# การศึกษาผลเฉลยของระบบสมการเพลล์
## $x^2\text{-}8t^2y^2=1$ และ $pz^2\text{-}t^2y^2=\text{-}1$
## An Investigation of the Solutions of the System of Pell Equations
## $x^2 - 8t^2y^2 = 1$ and $pz^2 - t^2y^2 = -1$

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## บทคัดย่อ

การวิจัยนี้มีวัตถุประสงค์เพื่อศึกษาผลเฉลยที่เป็นจำนวนเต็มบวกของระบบสมการเพลล์

$$x^2 - 8t^2y^2 = 1 \text{ และ } pz^2 - t^2y^2 = -1$$

โดยที่ $p$ เป็นจำนวนเฉพาะ และ $t, x, y, z$ เป็นจำนวนเต็มบวก อีกทั้งต้องเป็นไปตามเงื่อนไขว่า $gcd(x,y) = 1$ และ $gcd(t,x) = 1$ วิธีการวิจัยประกอบด้วยการปรับสมการให้อยู่ใน รูปแบบที่เหมาะสมต่อการวิเคราะห์เชิงโครงสร้าง พร้อมใช้แนวคิดทางทฤษฎีจำนวน เช่น คุณสมบัติของ จำนวนเฉพาะ สมการกำลังสี่ และทฤษฎีความผกผันกำลังสอง เพื่อแยกกรณีและพิสูจน์อย่างเป็นระบบ ผลการวิจัย พบว่า ระบบสมการดังกล่าวไม่มีคำตอบจำนวนเต็มบวก กล่าวคือ ไม่มีจำนวนเต็มบวก $x, y, z$ ใดที่ทำให้ระบบสมการนี้เป็นจริงพร้อมกัน การศึกษานี้ช่วยเสริมสร้างความเข้าใจเกี่ยวกับ พฤติกรรมของสมการแบบเพลล์ และส่งเสริมแนวทางใหม่ในการวิเคราะห์สมการเชิงโครงสร้างที่มีความ ซับซ้อนในคณิตศาสตร์เชิงทฤษฎี อีกทั้งยังสามารถนำไปประยุกต์ใช้ในการศึกษาปัญหาทางคณิตศาสตร์ อื่น ๆ ได้ในอนาคต

**คำสำคัญ**: ระบบสมการเพลล์ ผลเฉลยพร้อมกัน สมภาค สมการไดโอแฟนไทน์

## ABSTRACT

This research aims to study the positive integer solutions of the Pell-type system of equations

$$x^2 - 8t^2y^2 = 1 \text{ and } pz^2 - t^2y^2 = -1$$

where $p$ is a prime number, and $t, x, y, z$ are positive integers satisfying the conditions $gcd(x, y) = 1$ and $gcd(t, x) = 1.$ The methodology involves transforming the system into a form suitable for structural analysis. Number-theoretic concepts such as the properties of prime numbers, fourth power residues, and the quadratic reciprocity theorem are employed to classify cases and provide rigorous proofs. The results reveal that the system has no positive integer solutions; that is, there are no positive integers $x, y, z$ that simultaneously satisfy both equations. This study enhances the understanding of the behavior of Pell-type equations and promotes new approaches for analyzing complex structural systems in theoretical number theory. Moreover, the findings have the potential for application in future mathematical research.

**Key Words**:  Pell's equation system, Simultaneous solutions, Congruences, Diophantine equations

## Introduction

The Pell equation is one of the most significant equations in number theory and algebra, and it has been extensively studied regarding its potential solutions. Numerous mathematicians have undertaken the task of proving and exploring the possibilities of solutions in various forms. For instance, Bennett (1998) established the statement *"if a and b are distinct positive integers, the system of equations $x^2 - az^2 = 1$ and $y^2 - bz^2 = 1$ could yield at most three solutions."* J.H. Chen (2001) demonstrated that a particular form of the Pell equation $a^2x^4 - By^2 = 1$ had at most one solution, while P. Yuan (2004) proved that $x^2 - 4m(m + 1)y^2 = y^2 - bz^2 = 1$ had a unique positive integer solution.

In 2014, X. Ai, J. Chen, S. Zhang, and H. Hu investigated Pell system of equations

$x^2 - 24y^2 = 1$ and $y^2 - pz^2 = 1$ where p is a prime number, revealing that such system of equations yielded exactly two solutions. Simultaneously, L. Tao (2015) showed that certain forms of the Pell equations $x^2 - 24y^2 = 1$ and $y^2 - 2pz^2 = 1$ did not have integer solutions. In 2018, Q. Yunyun (2018) examined specific Pell system of equations $x^2 - (a^2 - 1)y^2 = 1$ and $y^2 - pz^2 = 1$ for $a = 2, 3$ and concluded that these equations similarly fail to yield integer solutions. These studies highlight the ongoing importance and continuity of research regarding the Pell equation. Consequently, this research aims to explore the solutions of Pell system of equations $x^2 - 8t^2y^2 = 1$ and $pz^2 - t^2y^2 = -1$ where $p$ is a prime number and $t, x, y, z$ are positive integers, we focus on fundamental concepts in number theory, such as congruences and quartic residues.

## Preliminaries

This section discusses the important definitions and theorems used in the proofs of the lemma and the main theorems of this paper, without providing proofs of these theories. Those who are interested can study more from the reference documents.

