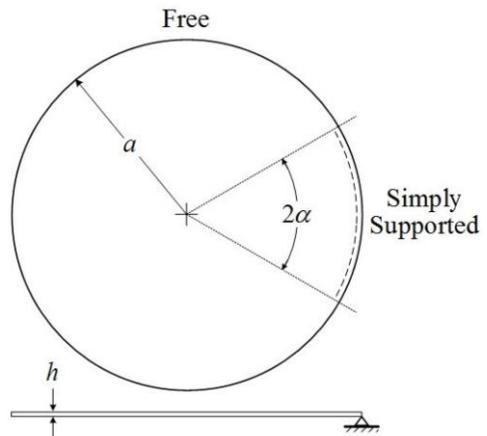
nal vibration of a rectangular disc. The method was presented for a simple rectangular disc to eliminate the modal interaction and decouple the longitudinal mode from other modes. Sari et al.[24] studied the vibration problem of nonlocal axially functionally graded Timoshenko beams based on the Eringen’s theory of nonlocal elasticity. The method of Cheby-shev spectral collocation was used to determine the natural frequency parameters and their associated vibration mode shapes. The results were showed that material grad-ing, geometry grading, and nonlocal resi-duals have a great influence on frequency and mode shape veering. Zhang et al.[25] analyzed the frequency veering characteristics and strain energy distribution of the tuned bladed disk using the finite ele-ment reduced-order models. Zinkovskii and Tokar [26] conducted computational and experimental investigations on the influence of the parameters of local damage in the form of a rectangular notch on the surface of a cantilever rod with a constant cross section as a simplest model of the open edge crack. Frequency spectrum and flexural modes of vibration were also carried out. Shaat [27] developed a new model for nanobeams making of functionally graded materials with engineering surface that considered a different material phase having a surface texture. The natural frequencies and their mode shapes of the beams were investigated depending on the surface’s texture and mechanical properties. It was, however, observed that natural frequencies may decrease or increase due to the effects of surface roughness.

3. NUMERICAL PRESENTATIONS

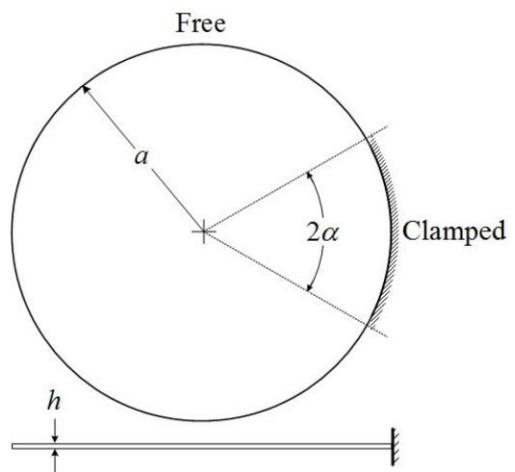
Two cases of vibratory circular plates with mixed edge conditions are considered. The first is a partially simply supported plate and the second is partially clamped plate as shown in Figure 1. The plates have a uniform thickness of h with a radius of a ,

and the angle 2α is measured with respect to the plate center providing the circumferential support length.

The vibration analysis is made in the same manner with the previous work [4] by the use of the well-known ANSYS computer finite element program [28]. Numerical results are carried out from a dense net of 12720 elements with 73482 degrees of freedom by making use of quadrilateral shape of SHELL181 element type [29] for the representative finite element model.



(a) Partially simply supported circular plate



(b) Partially clamped circular plate

Figure 1. Configuration of circular plates with mixed edge conditions.

The natural frequencies (f) obtained from the finite element program are conveniently expressed in terms of dimensionless form of the frequency parameters (λ^2) as defined by [30]

$$\lambda^2 = 2\pi f a^2 \sqrt{\rho/D} \quad (1)$$

and the flexural rigidity of the plate (D) can be expressed as

$$D = Eh^3/12(1-\nu^2) \quad (2)$$

in which ρ stands for the mass density per unit area of the plate surface, E and ν are the material properties, namely Young's modulus and Poisson's ratio, respectively.

It can, however, be noted that all numerical results presented here are given for the Poisson's ratio taken as 0.3. This is due to the equation condition of the free boundary of the plate that involving with the term of Poisson's ratio.

In order to graphically detect the occurrence of frequency curve veering phenomenon, the system parameter (geometric parameter) to be used in the present study, which is the angle α , is varied from 0° to 180° (deg) with the increment angle of 1° in each finite element model of circular plates. Thus, 361 finite element models are numerically conducted for two different cases of circular plate with mixed edge conditions. The variations of frequency parameter for the first twenty modes of vibration are then graphically presented in Figures 2 and 3 that corresponded to the partially simply supported and partially clamped circular plates as depicted in Figures 1(a) and 1(b), respectively.

4. DISCUSSION

As is immediately seen in Figures 2 and 3 that the occurrence of frequency curve veering phenomena can clearly be detected in many regions between two ad-

acent modes of vibration. These phenomena can be described in the following.

Consider first the lower modes of vibratory circular plate with partially simply supported edge such as mode 2 and mode 3 as illustrated in Figure 2, the transition zone or veering region is found for the angle (α) to be taken as 88° approximately. Next, for the case of mode 4 and mode 5, the number of veering regions is increased and found to have 3 regions for the angles being nearly 22° , 97° , and 119° in accordance with the increasing of frequency parameter. These frequency curve veering phenomena also appeared in the vibration problem of circular plates having partially clamped edge as presented in Figure 3. The first veering region exists when α is equal to be 102° approximately for two adjacent modes between mode 2 and mode 3.

