Solution of the exponential Diophantine Equation $n^x + (2p - 1)^y = z^2$

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Abstract

This study investigates the exponential Diophantine equation

$$n^x + (2p - 1)^y = z^2,$$

where p is a prime number and n, x, y, z are non-negative integers, subject to the modular condition

$$n \equiv 5 (\text{mod } 12) \text{ with } \gcd(n, 2p - 1) = 1.$$

The primary objective is to determine all non-negative integer solutions of this equation by employing quadratic residue theory, modular arithmetic, and its connections to Pell-type equations.

The results demonstrate that the equation admits a unique non-negative integer solution given by

$$(n, p, x, y, z) = (n, 2, 0, 1, 2).$$

For all other values of p, no non-negative integer solutions exist, and it can be rigorously proven that z cannot be a perfect square outside this solution. These findings provide a clear classification of the solution set structure and offer theoretical insights beneficial for further studies on exponential Diophantine equations, including potential applications in computational number theory and cryptographic systems.

Keywords: Diophantine equations, congruence, integer solutions, number theory

Introduction

The study of Diophantine equations holds enduring theoretical importance and serves as a cornerstone in advancing applied mathematics. Systematic investigation of these equations reveals fundamental solution patterns, establishes general problem-solving frameworks, and extends classical results—thereby providing a robust foundation for both pure and applied research. Insights into the integer solutions and their structures empower researchers to develop methods applicable to more complex equations and contribute to future mathematical discoveries.

Over the past decade, researchers have devoted considerable effort to analyzing various forms of exponential Diophantine equations, focusing on uncovering the properties and structure of integer solutions through systematic strategies. Achievements in this domain are noteworthy, as illustrated by several pivotal studies:

In 2004, Catalan's Conjecture which had been proposed in 1844 by Catalan [2], stating that the only solution $(a,b,x,y) = (3,2,2,3)$ satisfies the equation $a^x - b^x = 1$, with min $\{a,b,x,y\} > 1$ was finally proven by Mihăilescu [7].

In 2011, Suvarnamani [11] considered the equation in the form $2^x + p^y = z^2$ and found that solutions of this equation follows the value of p for example, $(3, 0, 3)$ is a solution for $p > 2$, besides, $(4, 2, 5)$ is another solution to the equation for $p = 3$.

Later in 2012, Sroysang [9] proved that the Diophantine equation $3^x + 5^y = z^2$ has a unique non-negative integer solution. The solution (x, y, z) is $(1, 0, 2)$.

In 2014, Sroysang [10] showed that the Diophantine equations $7^x + 31^y = z^2$ has no non-negative integer solution.

Additionally, in 2018, Kumar, Gupta and Kishan [6] showed that the Non-Linear Diophantine equation $p^x + (p+6)^y = z^2$, when p and $p+6$ both are primes, has no solution in non-negative integers. Moreover, Fernando [5] demonstrated that the Diophantine equation $p^x + (p+8)^y = z^2$ when $p > 3$ and $p+8$ are primes, admits no solution (x, y, z) in positive integers. In 2020, Burshtein [1] proved that the Diophantine equation $p^x + (p+5)^y = z^2$, when $p + 5 = 2^{2u}$ where x, y, z and u are positive integer, has no solution (x, y, z) in positive integers. In 2021, N. Viriyapong and C. Viriyapong [12] studied a Diophantine equation $n^x + 13^y = z^2$ which has exactly one solution $(n, x, y, z) = (2,3,0,3)$, where x, y and z are non-negative integers and n is a positive integer with $n \equiv 2 (\mathrm{mod}\ 39)$ and $n + 1$ is not a square number. In the same year. Tangjai and Chubthaisong [13] investigated non-negative integer solutions of the Diophantine equation $3^x + p^y = z^2$, where $p \equiv 2 (\mathrm{mod}\ 3)$. They found that for $y = 0$, the unique solution is $(p, x, y, z) = (p, 1,0,2)$, and for y not divisible by 4, the unique solution is $(p, x, y, z) = (2,0,3,3)$. Later in 2022, Pakapongpun and Chattae [8] had demonstrated how to find the solution of the equation $p^x + 7^y = z^2$, it was found that there was a unique solution for the equation, $(x, y, z) = (3,0,3)$ when $p \equiv 2 (\mathrm{mod}\ 6)$. In 2023 Tadee and Siraworakun [14] studied the Diophantine equation $p^x + (p+2q)^y = z^2$ where p, q and $p + 2q$ are prime numbers and showed that the equation has no positive integer solution.