**Definition 2.1** Let $n$ be a positive integer. For integers $a$ and $b$, we say that $a$ is congruent to $b$ modulo $n$, denoted as $a \equiv b \pmod{n}$, if $n$ divides $(a - b)$ (Smith,2020).

**Definition 2.2** Let $a$ be an integer and $n$ be a positive integer. We say that $a \equiv r \pmod{n}$ if $r$ is the remainder when $a$ is divided by $n$, where $0 \leq r < n$.

**Definition 2.3** Let $n$ be a positive integer. If there exists an integer $x$ such that $x^4 \equiv a \pmod{n}$, then $a$ is called a biquadratic residue modulo $n$.

**Example:** The fourth power residues modulo $6$ for $x = 0, 1, 2, 3, 4, 5$ are shown as follows. Because

$$0^4 \equiv 0 \pmod{6}, \qquad 1^4 \equiv 1 \pmod{6}$$
$$2^4 \equiv 4 \pmod{6}, \qquad 3^4 \equiv 3 \pmod{6}$$
$$4^4 \equiv 4 \pmod{6}, \qquad 5^4 \equiv 1 \pmod{6}$$

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to analyze the Diophantine equation $x^4 \equiv 5 (\text{mod } 6)$, we begin by considering the possible residues modulo $6$. The residues $x^4 (\text{mod } 6)$ for $x \in \{0,1,2,3,4,5\}$ yield the set of possible results $\{0,1,3,4\}$. Since $5$ is not in this set, the congruence $x^4 \equiv 5 (\text{mod } 6)$ has no solutions.

For $n = 8$ similarly we evaluate $x^4$ for $x = 0, 1, 2, 3, 4, 5, 6, 7$ as follow:

$$0^4 \equiv 0 \ (mod \ 8), \qquad\qquad 1^4 \equiv 1 \ (mod \ 8)$$
$$2^4 \equiv 0 \ (mod \ 8), \qquad\qquad 3^4 \equiv 1 \ (mod \ 8)$$
$$4^4 \equiv 0 \ (mod \ 8), \qquad\qquad 5^4 \equiv 1 \ (mod \ 8)$$
$$6^4 \equiv 0 \ (mod \ 8), \qquad\qquad 7^4 \equiv 1 \ (mod \ 8)$$

*Thus, the resulting residues are* $\{0, 1\}$*. Since* $4$ *is not in this set, the congruence* $x^4 \equiv 4(mod \ 8)$ has no solutions.

For $n = 9$ similarly we evaluate $x^4$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8$ as follow:

$$0^4 \equiv 0 \ (mod \ 9), \qquad\qquad 1^4 \equiv 1 \ (mod \ 9)$$
$$2^4 \equiv 7 \ (mod \ 9), \qquad\qquad 3^4 \equiv 0 \ (mod \ 9)$$
$$4^4 \equiv 4 \ (mod \ 9), \qquad\qquad 5^4 \equiv 4 \ (mod \ 9)$$
$$6^4 \equiv 0 \ (mod \ 9), \qquad\qquad 7^4 \equiv 7 \ (mod \ 9),$$
$$8^4 \equiv 1 \ (mod \ 9).$$

Thus, the resulting residues are $\{0, 1, 4, 7\}$. Since $9$ is not in this set, the congruence $x^4 \equiv 9(mod \ 9)$ has no solutions.

**Definition 2.4** A positive integer that is not a perfect square is referred to as a non-square integer. The positive solution of a Diophantine equation is called the fundamental solution if $(x, y)$ are positive integers that satisfy the equation for any positive solution (Hardy & Wright, 2008).

**Definition 2.5** A Pell's equation is a specific of Diophantine equation of the form $x^2 - Dy^2 = 1$ where $D$ is a non-square positive integer, and $x$ and $y$ are integers, which are the integer solution $(x, y)$ that satisfies this relationship.

**Example 1:** Consider the Diophantine equation $x^2 - 3y^2 = 1$.
The positive integer solutions include $(x, y) = (2,1)$. In this case, $(2,1)$ is regarded as a fundamental solution because it is the smallest positive solution that satisfies the equation. Moreover, since $D = 3$ is a positive integer that is not a perfect square, this equation is classified as a Pell's equation according to the definition.

**Example 2:** Consider the equation $x^2 - 2y^2 = 1$.

This is known as Pell's equation. One fundamental solution is $(3,2)$. Since $3^2 - 2(2^2) = 9 - 8 = 1,$ other solutions can be generated from this fundamental solution using recurrence relation.

## Main Theorem

**Theorem 3.1** The Pell-type system of equations

$$\begin{cases} x^2 - 8t^2y^2 = 1 \\ pz^2 - t^2y^2 = -1 \end{cases}$$

has no integer solutions $(x, y)$, when $p$ is a prime numbe, $t, x, y, z$ are positive integers, and $gcd(x, y) = 1, gcd(t, x) = 1$.