It is interesting to notice in both Figures 2 and 3 for three limiting cases that when $\alpha = 0^\circ$, both circular plates as shown in Figure 1 become a problem of completely free plate. Since α is equal to 180° , the problems of circular plates are to be of fully simply supported and fully clamped plates as referred in Figure 1. As is true for another most interesting aspect of the phenomena that can be observed when looking back to the characteristics of modal pattern for plate vibrations is the occurrence of mode localization. However, it has been stated in the previous literature that the characteristics of vibration modes are very sensitive to small parameter variations when the frequency loci veer away from the veering region.

Thus, in order to examine the localized mode phenomena, situations of mode localization can graphically be explained as demonstrated in Figures 4 and 5 in correspondence with the cases of partially simply supported and partially clamped circular plates, respectively, for two adjacent modes 2 and 3.

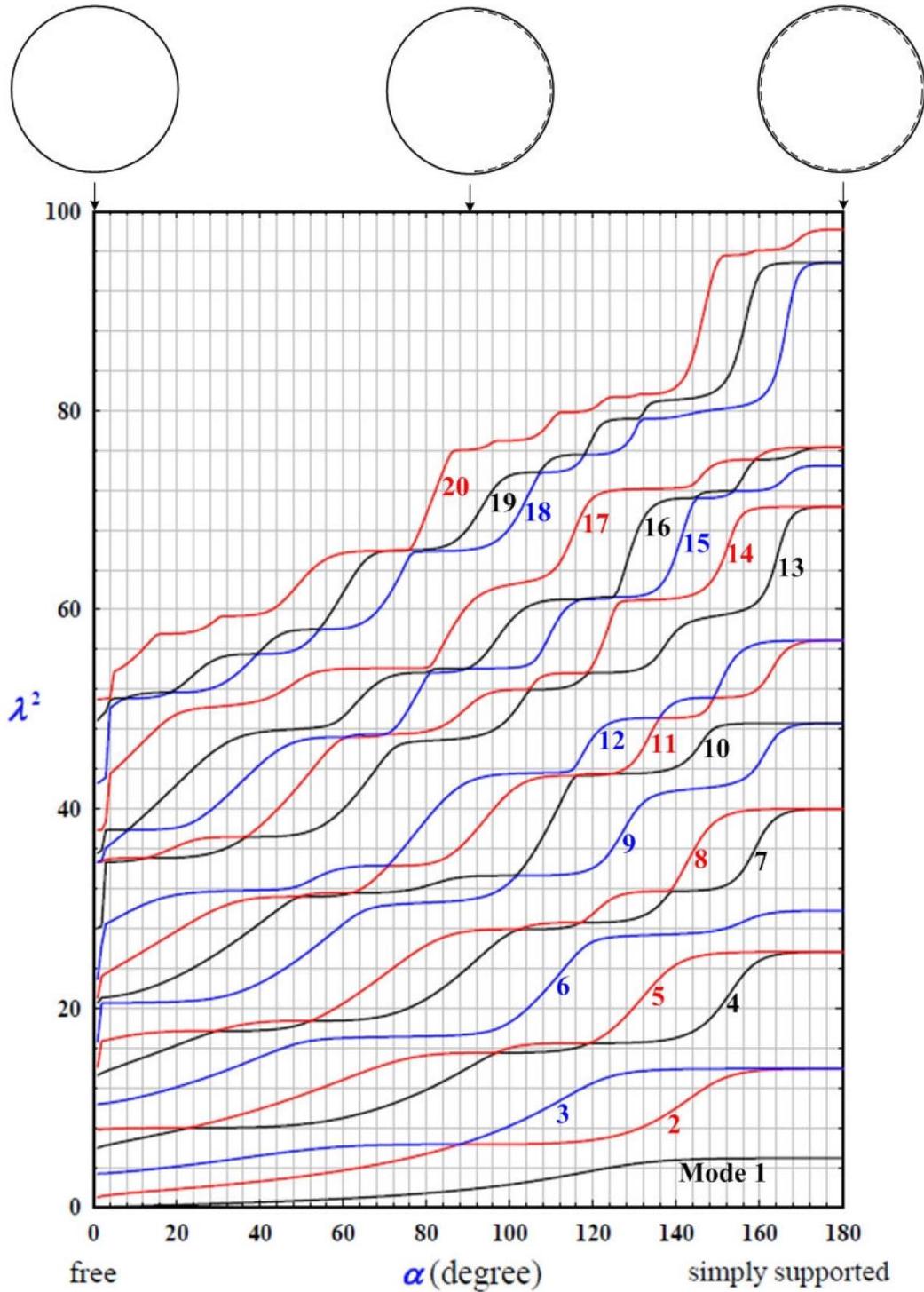


Figure 2. Variation of frequency parameter with the half angle (α) of circumferential support length for partially simply supported circular plates.