After reviewing previous research on exponential Diophantine equations, it is evident that this topic remains both challenging and highly intriguing, particularly in the search for non-negative integer solutions. Several earlier studies have demonstrated that certain forms of these equations possess either a unique solution or no solution at all under specific conditions. Motivated by these findings, this study aims to investigate all non-negative integer solutions of the Diophantine equation

$$n^x + (2p-1)^y = z^2 \qquad (1)$$

where p is a prime number and n, x, y, z are non-negative integers, under the modular condition $n \equiv 5 (\mathrm{mod}\ 12)$ and $\gcd(n, 2p-1) = 1$.

The principal objectives are to find all nonnegative integer solutions under these conditions, analyze the structural features imposed by modular and coprimality constraints, and close an existing gap in the literature since this particular formulation has yet to be thoroughly explored. Beyond its theoretical contributions, this line of research has potential implications in cryptography. Modern encryption schemes such as RSA, ElGamal, and Diffie–Hellman rely on the computational difficulty of solving exponential equations modulo large primes. Understanding the intricate behavior and structure of such equations can inform the development or security assessment of cryptographic protocols in the future.

1. Preliminaries

Definition 1. [4] Let a and b be two integers such that $b \neq 0$. We say that a divides b, and write $a|b$, if $b = ac$ for some integer c.

Definition 2. [4] Let a, b and m be three integers such that $m \geq 1$. We say that a is congruent to b modulo m, and write $a \equiv b(\text{mod } m)$, if $m|a - b$.

Definition 3. [4] Let n be a positive integer and a be an integer such that $(a, n) = 1$. It can be explained that a is the quadratic residue of n if it is an integer. $x \in \{1,2,3, \dots, n-1\}$ that makes $x^2 \equiv a(\text{mod } n)$ have a solution, but if $x^2 \not\equiv a(\text{mod } n)$ has no solution, we can say that a is not the quadratic non-residue of n.

Theorem 1. $(a, b, x, y) = (3,2,2,3)$ is a unique solution of the Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers with min $\{a, b, x, y\} > 1$.

Proof see Mihăilescu [7].

Lemma 1. [4] Let a, b, c, d and m are integers such $m \geq 1$.. Then the following statement hold.

1. $a \equiv a(\text{mod } m)$.
2. If $a \equiv b(\text{mod } m)$, then $b \equiv a(\text{mod } m)$.
3. If $a \equiv b(\text{mod } m)$ and $c \equiv d(\text{mod } m)$, then $ac \equiv bd(\text{mod } m)$.
4. If $a \equiv b(\text{mod } m)$ and $c \equiv d(\text{mod } m)$, then $a + c \equiv b + d(\text{mod } m)$.
5. If $a \equiv b(\text{mod } m)$, then $a^k \equiv b^k(\text{mod } m)$ for all integer $k \ngeq 0$.

Lemma 2. Let $z \in \mathbb{Z}$ is a positive integer. Then $z^2 \equiv 0,1,4,9(\text{mod } 12)$.

Proof. Since z is a positive integer, Then $z \equiv r(\text{mod } 12)$ for $r \in \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$.

Case 1: $z \equiv 0(\text{mod } 12)$. Then $z^2 \equiv 0(\text{mod } 12)$.

Case 2: $z \equiv 1(\text{mod } 12)$. Then $z^2 \equiv 1(\text{mod } 12)$.

Case 3: $z \equiv 2(\text{mod } 12)$. Then $z^2 \equiv 4(\text{mod } 12)$.

Case 4: $z \equiv 3(\text{mod } 12)$. Then $z^2 \equiv 9(\text{mod } 12)$.

Case 5: $z \equiv 4(\text{mod } 12)$. Then $z^2 \equiv 4(\text{mod } 12)$.

Case 6: $z \equiv 5(\text{mod } 12)$. Then $z^2 \equiv 1(\text{mod } 12)$.

Case 7: $z \equiv 6(\text{mod } 12)$. Then $z^2 \equiv 0(\text{mod } 12)$.

Case 8: $z \equiv 7(\text{mod } 12)$. Then $z^2 \equiv 1(\text{mod } 12)$.

Case 9: $z \equiv 8(\text{mod } 12)$. Then $z^2 \equiv 4(\text{mod } 12)$.

Case 10: $z \equiv 9(\text{mod } 12)$. Then $z^2 \equiv 9(\text{mod } 12)$.