**Proof.** Consider the Pell's system of equation given by:

$$\begin{cases} x^2 - 8t^2y^2 = 1 & (1) \\ pz^2 - t^2y^2 = -1 & (2) \end{cases}$$

when $p$ is a prime number and $t, x, y$ and $z$ are positive integers. This equation can be rewritten as

$$\begin{cases} x^2 - 8t^2y^2 = 1 \\ x^2 + pz^2 = 9t^2y^2 \end{cases} \qquad (3)$$

By adding equations (1) and (2), we obtain equation (3). This equation can then be expressed in a new form as follows:

$$\begin{cases} x^2 - 8t^2y^2 = 1 \\ pz^2 = (3ty + x)(3ty - x) \end{cases} \qquad (4)$$

Let $d = gcd(3ty + x, 3ty - x)$.

Thus, $d|(3ty + x)$ and $d|(3ty - x)$.

Then, there exist integers $m$ and $n$ such that

$$3ty + x = dm \text{ and } 3ty - x = dn.$$

Adding and subtracting these two equations, we get $6ty = d(m + n)$ and $2m = d(m - n)$.

Therefore $d|6ty$ and $d|2x$.

Now, consider that $(x, y) = 1$ and $(t, x) = 1$. if $3|x$, then $gcd(3ty, x) = 3$.

Suppose that $3|x,$ then clearly $3|3ty,$ and since $x$ is divisible by $3,$ we have $gcd(3ty, x) = 3$.

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On the other hand, if $3 \nmid x,$ then $gcd(3ty, x) = 1$.

Therefore, the possible values of $gcd(3ty, x)$ are either $1$ or $3$.

Since $d|6ty$ and $d|2x$, the possible values of $d$ must divide both $6$ and $2$.

The common divisors of $6$ and $2$ are $gcd(6, 2) = \{1, 2\}$.

But from the previous reasoning about $gcd(3ty, x)$,

we also consider the case when $gcd(3ty, x) = 3$.

Therefore, the possible values of $d$ must divide both $6$ and $3$, i.e.,

$$d \in \{1, 2, 3, 6\}$$

Therefore, we conclude that $d$ can be $1, 2, 3$ or $6$. From equation (4), we have

$$\left(\frac{3ty + x}{d}\right)\left(\frac{3ty - x}{d}\right) = \frac{pz^2}{d^2} \tag{5}$$

This implies

$$\left(\frac{3ty+x}{d}\right)\left(\frac{3ty-x}{d}\right) = mn \tag{6}$$

From equation (5), we have

$$d(dmn) = pz^2$$

Which implies that

$$d \mid pz^2$$

Since $p$ is a prime number, the possible prime factorization of $pz^2$ implie that $d$ must be a divisor of $p, z,$ or $z^2$. In particulae, we focus on the case $d|p,$ which implies $d = 1$ or $d = p,$ due to the primality of $p.$ In order to analyze the structure of the equation, we consider the implications of $d^2$ in relation to $p$ and $z^2$. Among all possible factorizations, we focus on the meaningful aigebraic cases that lead to integer values of all variables:

Case 1: $d^2 = p,$ which implies $d = \sqrt{p}$ (requires $p$ to be a square), and hence $mn = z^2$.

Case 2: $d^2 = z^2$, which implies $d = z,$ and thus $mn = p.$.

Orther algebraic combinations, such as $d^2 = pz,$ do not result in perfect squares or contradict the integrality of $d, z,$ or $p,$ and are therefore excluded from consideration. These two cases represent the only feasible scenarios that preserve the integer nature of all variables and align with the original equation structure.

**Case 1:** $d^2 = p$ and $mn = z^2$

since $p$ is a prime number $(p > 1)$. the condition $d^2 = p$ implies that $d = \sqrt{p}$.

However, prime numbers like $2, 3, 5,$ etc., are not perfect squares, meaning their square roots are not integers. Thus, $d$ cannot be an integer.

**Case 2:** $d^2 = z^2$ and $mn = p$.

In this scenario, we have $d|z$ and $gcd(m, n) = 1$.

We let

$$z = duv \tag{7}$$

where $u$ and $v$ are integers such that $(u, v) = 1$.

Substituting equation (5) into equation (6), we have

$$\left(\frac{3ty + x}{d}\right)\left(\frac{3ty - x}{d}\right) = pu^2 v^2.$$

Given the equation and the conditions $mn = pu^2 v^2$ and $\gcd(m, n) = 1,$ it follows that $m = pu^2$ and $n = v^2,$ where $m$ and $n$ are integers satisfying $gcd(m, n) = 1$. Since $\frac{3ty+x}{d}$ and $\frac{3ty-x}{d}$ have a product equal to $pu^2 v^2$ and exhibit symmetry, there are two possible Scenario for assigning the values $pu^2$ and $v^2$:

Scenario 1: $\qquad \frac{3ty+x}{d} = pu^2$ and $\frac{3ty-x}{d} = v^2$

Scenario 2: $\qquad \frac{3ty+x}{d} = v^2$ and $\frac{3ty-x}{d} = pu^2$

Since $d|6ty,$ we observe that $\dfrac{6ty}{d} = \frac{3ty+x}{d} + \frac{3ty-x}{d} = pu^2 + v^2$

where $t, x, y$ and $z$ are positive integers..

Since $pu^2 + v^2 \in \mathbb{Z},$ and $d|6ty,$ this implies that $6t|d(pu^2 + v^2)$.

