An approximation to the average run length of cumulative sum control chart for long memory under fractionally integrated process with exogenous variable

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Abstract

Cumulative sum (CUSUM) control chart is widely used in industries for the detection of small and moderate shifts in the process. Evaluation of the Average Run Length (ARL) plays an important role in the performance comparison of the control chart. Approximated ARL with the Numerical Integral Equation (NIE) method calculated by solving the system of linear equations and concept of integration based on the partition and summation the area under the curve of a function. The main purpose of this paper is to approximate the ARL of the CUSUM control chart using the Numerical Integral Equation (NIE) method for long-memory process in case of exponential white noise. The NIE method approximate solutions are derived by the Gauss-Legendre quadrature rule technique. For the long memory, the process is derived from the fractionally integrated with the exogenous variable model, which details the process depends on fractional differencing. This ARL approximation using NIE method is shown to be in good alternative compared with the explicit formula. An obvious extension is to other control charts for long memory under the fractionally integrated with the exogenous variable process, and hopefully, this work will encourage real-world applications such as finance economics and agriculture.

Keywords: Numerical Integral Equation (NIE) method, Exponential white noise, Long memory, CUSUM control chart, Average Run Length (ARL)

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Introduction

The Cumulative sum (CUSUM) control chart is one of the most applied tools in statistical process control (SPC) using in control chart. Monitoring the detection of small process shifts and controling the quality of products from manufacturing processes, CUSUM chart has been broadly applied at this stage. Page [1] initially proposed a review of CUSUM chart which has been studied in numbers of literatures particularly in Gan [2], Luceno and Puig-Pey [3], and Wu and Wang [4]. Evaluation and comparison of the performance the control chart based on the Average Run Length (ARL) were extensively involved in using measurement for CUSUM chart. The average number of observations was measured by ARL taken before the signals. Before raising of a false out-of-control alarm which is measured in terms of a false-alarm rate measurement, the average number of observations from the in-control process was generally fixed and referred to the in-control ARL abbreviated by ARL_0. On the other hand, there is a requirement of out-of-control ARL abbreviated by ARL_1 which is the average number of observations representing the detection power of the control chart see Ryu et al. [5]) for detecting a process mean shift. The performance of CUSUM control chart with the autocorrelation has been studied in many aspects; for example, Johnson and Bagshaw [6], Lu and Reynolds [7], and Kim et al. [8].

The assessment made to exponential white noise and time series has been completely conducted. The process of ARMA (1,1) is to denote the autoregressive moving-averageprocess order (1,1). According to Jacob and Lewis [9], it is shown that after the observation, white noise would beexponentially distributed.The Bayesian analysis of the autoregressive model order 1 denoted by AR (1) with exponential distribution was conducted by Mohamed and Hocine [10] after 26 years later. Additionally, the exponential white noise was employed to make development to the application of Bayesian for threshold autoregressive model analysis (Pereira and Turkrman [11]).

The Autoregressive Fractionally Integrated Moving Average (ARFIMA) model was the first presented by Granger and Joyeux [12], and Hosking [13] to describe and fit the long memory. The long-memory process is involved in a number of applications including finance and economics, environmental sciences and engineering. The long memory is said to be fractionally integrated, or FI(d), if it is integrated of order d, with d not necessarily integer and $0<d<0.5$ (Granger and Joyeux [12], Hosking [13]). Typically, the real economic variables are fractionally integrated or trend stationary (see Marmol and Velasco [14]).