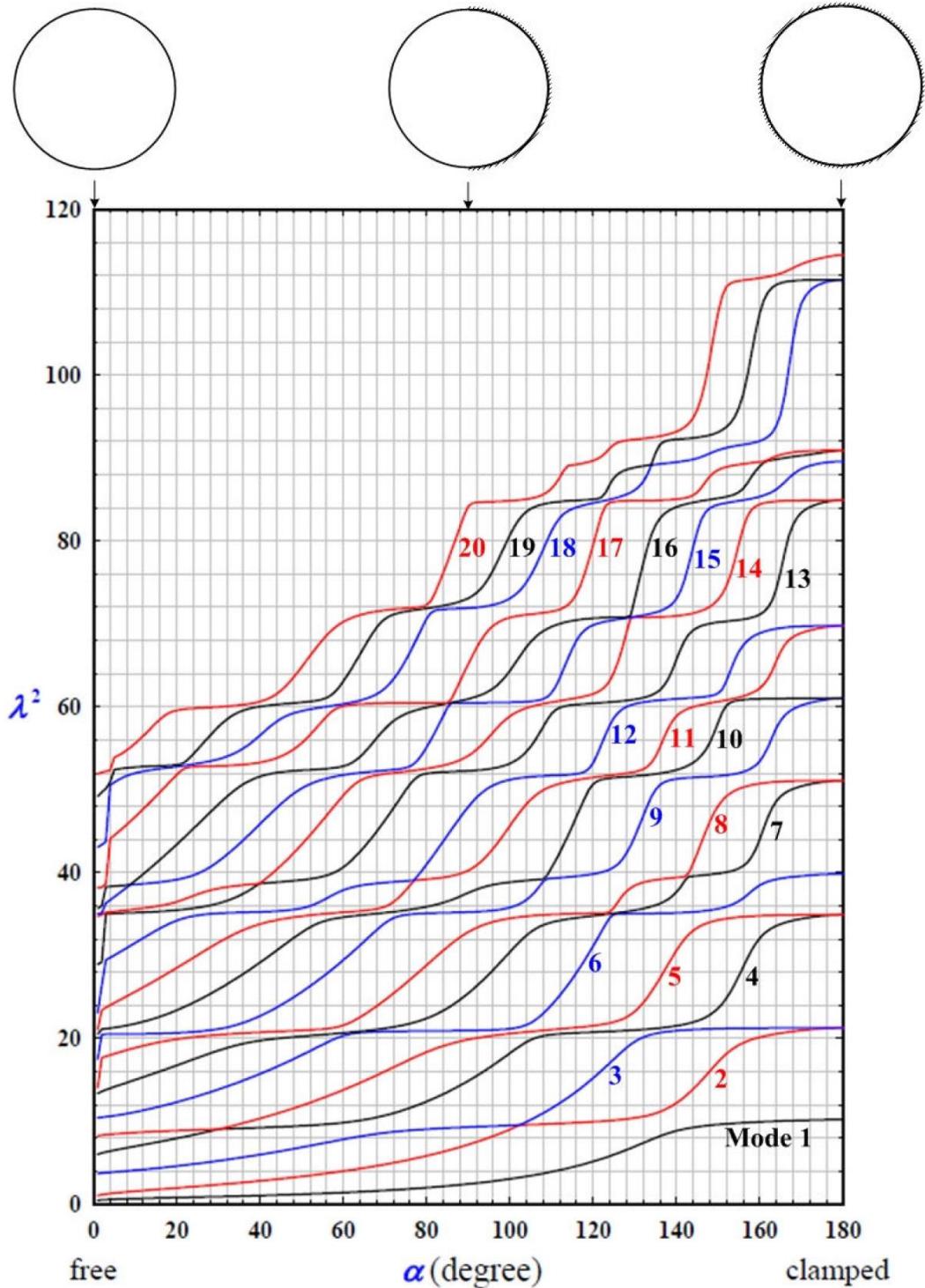


Figure 3. Variation of frequency parameter with the half angle (α) of circumferential support length for partially clamped circular plates.

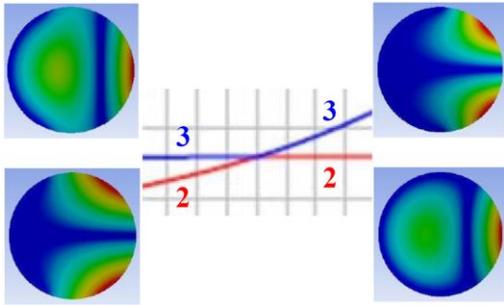


Figure 4. Enlarged view of frequency curve veering between mode 2 and mode 3 for partially simply supported circular plates.

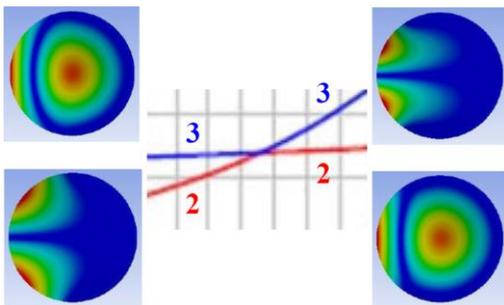


Figure 5. Enlarged view of frequency curve veering between mode 2 and mode 3 for partially clamped circular plate.

It is explicitly observable that the vibratory mode shapes associated with frequencies of each locus before veering are switched rapidly after veering. This can be confirmed that phenomena of mode localization are sensitive and reliable veering of frequency loci depending on the changes of system parameter. Some modal patterns for the plate vibrations are also given and presented in Figures 6 to 18.

5. CONCLUSIONS

This study considers the problem of linear vibratory behaviors of circular plates having mixed edge conditions. Based upon the obtainable numerical results, the conclusions can be drawn in the followings.

(i) Changing a length of partial support causes two adjacent natural frequencies to become close together. When two

frequency loci of the curves approach each other, they abruptly diverge away from each other in order to the avoidance of crossing. This phenomenon is called the frequency curve veering, and can easily be observed by plotting the variation of frequency parameter against the angle over the circumferential support length.

(ii) Considering the characteristics of vibration modes, there is an interchange between vibration modes when the frequency loci veer away from the veering region. This is called the mode localization phenomenon.