Hence, we assume that the parameters $d, u, v, t \in \mathbb{Z}^+$ are chosen such that $d(pu^2 + v^2)$ is divisible by $6ty,$ ensuring $y = d\left(\frac{pu^2+v^2}{6t}\right)$ is a positive integer.

From the two scenarios, we obtain

$$x = \pm d\left(\frac{pu^2 - v^2}{2}\right) \text{ and } y = d\left(\frac{pu^2 + v^2}{6t}\right).$$

Substitute $x$ and $y$ into equation (1),

we have

$$d^2\left(\frac{pu^2 - v^2}{2}\right)^2 - 8t^2 d^2\left(\frac{pu^2 + v^2}{6t}\right)^2 = 1.$$

$$\frac{d^2}{4}(p^2u^2 - 2pu^2v^2 + v^4) - 8\frac{d^2}{36}(p^2u^4 + 2pu^2v^2 + v^4) = 1$$

$$\left(\frac{d^2}{4} - \frac{8d^2}{36}\right)p^2u^4 + \left(\frac{-d^2}{4} - \frac{8d^2}{36}\right)(2pu^2v^2) + \left(\frac{d^2}{4} - \frac{8d^2}{36}\right)v^4 = 1.$$

Then

$$d^2(p^2u^4 - 17(2pu^2v^2) + v^4) = 36.$$

$$p^2u^4 + (-17)(2pu^2v^2) + v^4 = \frac{36}{d^2}.$$

That is

$$(pu^2 - 17v^2)^2 - 288v^4 = \frac{36}{d^2}.$$

Let

$$\beta = \left|\frac{d(pu^2 - 17v^2)}{6}\right|$$

Therefore

$$\beta^2 - 8d^2v^4 = 1 \tag{8}$$

From equation (8), we can consider four cases based on the values of $d = 1, 2, 3$ and $6$, utilizing the concept of fourth power remainders and the definition of the solution set for Diophantine equations to analyze the solutions for each sub-case as follows.

**Case 2.1:** $d = 1.$ From equation (8) we have

$$\left(\frac{\beta + 1}{2}\right)\left(\frac{\beta - 1}{2}\right) = 2v^4 \tag{9}$$

The equation can be rearranged for clarity by letting $v = st$ to simplify the analysis as follows.

$$\left(\frac{\beta + 1}{2}\right)\left(\frac{\beta - 1}{2}\right) = 2s^4t^4.$$

We can consider two cases because the equation has two factors $\frac{\beta+1}{2}$ and $\frac{\beta-1}{2}$. These factors can be related in different ways, which leads to two possible scenarios.

**Case 2.1.1**

Assume:

$$\begin{cases} \frac{\beta+1}{2} = 2s^4 \\ \frac{\beta-1}{2} = t^4 \end{cases} \quad \text{or} \quad \begin{cases} \frac{\beta+1}{2} = 2t^4 \\ \frac{\beta-1}{2} = s^4 \end{cases}$$

From this, we can derive

$$\frac{\beta+1}{2} - \frac{\beta-1}{2} = 2s^4 - t^4 = 1.$$

Or

$$\frac{\beta+1}{2} - \frac{\beta-1}{2} = 2t^4 - s^4 = 1.$$

which is a Diophantine equation. The only integer solution is $(s,t) = (1,1)$ when $s$ and $t$ are positive integers.

From $\beta = \left| \frac{d(pu^2 - 17v^2)}{6} \right|$, we get $\beta = 3$ and $pu^2 = 35 = (5)(7)$.

Thus $p = 5$ and $u = \pm\sqrt{7}$ or $p = 7$ and $u = \pm\sqrt{5}$. Therefore, $z = \pm\sqrt{7}$ or $z = \pm\sqrt{5}$.

Since the values obtained do not yield any integer solutions, we conclude that this case has no integer results.

**Case 1.1.2**

Assume:

$$\begin{cases} \frac{\beta+1}{2} = s^4 \\ \frac{\beta-1}{2} = 2t^4 \end{cases} \tag{10}$$

Or

$$\begin{cases} \frac{\beta+1}{2} = t^4 \\ \frac{\beta-1}{2} = 2s^4 \end{cases} \tag{11}$$

From this, we can derive

$$\frac{\beta+1}{2} - \frac{\beta-1}{2} = s^4 - 2t^4 = 1.$$

Or

$$\frac{\beta+1}{2} - \frac{\beta-1}{2} = t^4 - 2s^4 = 1.$$

Let $u = s^2$ and $w = t^2$ for positive integers $w$ and $u$. This gives the equation:

$$s^4 - 2t^4 = u^2 - 2w^2 = 1.$$

Or

$$t^4 - 2s^4 = w^2 - 2s^2 = 1$$
$$u_n + w_n\sqrt{2} = \left(3 + 2\sqrt{2}\right)^n.$$

Or

$$w_n + u_n\sqrt{2} = \left(3 + 2\sqrt{2}\right)^n.$$

Since $3 + 2\sqrt{2}$ is the fundamental solution to the equation $u^2 - 2w^2 = 1$ and $w^2 - 2s^2 = 1$ we have $u_1 = s^2 = 3$ and $w_1 = t^2 = 2$.

Qr $w_1 = t^2 = 3$ and $u_1 = s^2 = 2$.