According to the relevant research studies, several methods, for instance, Monte Carlo simulations (MC), Markov Chain approach (MCA), Martingale approach, Numerical Integral Equation (NIE) method and explicit formulas, could be employed to evaluate the Average Run Length. Some case studies would be illustrated as examples in this section. Integral equation and Markov Chain approach application to study EWMA and CUSUM charts for ARL evaluation in the case of AR(1) process with additional random error were conducted by Vanbrackle and Reynold [15]. Derivations of analytical formulas of the Average Run Length (ARL) and the average delay (AD) with Martingale approach in the case of Gaussian and a few non-gaussian distributions were illustrated by Sukparungsee and Novikov [16]. Numerical Integral Equation (NIE) method of ARL for CUSUM chart for a stationary first order autoregressive, AR(1) process, with exponential white noise was employed by Busaba et al. [17]. Furthermore, the same methods were presented by Phanyaem et al. [18] with major focus on analytical explicit formulas found in ARL_0 and ARL_1 with the NIE method on CUSUM chart for ARMA(1,1) process with exponential white noise. Next, Phanyaem et al. [19] proposed the developed NIE method to calculate ARL of autoregressive and moving average process, ARMA(p,q) process with exponential white noise on EWMA and CUSUM charts as well as to compare ARL between EWMA and CUSUM charts. Moreover, Paichit et al. [20] derived analytical formulas and use numerical methods to find ARL of CUSUM control chart for ARX(1) processes with exponential white noise for detecting a change in the process mean. Finally, Peerajit et al. [21] also presented that observations are long memory processes with non-seasonal and seasonal ARFIMA model with exponential white noise when the NIE method is applied for ARL approximation on CUSUM chart.

The two main goals of this paper are to approximate ARL by using Numerical Integral Equation (NIE) method based on Gauss-Legendre quadrature rule technique of CUSUM control chart for long memory process under fractionally integrated with the exogenous variable model in case of exponential white noise and to compare between NIE method and explicit formula with rest to the performance of ARL. Other sections of the paper are organized as follows. Section 2 is the brief introduction to the preliminaries, CUSUM control chart, characteristics of Average Run Length and the process known as long memory with the fractionally integrated with the exogenous variable process which was applied to this paper. The two subsequent sections are devoted to the approximation of ARL. Sections 3 and 4 are the approximation of ARL with the use of Numerical Integral Equation (NIE) method for long memory process under fractionally integrated with the exogenous variable model, and the comparison of numerical results respectively. Concluding remarks are provided in Section 5.

Material and Methods

Preliminaries

In the following sections, CUSUM control chart, characteristics of Average Run Length and, long memory process under fractionally integrated with the exogenous variable model used in the paper will be described, that is, the approximation of ARL are computed by Numerical Integral Equation (NIE) method.

1. CUSUM control chart

The upper-sided CUSUM chart is defined as the chart under the assumption $\{C_t, t = 1, 2, \ldots\}$ as a sequence of independent and identically distributed (i.i.d) continuous random variables with common probability density function. The CUSUM chart's statistics is expressed by the recursion:

$$C_0 = u,$$
$$C_t = \max\{C_{t-1} + Y_t - k, 0\}, \; t = 1, 2, \ldots, \tag{1}$$

where the chart's parameter Y_t is a sequence of the generalized fractionally integrated with exogenous variable processes in case of exponential white noise. The starting value $C_0 = u$ is an initial value and the constant *k* is called the reference value.

More formally, the CUSUM control chart in Equation (1) is characterized by the stopping time (τ_h) which can be written as:

$$\tau_h = \inf\{t > 0;\, C_t > h\}, \; u \le h, \tag{2}$$

where *h* is a constant parameter known as upper control limit (UCL) of CUSUM control chart.

2. Characteristics of Average Run Length

Let $E_m(\cdot)$ denote the expectation of stopping time for a fixed change point *m* under distribution $F(x, \lambda)$ as follows:

$$\mathrm{ARL} = \begin{cases} E_{m=\infty}(\tau_h) = \gamma, & \text{in-control ARL}\,(\mathrm{ARL}_0) \\ E_{m=1}(\tau_h \mid \tau_h \ge 1), & \text{out-of-control ARL}\,(\mathrm{ARL}_1), \end{cases}$$

where γ is assumed be large enough.

Let $L(u)$ be the ARL of CUSUM chart for long memory under fractionally integrated with the exogenous variable model with initial value u, can be written as,

$$\mathrm{ARL} = L(u) = E_\infty(\tau_h),$$

where the initial value $u \in [0, h]$.

3. The long memory and generalized FIX(*d*, *X*) process with exponential white noise

3.1 Long Memory

The process has long memory (see. Baillie [22], and Beran et al. [23]), if its autocorrelation function (ACF) has power-law decay: $\rho(.)$ For example,

$$\rho(j) \sim c_p.K^{2d-1} \text{ as } j \to \infty,$$

where c_p is finite non-zero constant, $\rho(j)$ is the autocorrelation function (ACF) at lag *j*, and *d* are restricted to the range of (0, 0.5). In addition, the parameter *d* is the memory parameter. If the parameter *d* = 0, the process will not exhibit long memory. While *d* is in the range of (- 0.5, 0), it is said to be anti-persistent (see. Baillie [22], and Beran et al. [23]).