To the best of the author's knowledge, the presented results have not previously been reported in the open scientific or technical literature for the higher frequencies of circular plates with mixed edge conditions. However, it is interesting to note from a search of the published literature in the past that there is no study on the frequency curve veering of vibratory plates with mixed boundary conditions up to the present time.

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7. APPENDIX

This section provides the lists of some vibration modal patterns of circular plates for the first twenty modes. When α is set to be 0° , both circular plates as presented in Figure 1 become a problem of completely free plate and its modal patterns is given in Figure 6. It can be noted for the first six modes of vibration as shown in Figure 6 that they are the rigid body motions

of the plate due to the use of SHELL181 element type. This element has six degrees of freedom at each node of quadrilateral element. Thus, modes 1 to 3 are the rigid body translations in the x , y , and z directions. For the modes 4 to 6, they are corresponded to the rigid body rotations about the x , y , and z -axis.

Since α is taken to be 180° , the problems are to be fully simply supported and fully clamped circular plates and their modal patterns are given in Figures 12 and 18, respectively. It can also be observed that the vibratory modes for these two plate configurations have the similar modal patterns. An another observation is found that there are some repeated patterns of vibration mode for the cases of completely free, fully simply supported, fully clamped plates as shown in Figures 6, 12, and 18, respectively, but not for the plates having partial supports. These behaviors are due to the mode degeneracies [30].

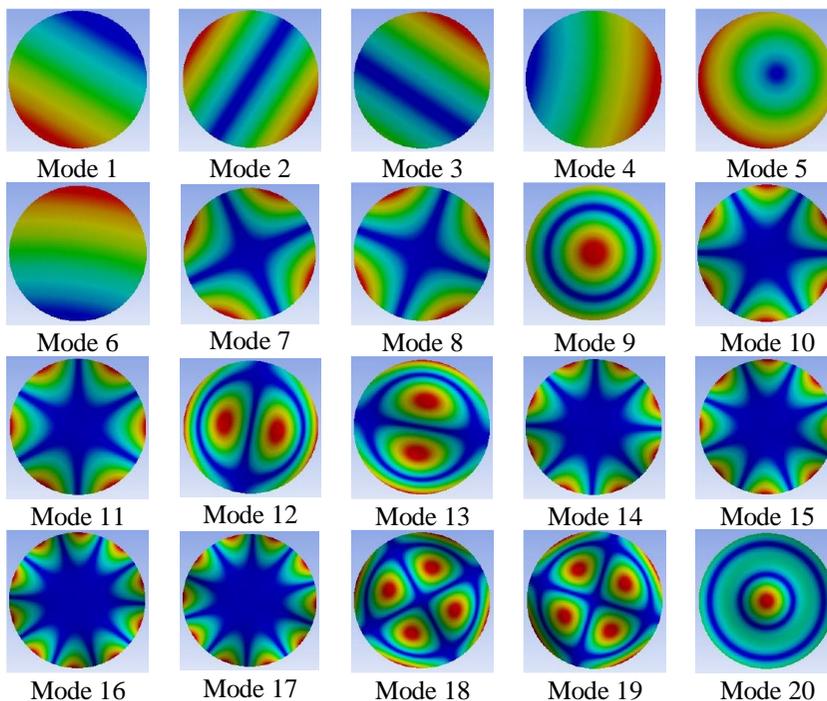


Figure 6. Vibration patterns for completely free circular plate ($\alpha = 0^\circ$).

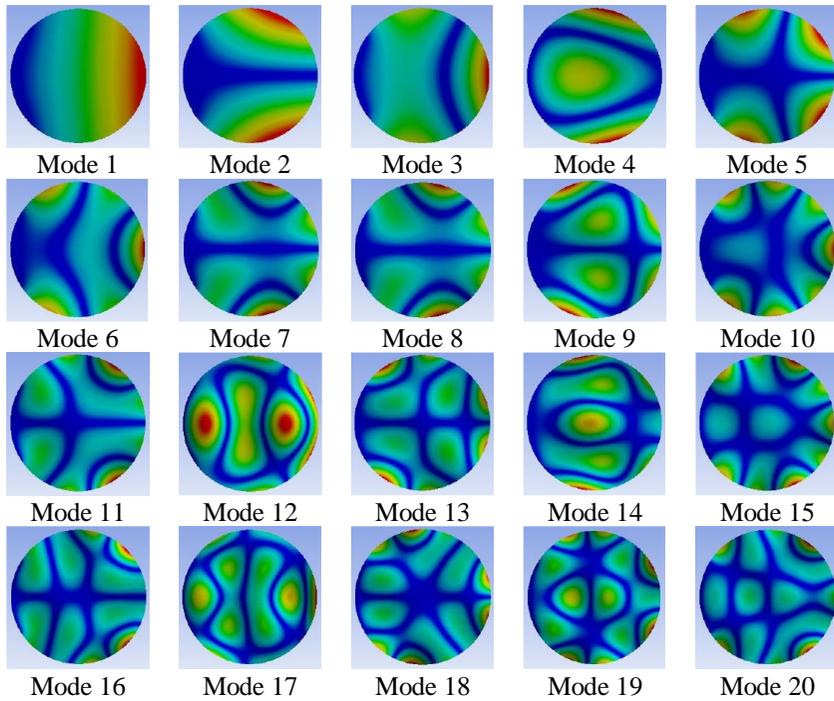


Figure 7. Vibration patterns for partially simply supported circular plate ($\alpha = 30^\circ$).