Thus, $s = \pm\sqrt{3}$ and $t = \pm\sqrt{2}$. Or $s = \pm\sqrt{2}$ and $t = \pm\sqrt{3}$.

Therefore, equation (10) and (11) has no integer solution.

It can be observed that swapping the values of $s^4$ and $t^4$ between $\left(\frac{\beta+1}{2}\right)$ and $\left(\frac{\beta-1}{2}\right)$ does not change the final result. Therefore, in cases 2 to 4, the proof will consider swapping only the constants, as detailed below.

### Case 2.2: $d = 2$

From equation (8), we obtain:

$$\beta^2 - 32v^4 = 1.$$

This implies:

$$\left(\frac{\beta + 1}{2}\right)\left(\frac{\beta - 1}{2}\right) = 8v^4.$$

Since $v = st,$ We can express it as:

$$\left(\frac{\beta + 1}{2}\right)\left(\frac{\beta - 1}{2}\right) = 8s^4t^4.$$

We will divide this into four sub-cases as follows:

### Case 2.2.1

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = 4s^4 \\ \dfrac{\beta - 1}{2} = 2t^4 \end{cases} \tag{12}$$

From this, we can derive

$$\frac{\beta + 1}{2} - \frac{\beta - 1}{2} = 4s^4 - 2t^4 = 1.$$

Let $r = 2s^2$ and $w = t^2$ for positive integers $w$ and $u$. This gives the equation:
$$r^2 - 2w^2 = 1,$$
The general solution to the equation $r^2 - 2w^2 = 1$ *is* given by:
$$r_n + w_n\sqrt{2} = \left(3 + 2\sqrt{2}\right)^n.$$
Since $3 + 2\sqrt{2}$ is the fundamental solution to the equation $r^2 - 2w^2 = 1,$ we have:
$r_1 = 2s^2 = 3$ and $w_1 = t^2 = 2$.

Thus, $s = \sqrt{\dfrac{3}{2}}$ and $t = \sqrt{2}$

Therefore, equation (12) has no integer solution.

### Case 2.2.2

Assume:
$$\begin{cases} \dfrac{\beta + 1}{2} = 2s^4 \\ \dfrac{\beta - 1}{2} = 4t^4 \end{cases} \tag{13}$$

From this, we can derive
$$\frac{\beta + 1}{2} - \frac{\beta - 1}{2} = 2s^4 - 4t^4 \tag{14}$$

From equation (14), we obtain:
$$2s^4 \equiv 1 (mod\ 4).$$

This means: $8s^4 \equiv 4(mod\ 8)$,

Which is impossible since $8|8s^4$. This means that $4$ cannot be a remainder of $8s^4$ modulo $8$.

Therefore, equation (13) has no integer solution.

### Case 2.2.3

Assume:
$$\begin{cases} \dfrac{\beta + 1}{2} = 8s^4 \\ \dfrac{\beta - 1}{2} = t^4 \end{cases} \tag{15}$$

From this, we can derive
$$\frac{\beta + 1}{2} - \frac{\beta - 1}{2} = 8s^4 - t^4 = 1 \tag{16}$$

From equation (16), we obtain:

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$$t^4 \equiv -1 (mod\ 8).$$

Since $-1 \equiv 7(mod\ 8)$. This leads to:

$$t^4 \equiv 7(mod\ 8).$$

Since $7$ is not a fourth power residue modulo $8$, Therefore, equation (15) has no integer solution.

### Case 2.2.4

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = s^4 \\ \dfrac{\beta - 1}{2} = 8t^4 \end{cases} \tag{17}$$

From (17), we derive:

$$\frac{\beta + 1}{2} - \frac{\beta - 1}{2} = s^4 - 8t^4 = 1 \tag{18}$$

Let $u = s^2$ and $w = t^2$ for positive integers $w$ and $u$.
This gives the equation:

$$u^2 - 8w^2 = 1.$$

Since $3 + \sqrt{8}$ is the fundamental solution to the equation $u^2 - 8w^2 = 1$, we have: The general solution to the equation
$u^2 - 8w^2 = 1$ is given by:

$$u_n + w_n\sqrt{8} = \left(3 + \sqrt{8}\right)^n.$$

$u_1 = s^2 = 3$ and $w_1 = t^2 = 1$.

Thus, $s = \pm\sqrt{2}$ and $t = \pm 1$

Therefore, equation (17) has no integer solution.

### Case 2.3: $d = 3$

From equation (8), we obtain:

$$\beta^2 - 72v^4 = 1.$$

The equation can be rearranged for clarity by letting
$v = st$ to simplify the analysis as follows.

$$\left(\frac{\beta + 1}{2}\right)\left(\frac{\beta - 1}{2}\right) = 18s^4t^4.$$

We can consider six cases because the equation has two factors $\frac{\beta+1}{2}$ and $\frac{\beta-1}{2}$. These factors can be related in different ways, which leads to six possible scenarios.

### Case 2.3.1

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = 6s^4 \\ \dfrac{\beta - 1}{2} = 3t^4 \end{cases} \tag{19}$$

From equation (19), we can derive:

$$\frac{\beta + 1}{2} - \frac{\beta - 1}{2} = 6s^4 - 3t^4 = 1 \tag{20}$$

From equation (20), we obtain the congruence

$$3t^4 \equiv -1 \pmod 6 \text{ or } 3t^4 \equiv 5 \pmod 6$$

Now we analyze possible values of $t \pmod 6$. We compute $t^4 \pmod 6$ for $t = 0, 1, 2, 3, 4, 5$.