Remark: The symbol ~ means that the ratios of the left and the right hand sides are finite when *K* tends to be equal to infinity (see. Baillie [22], Beran et al. [23]).

3.2 Generalized FIX(d, X) Process

The theoretical long memory process related to fractionally integrated process. Especially in the case of the ARFIMAX(*p*, *d*, *q*, *X*) process is when *p* and *q* are selected zero to model no effects. This process is called the pure fractionally integrated model with the exogenous variable, denoted by ARFIMAX(0, *d*, 0, *X*) or abbreviate FIX(d^*, *X*), which will be the main goal of in this paper, and it can be written as:

$$\left(1-L\right)^{d}(Y_t-\beta X_i)=\mu+\varepsilon_t, \tag{3}$$

where Y_t is a sequence of FIX(d^*) process, ε_t is i.i.d white noise process assumed with exponential distribution , $\varepsilon_t \sim Exp(\lambda), L$ is the lag-operator, X_i is a exogenous variable and β is a coefficent, *d* is the fractional which represents the degree of fractional difference (or fractional integration) operator. In fractional, values of *d* is interesting in the context of long memory process that are restricted to the range of (0, 0.5), which is not an integer.

The expression of the operator $(1-L)^d$ can be defined in a natural way by using binomial expansion for any real number *d* with Gamma function:

$$(1-L)^d = 1-dL+\frac{d(d-1)L^2}{2!}-\frac{d(d-1)(d-2)L^3}{3!}+\dots . \tag{4}$$

Obviously, the generalized fractionally integrated with exogenous variable (abbreviation FIX(*d*, X)) process for long memory (Y_t) with exponential white noise which is used for CUSUM chart in Equation (1), namely:

$$Y_t=\sum_{i=1}^{r}\beta_i X_i+dY_{t-1}-\frac{d(d-1)Y_{t-2}}{2!}+\frac{d(d-1)(d-2)Y_{t-3}}{3!}-\dots+\varepsilon_t, \tag{5}$$

where $\varepsilon_t \sim Exp(\alpha)$. It is assumed that the initial value of ε_t and fractional integration process equals 1. The initial value of generalized FIX(*d*, 1) process is assigned as $Y_{t-1}, Y_{t-2}, \dots, Y_{t-k}, Y_{t-(k+1)}, \dots$, equals 1.

Numerical Integral Equation (NIE) Method of ARL of CUSUM Control Chart

The idea of the derived an integral equation for computing its performance ARL and introduced the CUSUM chart was suggested by Page [1]. Moreover, Page [24] the midpoint rule was employed to solve the ARL from the integral equation. Similarly, the integral equations for the approximate ARL performance of upper-sided CUSUM chart can be demonstrated as

$$\begin{aligned} L(u) \;=\; & 1+L(0)F\left(k-u-\sum_{i=1}^{r}\beta_i X_i-dY_{t-1}+\frac{d(d-1)Y_{t-2}}{2!}-\frac{d(d-1)(d-2)Y_{t-3}}{3!}+\dots-\varepsilon_t\right) \\ & +\int_0^h L(z)f(z+k-u-\sum_{i=1}^{r}\beta_i X_i-dY_{t-1}+\frac{d(d-1)Y_{t-2}}{2!}-\frac{d(d-1)(d-2)Y_{t-3}}{3!}+\dots-\varepsilon_t)dz, \end{aligned} \tag{6}$$

is formed as a Fredholm integral equation of the second kind, and $F(u)=1-e^{-\lambda u}$ whereas $f(u)=\lambda e^{-\lambda u}$

Apparently, when the final term of Equation (6) is applied to quadrature rule, the integral can be approximated by the sum of rectangle as shown below:

$$\int_0^h L(z) f(z+k-u-\sum_{i=1}^{r}\beta_i X_i - dY_{t-1} + \frac{d(d-1)Y_{t-2}}{2!} - \frac{d(d-1)(d-2)Y_{t-3}}{3!} + \ldots - \varepsilon_t)dz, \qquad (7)$$

where the integral f value is chosen by base h/m the heights is maintained at the midpoints of intervals of length and h/m beginning at zero. The interval $[0,h]$ is divided into partitions $0 \le a_1 \le \ldots \le a_m \le h$ and $w_j = h/m \ge 0$ are sets of constant weights.