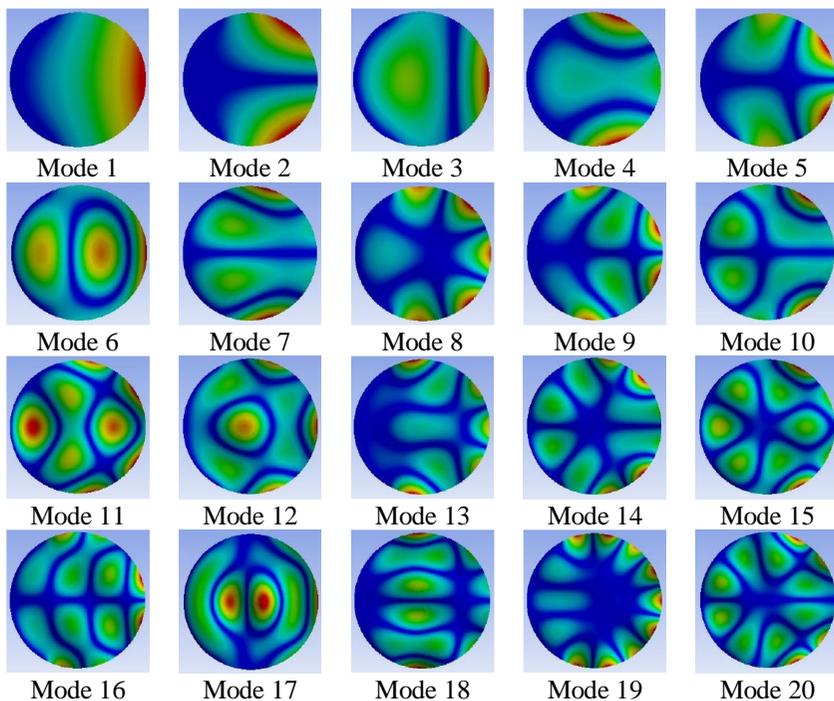


Figure 8. Vibration patterns for partially simply supported circular plate ($\alpha = 60^\circ$).

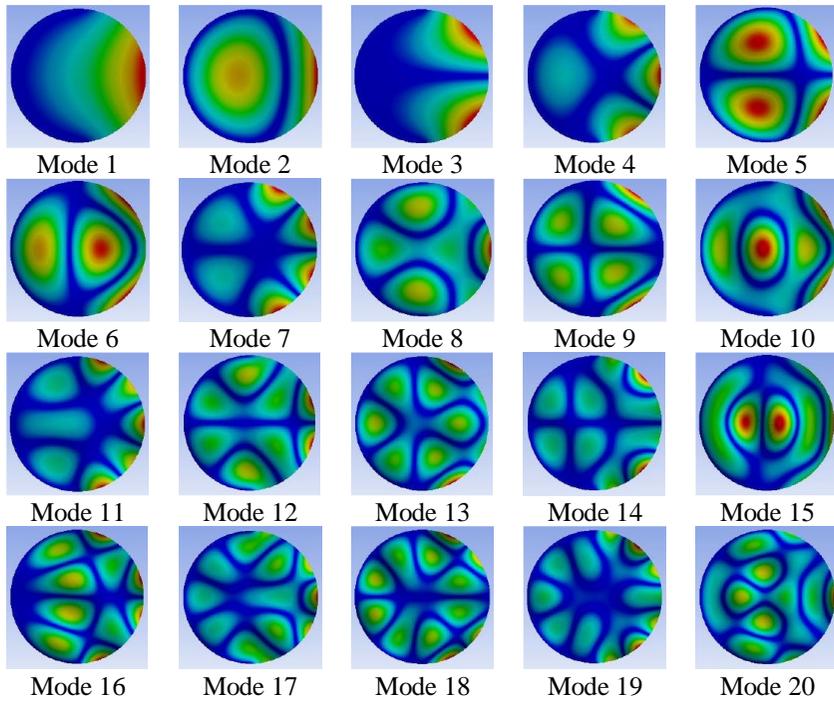


Figure 9. Vibration patterns for partially simply supported circular plate ($\alpha = 90^\circ$).

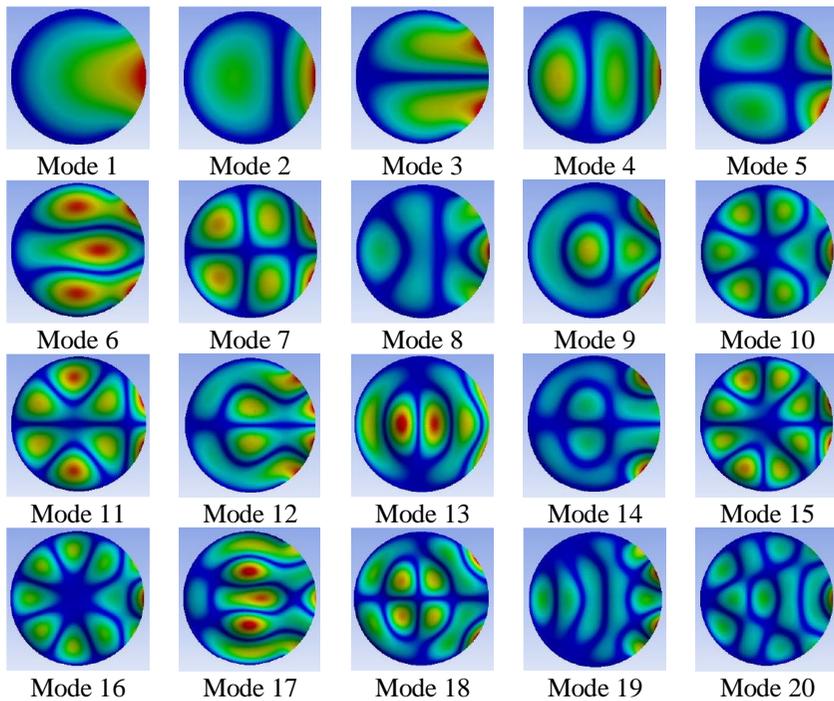


Figure 10. Vibration patterns for partially simply supported circular plate ($\alpha = 120^\circ$).