The resulting values are $t^4 \pmod 6 \in \{0, 1, 3, 4\}$.

Thus, $3t^4 \equiv 5 \pmod 6$ is not possible. This contradiction implies that our assumption leads to no integer solution. Therefore, equation (19) has no integer solution.

### Case 2.3.2

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = 3s^4 \\ \dfrac{\beta - 1}{2} = 6t^4 \end{cases} \tag{21}$$

From equation (21), we can derive:

$$\frac{\beta + 1}{2} - \frac{\beta - 1}{2} = 3s^4 - 6t^4 = 1 \tag{22}$$

From equation (22), we obtain the following congruence:

$$3s^4 \equiv 1 \pmod 6 \text{ or } 6t^4 \equiv 2 \pmod 6.$$

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Next, we analyze these congruences to determine the implications for $s$ and $t$:

$3s^4 \equiv 1 (mod\ 6)$ implies that $s^4$ must be congruent to $1 (mod\ 2)$ (since $3 \equiv 3 (mod\ 6)$ and $1$ is odd) $6s^4 \equiv 2 (mod\ 6)$ simplifies to $0 \equiv 2 (mod\ 6)$ which is impossible.

Thus, we conclude that the system of equations given by equation (21) does not yield any integer solutions.

### Case 2.3.3

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = 9s^4 \\ \dfrac{\beta - 1}{2} = 2t^4 \end{cases} \tag{23}$$

From equation (23), we can derive:

$$\frac{\beta + 1}{2} - \frac{\beta - 1}{2} = 9s^4 - 2t^4 = 1 \tag{24}$$

Let $w = t^2$ and $q = 3s^2$ for positive integers $w$ and $q$. This gives the equation:

$$q^2 - 2w^2 = 1,$$

The general solution to the equation $q^2 - 2w^2 = 1$ is given by $q_n + w_n\sqrt{2} = \left(3 + 2\sqrt{2}\right)^n$.

Since $3 + 2\sqrt{2}$ is the fundamental solution to the equation $q^2 - 2w^2 = 1$, we have: $q_1 = 3s^2 = 3$ and $w_1 = t^2 = 2$. Thus, $s = \pm 1$ and $t = \pm\sqrt{2}$.

Therefore, equation (23) has no integer solution.

### Case 2.3.4

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = 2s^4 \\ \dfrac{\beta - 1}{2} = 9t^4 \end{cases} \tag{25}$$

From equation (25), we can derive:

$$\frac{\beta + 1}{2} - \frac{\beta - 1}{2} = 2s^4 - 9t^4 = 1 \tag{26}$$

This simplifies to:

$$2s^4 \equiv 1 (mod\ 9)$$

$$10s^4 \equiv 5(mod\ 9)$$

Since $10 \equiv 1(mod\ 9)$, we can conclude:

$$s^4 \equiv 5(mod\ 9)$$

Now we analyze all possible values of $s^4(mod\ 9)$:

We find that $s^4(mod\ 9) \in \{0, 1, 4, 7\}$, so $5$ is not a valid value.

This is impossible because $5$ is not fourth power residue modulo $9$.

Thus, we conclude that equation (25) has no integer solutions.


### Case 2.3.5

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = 18s^4 \\ \dfrac{\beta - 1}{2} = t^4 \end{cases} \tag{27}$$

From equation (27), we can derive:

$$18s^4 - t^4 = 1 \tag{28}$$

Now, rearranging gives:

$$-t^4 - 1 \equiv 9(2s^4).$$

That implies:

$$-t^4 \equiv 1(mod\ 9).$$

Since $-1 \equiv 8(mod\ 9)$, we can write:

$$t^4 \equiv 8(mod\ 9).$$

This is impossible because $8$ is not a quadratic residue modulo $9$.

Therefore, we conclude that equation (27) has no integer solutions.


### Case 2.3.6

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = s^4 \\ \dfrac{\beta - 1}{2} = 18t^4 \end{cases} \tag{29}$$

From equation (29), we can derive

$$\frac{\beta+1}{2} - \frac{\beta-1}{2} = s^4 - 18t^4 = 1 \tag{30}$$

Let $u = s^2$ and $k = 9t^2$ for positive integers $W$ and $u$. This gives the equation:

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$$u^2 - 2k^2 = 1,$$

The general solution to the equation $u^2 - 2k^2 = 1$ is given by:

$$u_n + k_n\sqrt{2} = \left(3 + 2\sqrt{2}\right)^n.$$

Since $3 + 2\sqrt{2}$ is the fundamental solution to the equation $u^2 - 2k^2 = 1$, we have:

$u_1 = s^2 = 3$ and $k_1 = 9t^2 = 2$.

Thus, $s = \pm\sqrt{3}$ and $t = \pm\frac{1}{3}\sqrt{2}$.

Therefore, equation (29) has no integer solution.

**Case 2.4:** $d = 6$, from equation (8), we derive

$$\beta^2 - 288v^4 = 1.$$

This implies

$$\left(\frac{\beta + 1}{2}\right)\left(\frac{\beta - 1}{2}\right) = 72v^4.$$

Given that $v = st$,

we have $\left(\frac{\beta+1}{2}\right)\left(\frac{\beta-1}{2}\right) = 72s^4 t^4$.