Therefore, approximation for an integral by Equation (7) obtained from the summation form becomes

$$\sum_{j=1}^{m} w_j \tilde{L}(a_j) f(a_j + k - a_i - \sum_{i=1}^{r}\beta_i X_i - dY_{t-1} + \frac{d(d-1)Y_{t-2}}{2!} - \frac{d(d-1)(d-2)Y_{t-3}}{3!} + \ldots - \varepsilon_t) \qquad (8)$$

with $a_j = \frac{h}{m}\left(j - \frac{1}{2}\right) \quad ; j = 1, 2, \ldots, m.$

More formally, solve the system of *m* linear equations in the *m* unknowns, which are able to the approximated solution of $L(u)$ for the interval $[0,h]$ by replacing u by a_i in Equation (6) as follows:

$$\tilde{L}(a_i) = 1 + \tilde{L}(a_1) F\left(k - a_i - \sum_{i=1}^{r}\beta_i X_i - dY_{t-1} + \frac{d(d-1)Y_{t-2}}{2!} - \frac{d(d-1)(d-2)Y_{t-3}}{3!} + \ldots - \varepsilon_t\right) \qquad (9)$$

where $i = 1, 2, \ldots, m.$ This function $\tilde{L}(.)$ is the approximated ARL of NIE method using the Gauss-Legendre quadrature rule technique.

Evaluating Equation (9) at $\tilde{L}(a_1), \tilde{L}(a_2), \ldots, \tilde{L}(a_m)$ results in a system of *m* linear equations:

$$\begin{aligned}
\tilde{L}(a_1) &= 1 + \tilde{L}(a_1)\Big[F(k - a_1 - \sum_{i=1}^{r}\beta_i X_i - dY_{t-1} + \frac{d(d-1)Y_{t-2}}{2!} - \frac{d(d-1)(d-2)Y_{t-3}}{3!} + \ldots - \varepsilon_t) \\
&\quad + w_1 f(k - \sum_{i=1}^{r}\beta_i X_i - dY_{t-1} + \frac{d(d-1)Y_{t-2}}{2!} - \frac{d(d-1)(d-2)Y_{t-3}}{3!} + \ldots - \varepsilon_t)\Big] \\
&\quad + \sum_{j=2}^{m} w_j \tilde{L}(a_j) f(a_j + k - a_1 - \sum_{i=1}^{r}\beta_i X_i - dY_{t-1} + \frac{d(d-1)Y_{t-2}}{2!} - \frac{d(d-1)(d-2)Y_{t-3}}{3!} + \ldots - \varepsilon_t) \\
&\quad \vdots \\
\tilde{L}(a_m) &= 1 + \tilde{L}(a_1)\Big[F(k - a_m - \sum_{i=1}^{r}\beta_i X_i - dY_{t-1} + \frac{d(d-1)Y_{t-2}}{2!} - \frac{d(d-1)(d-2)Y_{t-3}}{3!} + \ldots - \varepsilon_t) \\
&\quad + w_1 f(a_1 + k - a_m - \sum_{i=1}^{r}\beta_i X_i - dY_{t-1} + \frac{d(d-1)Y_{t-2}}{2!} - \frac{d(d-1)(d-2)Y_{t-3}}{3!} + \ldots - \varepsilon_t)\Big] \\
&\quad + \sum_{j=2}^{m} w_j \tilde{L}(a_j) f(a_j + k - a_m - \sum_{i=1}^{r}\beta_i X_i - dY_{t-1} + \frac{d(d-1)Y_{t-2}}{2!} - \frac{d(d-1)(d-2)Y_{t-3}}{3!} + \ldots - \varepsilon_t)
\end{aligned}$$