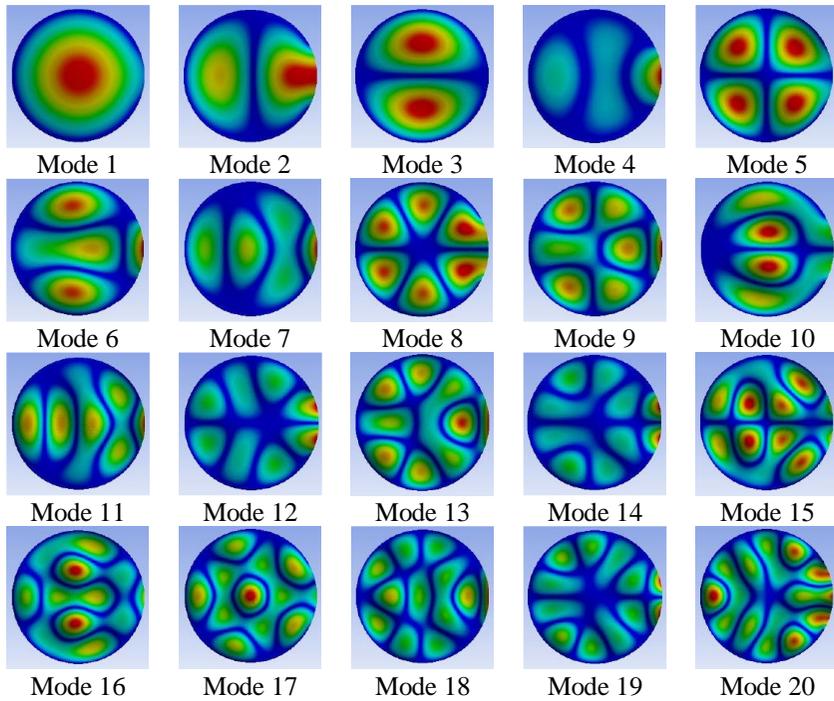


Figure 11. Vibration patterns for partially simply supported circular plate ($\alpha = 150^\circ$).

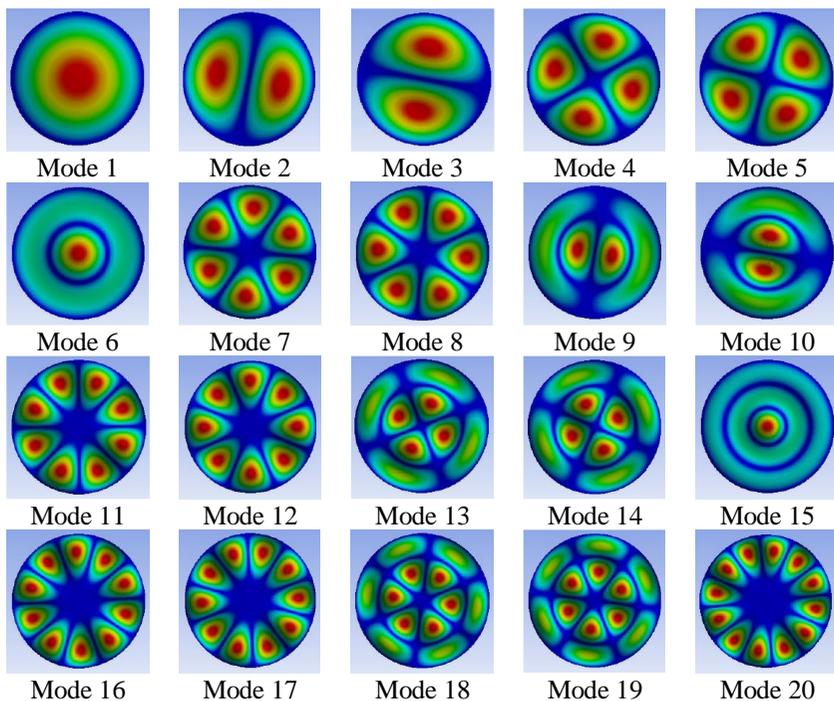


Figure 12. Vibration patterns for fully simply supported circular plate ($\alpha = 180^\circ$).