We can divide the analysis into $8$ sub-cases as follows:

**Case 2.4.1**

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = 9s^4 \\ \dfrac{\beta - 1}{2} = 8t^4 \end{cases} \tag{31}$$

From equation (31), we can derive

$$\frac{\beta+1}{2} - \frac{\beta-1}{2} = 9s^4 - 8t^4 = 1 \tag{32}$$

Given that $v = st$ and $z = duv$, we find that

$(s, t) = (1,1)$ is a solution for equation (32).

Now, substituting into the equation for $\beta$, we have:

$$\beta = \left|\frac{d(pu^2 - 17v^2)}{6}\right|$$

Assume that $\beta = 17$.

From $pu^2 = 34 = (17)(2)$ the possible values are:

a) If $p = 17$, then $u^2 = 2$, which implies $u = \pm\sqrt{2}$

b) If $p = 2$, then $u^2 = 17$, which implies $u = \pm\sqrt{17}$.

In both cases, $u \notin \mathbb{Z}$, and therefore $z = duv = d \cdot u \cdot v = d \cdot u \cdot 1 = \pm d\sqrt{2}$ or $\pm d\sqrt{17}$.

Therefore, we conclude that equation (31) has no integer solutions.

### Case 2.4.2

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = 8s^4 \\ \dfrac{\beta - 1}{2} = 9t^4 \end{cases} \tag{33}$$

From equation (33), we can derive

$$\frac{\beta+1}{2} - \frac{\beta-1}{2} = 8s^4 - 9t^4 = 1 \tag{34}$$

This leads to $8s^4 \equiv 1 \pmod 9$. Since $8s^4 \equiv -s^4 \pmod 9$,

we can rewrite this as $-8 \equiv 1 \pmod 9$.

Thus, we have $s^4 \equiv 8 \pmod 9$.

This is impossible because $8$ is not a quadratic residue modulo $9$.

Therefore, we conclude that equation (33) has no integer solutions.

### Case 2.4.3

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = 36s^4 \\ \dfrac{\beta - 1}{2} = 2t^4 \end{cases} \tag{35}$$

From equation (35), we can derive

$$\frac{\beta+1}{2} - \frac{\beta-1}{2} = 36s^4 - 2t^4 = \tag{36}$$

From equation (36). Let $h = 6s^2$ and $w = t^2$ for positive integers $h$ and $w$ This gives the equation

$h^2 - 2w^2 = 1$ is given by $h_n + w_n\sqrt{2} = \left(3 + 2\sqrt{2}\right)^n$.

Since $3 + 2\sqrt{2}$ is the fundamental solution to the equation $h^2 - 2w^2 = 1$,

we have: $h_1 = 6s^2 = 3$ and $w_1 = t^2 = 2$.

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Thus, $s = \pm\sqrt{\frac{1}{2}}$ and $t = \pm\sqrt{2}$.

Therefore, equation (35) has no integer solution

### Case 2.4.4

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = 2s^4 \\ \dfrac{\beta - 1}{2} = 36t^4 \end{cases} \tag{37}$$

From equation (37), we can derive

$$\frac{\beta + 1}{2} - \frac{\beta - 1}{2} = 2s^4 - 36t^4 = 1 \tag{38}$$

This leads to:

$2s^4 - 1 \equiv 1 \pmod{9}$ or $8s^4 \equiv 4 \pmod 9$.

Since $8s^4 \equiv -s^4 \pmod 9$,

we can rewrite this as $4 \equiv -5 \pmod 9$

Thus, we have $s^4 \equiv 8 \pmod 9$ is impossible.

Consequently, we conclude that equation (37) has no integer solutions.

### Case 2.4.5

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = 18s^4 \\ \dfrac{\beta - 1}{2} = 4t^4 \end{cases} \tag{39}$$

From equation (39), we can derive

$$\frac{\beta + 1}{2} - \frac{\beta - 1}{2} = 18s^4 - 4t^4 = 1 \tag{40}$$

This leads to $-4t^4 - 1 = 9(-2s^4)$.

Since $-4t^4 \equiv 1 \pmod 9$. We can also write this as $-8t^4 \equiv 2 \pmod 9$.

This simplifies to: $t^4 \equiv 2 \pmod 9$.

However, $2$ is not a quadratic residue modulo $9$. Therefore, this situation is impossible.

Consequently, we conclude that equation (39) has no integer solutions.

**Case 2.4.6**

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = 4s^4 \\ \dfrac{\beta - 1}{2} = 18t^4 \end{cases} \tag{41}$$

From equation (41), we can derive

$$\frac{\beta + 1}{2} - \frac{\beta - 1}{2} = 4s^4 - 18t^4 = 1 \tag{42}$$

Let $r = 2s^2$ and $f = 3t^2$ for positive integers $r$ and $f$.