Case 11: $z \equiv 10(\text{mod } 12)$. Then $z^2 \equiv 4(\text{mod } 12)$.

Case 12: $z \equiv 11 (\text{mod } 12)$. Then $z^2 \equiv 1 (\text{mod } 12)$.

Thus, for every $z \in \mathbb{Z}$ is a positive integer. Then $z^2 \equiv 0,1,4,9 (\text{mod } 12)$.

This completes the proof.

Lemma 3. Let $A \in \mathbb{Z}$, and let $n \in \mathbb{Z}^+$ be a positive integer. Then $(12A+9)^n \equiv 9 (\text{mod } 12)$ for all $n \geq 1$.

Proof. We will prove the lemma using mathematical induction on $n \in \mathbb{Z}^+$.

Base case: For $n = 1$, we have $(12A+9)^1 \equiv 12A+9 \equiv 9 (\text{mod } 12)$,

which satisfies the claim. Inductive hypothesis: Assume that for some $k \in \mathrm{Z}^+$, $(12A+9)^k \equiv 9 (\text{mod } 12)$.

By the inductive hypothesis, $(12A+9)^k \equiv 9 (\text{mod } 12)$, and since $12A+9 \equiv 9 (\text{mod } 12)$,

it follows that $(2A+9)^{k+1} \equiv (9)(9) \equiv 81 \equiv 9 (\text{mod } 12)$.

Thus, by the principle of mathematical induction, $(12A+9)^n \equiv 9 (\text{mod } 12)$, for all $n \in \mathbb{Z}^+$.

This completes the proof.

Lemma 4. Let A be a positive integer. Then for any positive integer n, we have:

$$(12A+5)^n = \begin{cases} 12M+1, \text{if } n \text{ is even.} \\ 12N+5, \text{if } n \text{ is odd} \end{cases}$$

For some positive integers M and N.

Proof. For some positive integers M and N, we divide the proof into two cases:

Case 1. Let n be a positive even number, i.e., $n = 2k$ for some positive integer k.

Since $12A+5 \equiv 5 (\text{mod } 12)$, we have $(12A+5)^{2k} = ((12A+5)^2)^k = (144A^2+120A+25)^k$.

Reducing modulo 12, note that $144A^2 \equiv 0 (\text{mod } 12)$, $120A \equiv 0 (\text{mod } 12)$ and $25 \equiv 1 (\text{mod } 12)$.

Therefore, $(12A+5)^2 \equiv 1 (\text{mod } 12)$. It follows that $(12\mathrm{A}+5)^{2\mathrm{k}} \equiv 1 (\text{mod } 12)$, can be written in the form $12M+1$ for some positive integer M.

Case 2. Let n be a positive odd integer, i.e., $n = 2k+1$ for some positive integer k.

Assume that $(12A+5)^{2k+1} = 12N+5$ for some positive integer N.

Then,
$$\begin{aligned}(12A+5)^{2(k+1)+1} &= (12A+5)^{2k+1}(12A+5)^2 \\ &= (12N+5)(144A^2+120A+25) \\ &= 12(144NA^2+120NA+25N+60A^2+50A+10)+5.\end{aligned}$$

Since $144NA^2+120NA+25N+60A^2+50A+10$ is a positive integer,

by the principle of mathematical induction, it follows that for every positive even integer $n > 0$, we have $(12A+5)^n = 12N+5$ for some positive integer N.

This completes the proof.

Lemma 5. Let n, x and z be non-negative integers. The Diophantine equation $n^x+1 = z^2$ has no solution in non-negative integers when $n \equiv 5 (\text{mod } 12)$.

Proof. Assume, for the sake of contradiction (n, x, z) is a solution in non-negative integers to the equation $n^x + 1 = z^2$. Since $n \equiv 5 (\text{mod } 12)$, it follows that $n^x \equiv 5^x (\text{mod } 12)$. By Lemma 4, we know that $5^x \equiv 1 (\text{mod } 12)$ (if x is even) or $5^x \equiv 5 (\text{mod } 12)$ (if x is odd) Therefore $z^2 \equiv 2 (\text{mod } 12)$ or $z^2 \equiv 6 (\text{mod } 12)$. This contradicts

Lemma 2, which states that no perfect square can be congruent to 2 or 6 $(\text{mod } 12)$. Hence, no solution exists under the given conditions.

This completes the proof.