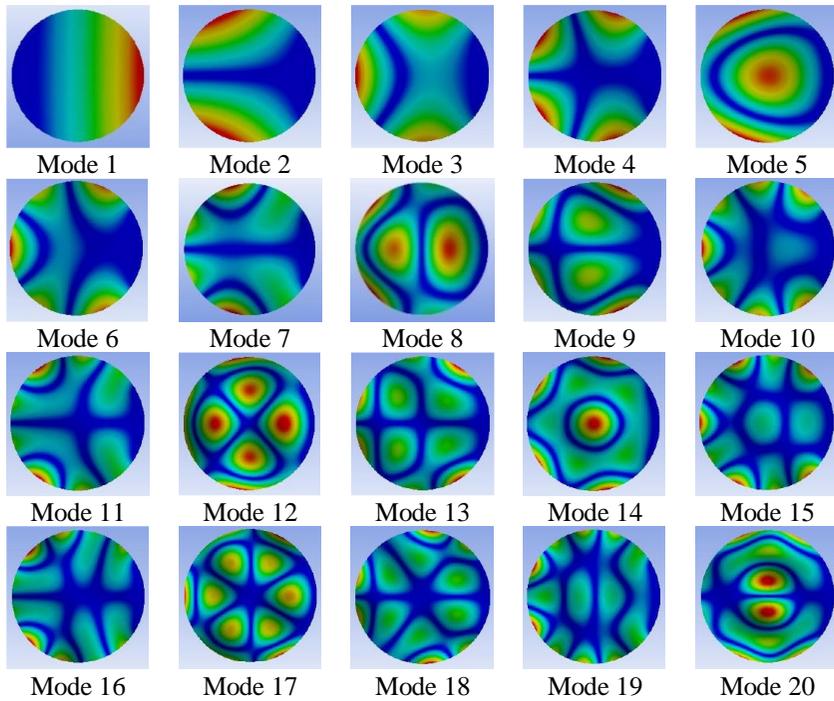


Figure 13. Vibration patterns for partially clamped circular plate ($\alpha = 30^\circ$).

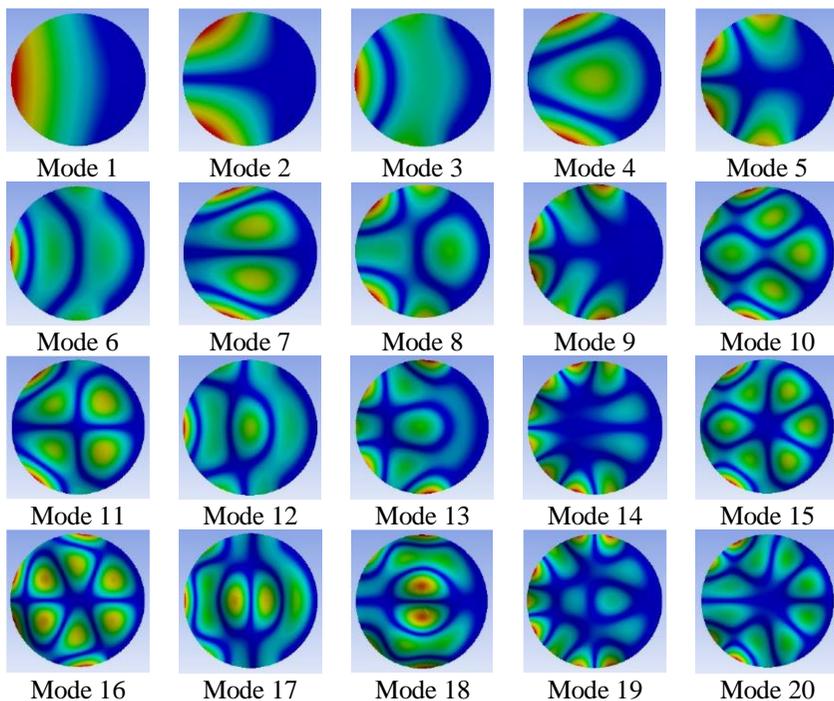


Figure 14. Vibration patterns for partially clamped circular plate ($\alpha = 60^\circ$).

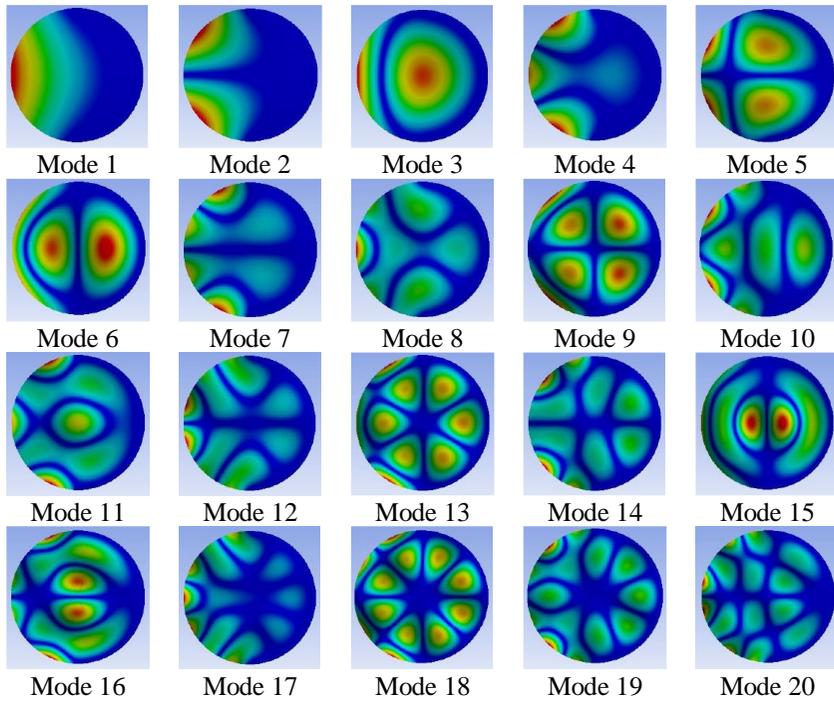


Figure 15. Vibration patterns for partially clamped circular plate ($\alpha = 90^\circ$).