This gives the equation $r^2 - 2f^2 = 1.$ The general solution to the equation $r^2 - 2f^2 = 1$ is given by:

$$r_n + f_n\sqrt{2} = \left(3 + 2\sqrt{2}\right)^n.$$

Since $3 + 2\sqrt{2}$ is the fundamental solution to the equation $r^2 - 2f^2 = 1,$

we have: $r_1 = 2s^2 = 3$ and $k_1 = 9t^2 = 2.$

Thus, $s = \pm\sqrt{\dfrac{3}{2}}$ and $t = \pm\sqrt{\dfrac{2}{9}}.$

Therefore, equation (41) has no integer solutions.

**Case 2.4.7**

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = 72s^4 \\ \dfrac{\beta - 1}{2} = t^4 \end{cases} \tag{43}$$

From equation (43), we can derive

$$\frac{\beta + 1}{2} - \frac{\beta - 1}{2} = 72s^4 - t^4 = 1 \tag{44}$$

Rearranging this gives:

$$-t^4 - 1 = 9(-8s^4)$$

This can also be written as:

$$t^4 + 1 = 72s^4$$

Next, we analyze this equation modulo 9:

$$t^4 \equiv -1 (mod\ 9)$$

Which can be expressed as: $t^4 \equiv 8(mod\ 9)$

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Since $8$ is not a residue for fourth powers modulo $9$, Therefore, equation (43) has no integer solutions.

**Case 2.4.8**

Assume:

$$\begin{cases} \dfrac{\beta + 1}{2} = s^4 \\ \dfrac{\beta - 1}{2} = 72t^4 \end{cases} \tag{45}$$

From equation (45), we can derive

$$\frac{\beta + 1}{2} - \frac{\beta - 1}{2} = s^4 - 72t^4 = 1.$$

This simplifies to:

$$s^4 - 1 = 72t^4 \tag{46}$$

We can factor the left side:

$$(s^2 - 1)(s^2 + 1) = 72t^4$$

Sine $72t^4$ is even, both $s^2 - 1$ or $s^2 + 1$.

Must also be even, which implies that $s^2$ must be odd as well.

Now, let us consider the equation modulo 3.

If $s^2 \equiv 0 (mod\ 3).$ then

$$s^2 - 1 \equiv 2 (mod\ 3).$$

And

$$s^2 + 1 \equiv 1 (mod\ 3).$$

The product becomes:

$$(s^2 - 1)(s^2 + 1) \equiv 2 \cdot 1 \equiv 2 (mod\ 3).$$

But

$$72t^4 \equiv 0 (mod\ 3).$$

This is a contradiction.

The analysis in the modulo systems shows that contradictions arise, which indicates that there are no integer values for sss and ttt that satisfy equation (45). Therefore, we conclude that the equation has no integer solution.

Conclusion: Main Theorem (Theorem 3.1)

The Pell's system of equations:

$$\begin{cases} x^2 - 8t^2y^2 = 1 \\ pz^2 - t^2y^2 = -1 \end{cases}$$

The objective is to explore possible integer solutions for $x, y$ and $z$ given that $p$ is a prime number, and $t, x, y$ and $z$ are positive integers. After deriving an equation from the combination of (1) and (2), we arrive at an analysis based on the gcd (greatest common divisor) of certain terms. This analysis further leads us to consider various cases for the value for $d = gcd(3ty + x, 3ty - x)$ and the implications for the system of equations. Each case $(d = 1, d = 2, d = 3$ and $d = 6)$ fails to yield any integer solutions. Therefore, the system has no integer solutions for any prime $p$.

## Conclusion

This study analyzes the system of Pell equations $x^2 - 8t^2y^2 = 1$ and $pz^2 - t^2y^2 = -1$ Where $p$ is a prime number and $t, x, y, z$ are positive integers. By applying number-theoretic methods such as modular analysis and quartic residues, we have demonstrated through an in-depth analysis that this system of equations does not admit positive integer solutions. This provides a clearer understanding of the characteristics and limitations of this type of Pell equation by exploring the properties of the variables and utilizing the theory of quartic residues. The results of this study are significant for the development of new approaches in the analysis of complex Diophantine equations. Additionally, it highlights the potential applications of number theory in various contexts, which can be useful for further studies or solving other mathematical problems in the future.

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# References

Ai, X., Chen, J., Zhang, S., & Hu, H. (2015). Complete solutions of the simultaneous Pell equations $x^2 - 24y^2 = 1$ and $y^2 - pz^2 = 1$. *Journal of Number Theory*, *147*(1), 103-108.

Bennett, M.A. (1998). On the number of solutions of simultaneous Pell equations. *Journal für die reine and angewandte Mathematik*, *498*(1), 173-199.

Chen, J.H. (2001). A note on the Diophantine equation $a^2 x^4 - By^2 = 1$. *Acta Arithmetica*, *96*(3), 205-212.

Smith, J. (2020). Congruences and their applications in number theory. *Number Theory Review*, *22*(3), 45-52.

Tao, L. (2015). Conjectured that for the system of simultaneous Pell equations $x^2 - 24y^2 = 1$ and $y^2 - pz^2 = 1$. *Pure Mathematical Sciences*, 5(1), 27-32.

Yuan, P. (2004). Unique positive integer solutions for a class of Pell-type equations. *Number Theory*, *26*(1), 33-40.

Yunyun, Q. (2018), On simultaneous Pell equations $x^2 - (a^2 - 1)y^2 = 1$ and $y^2 - pz^2 = 1$. *Journal of Number theory, 184*(1), 128-132.