The matrix form can be written as

$$\mathbf{L}_{m\times 1} = \mathbf{1}_{m\times 1} + \mathbf{R}_{m\times m}\mathbf{L}_{m\times 1}, \qquad (10)$$

where $\mathbf{L}_{m\times 1}=\left[\tilde{L}a_1),\tilde{L}(a_2),...,\tilde{L}(a_m)\right]^T$, $\mathbf{1}_{m\times 1}=[1,1,...,1]^T$ is a column vector of and one, $\tilde{L}(a_i)$ and $\mathbf{I}_m = diag(1,1,,1)$ is the unit matrix order *m* respectively. In addition, as $\mathbf{R}_{m\times m}$ is a matrix with the element, the $(m,m)^{th}$ can be written as

$$\mathbf{R}_{m\times m}=\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mm} \end{bmatrix}, \text{ with } r_{ij}=F(k-a_i-\sum_{i=1}^{r}\beta_i X_i - dY_{t-1}+\frac{d(d-1)Y_{t-2}}{2!}-\frac{d(d-1)(d-2)Y_{t-3}}{3!}+...-\varepsilon_t)$$

$$+w_j f(a_j+k-a_i-\sum_{i=1}^{r}\beta_i X_i - dY_{t-1}+\frac{d(d-1)Y_{t-2}}{2!}-\frac{d(d-1)(d-2)Y_{t-3}}{3!}+...-\varepsilon_t),$$ where $i,j=1,2,...,m.$ If the inverse $(\mathbf{I}_m-\mathbf{R}_{m\times m})^{-1}$ exists, then the unique solution of Equation is

$$\mathbf{L}_{m\times 1}=(\mathbf{I}_m-\mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times 1}. \quad (11)$$

The Numerical Integral Equation (NIE) method for $\tilde{L}(u)$ of CUSUM chart can be written:

$$\tilde{L}(u)=1+\tilde{L}(a_1)F(k-u-\sum_{i=1}^{r}\beta_i X_i - dY_{t-1}+\frac{d(d-1)Y_{t-2}}{2!}-\frac{d(d-1)(d-2)Y_{t-3}}{3!}+...-\varepsilon_t)$$

$$+\sum_{j=1}^{m}w_j\tilde{L}(a_j)f(a_j+k-u-\sum_{i=1}^{r}\beta_i X_i - dY_{t-1}+\frac{d(d-1)Y_{t-2}}{2!}-\frac{d(d-1)(d-2)Y_{t-3}}{3!}+...-\varepsilon_t), \quad (12)$$

with $w_j=\frac{h}{m}$, and $a_j=\frac{h}{m}\left(j-\frac{1}{2}\right)\ ; j=1,2,\ldots,m.$

Results and Discussion

This section is devoted to the comparison of performance obtained from numerical results between the approximation of ARL with the use of Numerical Integral Equation (NIE) method and the explicit formula for long memory with the use of the FIX(*d**, *X*) process. The performance with the ARL between NIE method and the explicit formula was measured in terms of absolute percent error formula $(APE\%)$

$$APE(\%)=\frac{|\tilde{L}(u)-L(u)|}{L(u)}\times 100\% \quad (13)$$

where $\tilde{L}(u)$ is approximated ARL from NIE method, $L(u)$ is ARL from explicit formula.

To construct the tables for ARL values, the author considers the case for the in-control ARL (ARL_0) target to be selected as 370. The out-of-control ARL (ARL_1) is calculated from (11) if $\delta \neq 0$, corresponding to a shift of size δ. The value of exponential parameter for in-control ARL ($\lambda_0=1$) and the value of exponential parameter for out-of-control ARL ($\lambda_1=(1+\delta)\lambda_0$), where δ is the magnitude of shift size; δ = 0.01, 0.03, 0.05, 0.30, 0.50 and 1.00 respectively. For the NIE method used the number of division points *m* = 800 nods for calculations by solving the system of linear equations. In addition, the two methods for this target and long memory with the use of the generalized FIX(*d*, *X*) process such as *d* = 0.17, 0.26, 0.35 and *X* = 1, respectively, are presented and compared.