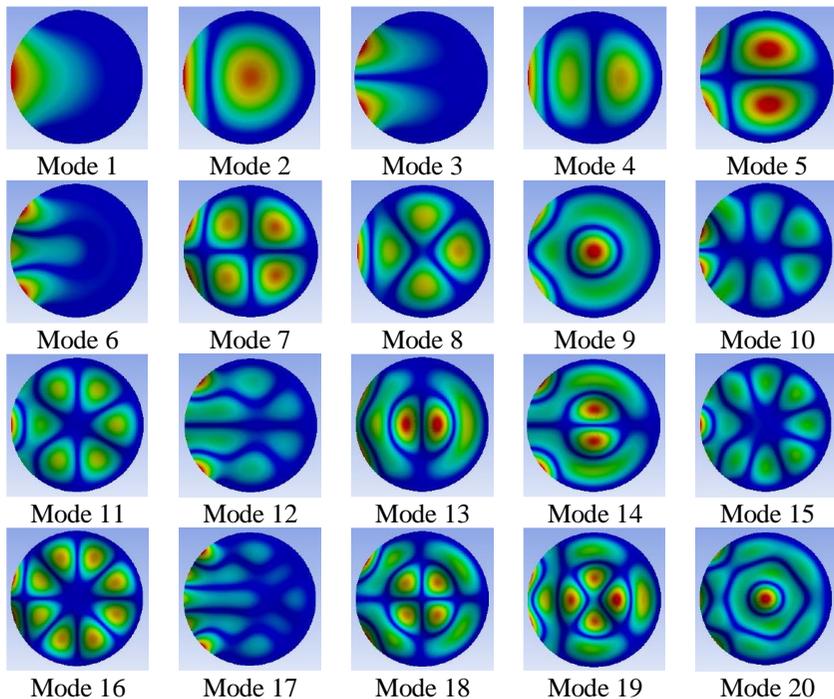


Figure 16. Vibration patterns for partially clamped circular plate ($\alpha = 120^\circ$).

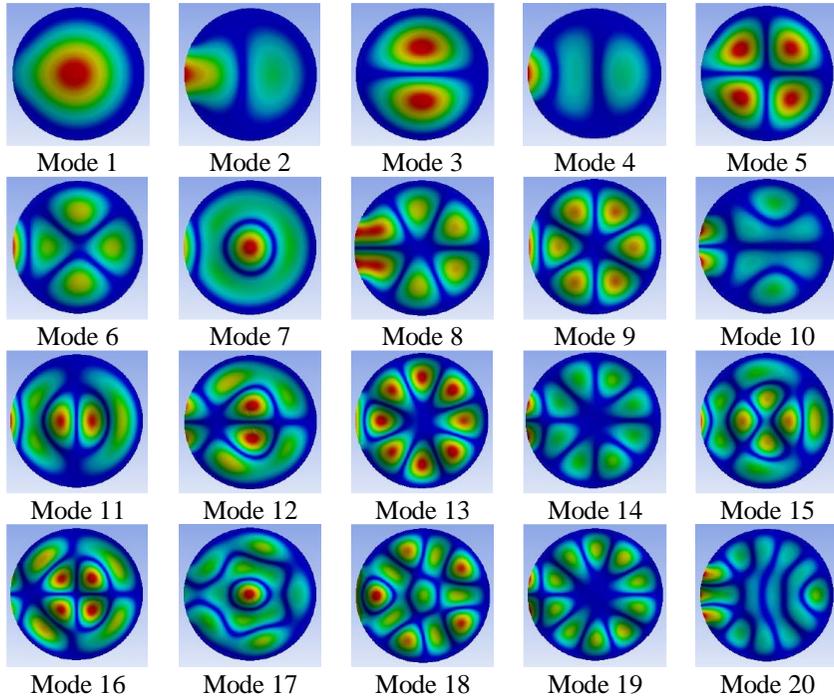


Figure 17. Vibration patterns for partially clamped circular plate ($\alpha = 150^\circ$).

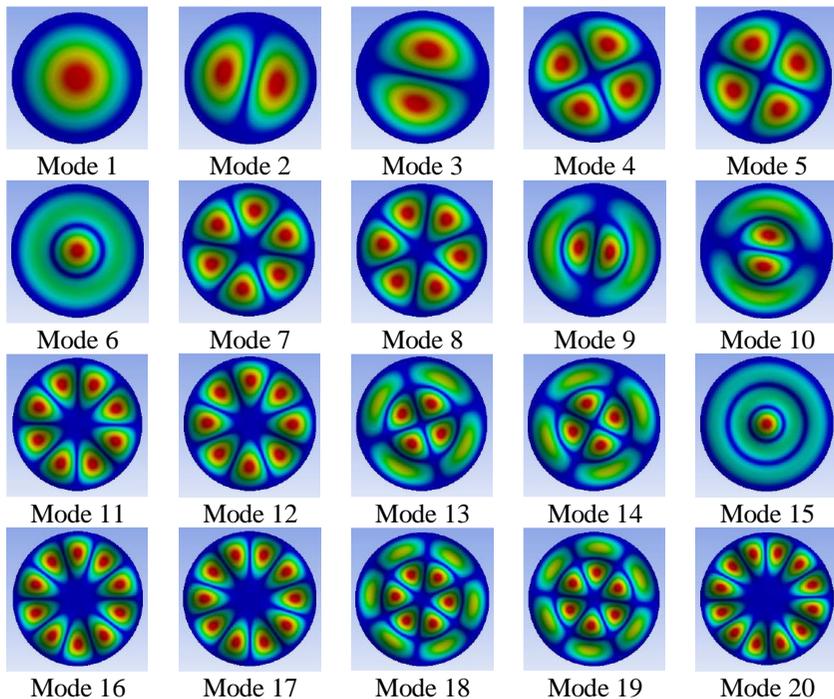


Figure 18. Vibration patterns for fully clamped circular plate ($\alpha = 180^\circ$).