Lemma 6. Let p be a prime number. Then the Diophantine equation $1 + (2\text{p} - 1)^{\text{y}} = \text{z}^2$ has a unique solution $(p, y, z) = (2,1,2)$ where y, z are non-negative integers.

Proof. Let p be a prime number and y, z are non-negative integers.

We divide the proof into two cases based on the value of y.

Case 1. $y = 0$, then $z^2 = 2$. It is impossible.

Case 2. $y \geq 1$, let p be an odd prime, and let y be an integer. Consider the Diophantine equation, $1 + (2p - 1)^y = z^2$.

We rewrite the equation $(2p - 1)^Y = z^2 - 1 = (z - 1)(z + 1)$. (2)

let $(2p - 1)^k = z - 1$ and $(2p - 1)^{y-k} = z + 1$, $k \geq 0$, substituting these expressions into equation (2), we obtain $(2p - 1)^k((2p - 1)^{y-2k} - 1) = 2$.

Since $2p - 1 \geq 3$, the only possible value for this equation to equal (2) is when $(2p - 1)^k = 1$, which implies $k = 0$. Substituting $k = 0$ into the equation yields $(2p - 1)^y - 1 = 2$, or equivalently $(2p - 1)^y = 3$. Thus, $y = 1, 2p - 1 = 3$, which implies $p = 2$.

Therefore, the solution is $(p, y, z) = (2,1,2)$.

Lemma 7. Let p be an odd prime then $p \equiv 1,3,5,7,11 (\text{mod } 12)$.

Proof. Every integer modulo 12 is congruent to one of the integers in the complete residue system modulo 12, namely $\{0,1,2,3,4,5,6,7,8,9,10,11\}$. We now examine each of these residue classes and determine whether it is possible for a prime number $p \geq 3$ to be congruent to each of them modulo 12.

Case 1. If $p \equiv 0 (\text{mod } 12)$, then p is divisible by 12, and hence divisible by both 3 and 4. Since a prime number has no divisors other than 1 and itself, p cannot be prime.

Case 2. If $p \equiv 2,4,6,8,10 (\text{mod } 12)$, Since 2, 4, 6, 8, and 10 are all divisible by 2, it follows that p is divisible by 2. As a prime number can only have 1 and itself as positive divisors, this implies that p cannot be a prime number.

Case 3. $p \equiv 9 (\text{mod } 12)$, since 9 is divisible by 3, this implies that p is divisible by 3. The only prime divisible by 3 is $p = 3$, but $3 \equiv 3 (\text{mod } 12)$, not 9. Hence, p cannot be congruent to 9 modulo 12.

Case 4. $p \equiv 3 (\text{mod } 12)$, this is only true for $p = 3$, which satisfies both $p \geq 3$ and primality.

Thus, the only congruence classes modulo 12 that a prime number $p \geq 3$ can belong to are $1,3,5,7,11 (\text{mod } 12)$.

This completes the proof.

Lemma 8. Let n is a non-negative integer. If $n \equiv 5 (\text{mod } 12)$, then $n \equiv 2 (\text{mod } 3)$ and $n \equiv 1 (\text{mod } 4)$.

Proof. Assume that n is a non-negative integer.

Such that $n \equiv 5 (\text{mod } 12)$, this means by definition of congruence: $n = 12k + 5$ for some integer k.

It follows that $n = 3(4k + 1) + 2$ and $n = 4(3k + 1) + 1$.

Thus, $n \equiv 2(\mathrm{mod}\ 3)$ and $n \equiv 1(\mathrm{mod}\ 4)$.

Main Results

Throughout our main results part, let p be an odd prime and let n, x, y, z be non-negative integers such that $n \equiv 5(\mathrm{mod}\ 12)$. We investigate the solutions of the Diophantine equation $n^x + (2p-1)^y = z^2$ by considering various cases based on the values of p and n^x. To ensure the logical continuity of the proofs, we present several supporting lemmas 4 and lemma 9 in this section, which will be used as auxiliary results in the main proofs that follow.

Lemma 9. If p be an odd prime such that $p \equiv 1,5,7,11(\mathrm{mod}\ 12)$. Then for any positive integer y, the following congruence holds $(2p-1)^y \equiv 1(\mathrm{mod}\ 12)$ or , $(2p-1)^y \equiv 9(\mathrm{mod}\ 12)$.

Proof. Let us consider the possible congruence classes of odd primes modulo 12. Since p is an odd prime, it cannot be divisible by 2 or 3, so the only possible values of $p\ (\mathrm{mod} 12)$ are $p \equiv 1,5,7,11(\mathrm{mod}\ 12)$.