The numerical results of the ARL obtained from the Numerical Integral Equation (NIE) method and the explicit formula given *a* = 3.0 for ARL_0 =370 and 500, as shown in Tables 1-3. As the magnitude of shift size (δ) of increases, the out-of-control ARL decrease more rapidly for both methods. The NIE method is also decreasing more than explicit formula only slightly. Additionally, both methods indicate that small shift ($\delta = 0.01 - 0.30$) and moderate shifts ($\delta = 0.50 - 1.00$) in every level of *d* of the long memory with FIX(*d, X*) process can be detected more quickly for ARL_0 = 370 and 500. According to the absolute percent error formula $(APE\%)$ is calculated as Equation (12), shown in the rows 3, and 6 of Tables 1-3. The results of the ARL is calculated in terms of absolute percent error as Equation (13), shown in rows 3 and 6. Considering each level of *d* of the long memory with FIX(*d, X*) process, it is found that when *d* is increased, the absolute percent error increases at the magnitude of small shifts ($\delta = 0.01 - 0.30$), while the absolute percent error for each level of *d* is similar when at the magnitude of moderate shifts ($\delta = 0.50 - 1.00$) for all ARL_0. It was found that the absolute percent error of two methods less than 0.25. Therefore, the results of approximated ARL by NIE method in terms of absolute percent error are shown to be a similar and a good agreement compared with the explicit formula.

Table 1 Comparison of ARL_1 values on FIX(0.17, 1) process between NIE method and explicit formula given *a* = 3.00

ARL_0	*h*		Shift size (δ)					
			0.01	0.03	0.05	0.30	0.50	1.00
370	3.951855	$L(u)$	346.219	304.318	268.809	79.1567	40.458	14.253
		$\tilde{L}(u)$	345.479	303.687	268.268	79.0452	40.414	14.244
		$(APE\%)$	0.214	0.208	0.202	0.141	0.109	0.063
500	4.301035	$L(u)$	465.880	406.127	355.892	96.868	47.478	15.787
		$\tilde{L}(u)$	464.791	405.209	355.113	96.7224	47.4239	15.7771
		$(APE\%)$	0.234	0.226	0.219	0.150	0.114	0.063

Bold is the minimum ARL_1.

Table 2 Comparison of ARL_1 values on FIX(0.26, 1) process between NIE method and explicit formula given *a* = 3.00

ARL_0	*h*		Shift size (δ)					
			0.01	0.03	0.05	0.30	0.50	1.00
370	3.7403486	$L(u)$	346.705	305.571	270.609	203.618	81.471	41.968
		$\tilde{L}(u)$	**345.985**	**304.954**	**270.078**	**203.245**	**81.356**	**41.921**
		$(APE\%)$	0.208	0.202	0.196	0.183	0.141	0.112
500	4.078512	$L(u)$	466.664	408.131	358.754	265.265	100.323	49.654
		$\tilde{L}(u)$	**465.603**	**407.230**	**357.985**	**264.736**	**100.171**	**49.595**
		$(APE\%)$	0.227	0.221	0.214	0.199	0.152	0.119

Bold is the minimum ARL_1.

Table 3 Comparison of ARL_1 values on FIX(0.35, 1) process between NIE method and explicit formula given *a* = 3.00

ARL_0	*h*		Shift size (δ)					
			0.01	0.03	0.05	0.30	0.50	1.00
370	3.5729135	$L(u)$	347.037	306.426	271.842	205.365	83.109	43.058
		$\tilde{L}(u)$	**346.337**	**305.825**	**271.322**	**204.997**	**82.992**	**43.010**
		$(APE\%)$	0.202	0.196	0.191	0.179	0.140	0.112
500	3.904302	$L(u)$	467.188	409.476	360.680	267.955	102.729	51.202
		$\tilde{L}(u)$	**466.154**	**408.595**	**359.925**	**267.431**	**102.573**	**51.140**
		$(APE\%)$	0.221	0.215	0.209	0.196	0.152	0.120

Bold is the minimum ARL_1.

Conclusion

This article proposes the observations of the long memory under fractionally integrated with the exogenous variable model where exponential white noise. These experimented was observed by applying the Numerical Integral Equation (NIE) method to approximate the ARL for CUSUM control chart. According to the aforementioned findings, the encouragement is made to the real-world situation. With the applications employed to the different processes of data as a solution to the economics, finance, and other issues, the successful outcome could be achieved with the NIE method and the explicit formulas of FIX(*d*, *X*) process with exponential white noise. The variety of data processes i.e. long memory process with seasonal fractional integration with exogenous variable could be extended to other observations. When observations are made to long memory processes with exponential white noise, the method could also be employed to the application of other control charts, for instance EWMA chart, and HWMA chart, the NIE method's results obtained from approximate the ARL could be further studied by other researchers of this field as the extension of the research Ramjee et al. [25].

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