We will compute $2p-1$ modulo 12 for each of these congruence classes.

Case 1. $p \equiv 1(\mathrm{mod}\ 12)$ then $(2p-1)^y \equiv 1(\mathrm{mod}\ 12)$.

Case 2. $p \equiv 5(\mathrm{mod}\ 12)$ then $(2p-1)^y \equiv 9^y \equiv 9(\mathrm{mod}\ 12)$.

Case 3. $p \equiv 7(\mathrm{mod}\ 12)$ then $(2p-1)^y \equiv 13^y \equiv 1(\mathrm{mod}\ 12)$.

Case 4. $p \equiv 11(\mathrm{mod}\ 12)$ then $(2p-1)^y \equiv 21^y \equiv 9^y \equiv 9(\mathrm{mod}\ 12)$. For all $y \geq 1$ (this can be proved by induction, if $y = k, 9^y = 9^k \equiv 9(\mathrm{mod}\ 12)$, then $9^{k+1} \equiv 9 \cdot 9 \equiv 81 \equiv 9(\mathrm{mod}\ 12)$).

In all four cases, $(2p-1)^y \equiv 1(\mathrm{mod}\ 12)$ or $(2p-1)^y \equiv 9^y \equiv 9(\mathrm{mod}\ 12)$.

Thus, $(2p-1)^y \equiv 1(\mathrm{mod}\ 12)$ or $(2p-1)^y \equiv 9^y \equiv 9(\mathrm{mod}\ 12)$. For all $y \in \mathbb{Z} > 0$, completing the proof.

Lemma 10. For any integer A and non-negative integer n, the following congruence holds:

$$(12A+3)^x \equiv \begin{cases} 1(\mathrm{mod}\ 12), x = 0, \\ 3(\mathrm{mod}\ 12), \text{if } x \text{ is odd}, \\ 9(\mathrm{mod}\ 12), \text{if } x \text{ is even}, x \geq 2 \end{cases}$$

Proof. Since $12A + 3 \equiv 3(\mathrm{mod}\ 12)$.

Hence $(12A+3)^x \equiv 3^x(\mathrm{mod}\ 12)$, and it suffices to determine the residue of 3^x modulo 12.

For $x = 0$, clearly $3^0 = 1 \equiv 1(\mathrm{mod}\ 12)$.

For $x = 1$, we have $3^1 = 3 \equiv 3(\mathrm{mod}\ 12)$, establishing the initial case.

Assume now that for some $n \geq 1$,

$$3^n \equiv \begin{cases} 3(\mathrm{mod}\ 12), n \text{ is odd}, \\ 9(\mathrm{mod}\ 12), n \text{ is even}. \end{cases}$$

Multiplying both sides by 3 gives

$$3^{n+1} \equiv 3 \cdot 3^n\ (\mathrm{mod}\ 12).$$

If n is odd, then $3^n \equiv 3$, so $3^{n+1} \equiv 9\ (\mathrm{mod}\ 12)$, as required for an even exponent.

If n is even, then $3^n \equiv 9$, so $3^{n+1} \equiv 27 \equiv 3(\mathrm{mod}\ 12)$, as required for an odd exponent.

Thus, the statement holds for $n+1$ whenever it holds for n. By the principle of mathematical induction, the claim is established for all $x \geq 0$.

Theorem 2. Let p be prime number and let n, x, y, z are non-negative integers, such that $n \equiv 5(\text{mod } 12)$ and $\gcd(n, 2p-1) = 1.$ Then the exponential Diophantine equation $n^x + (2p-1)^y = z^2$ has the unique solution $(n, p, x, y, z) = (n, 2,0,1,2).$

Proof. Let p be a prime number and n, x, y, z are non-negative integers, when $n \equiv 5(\text{mod } 12).$

We divide the proof into 4 cases:

Case 1. $x = 0$ and $y = 0,$ the equation becomes $z^2 = 2.$ It is impossible.

Case 2. $x = 0, y \geq 1.$ By Lemma 6, the solution to the equation (1) is $(n, p, x, y, z) = (n, 2, 0, 1, 2).$

Case 3. $y = 0$ and $x \geq 1.$ By Lemma 5, there is no solution.

Case 4. $x \geq 1, y \geq 1.$

Case 4.1 x is odd and $y \geq 1$

Let $p = 2.$ Then $(2p-1)^y = 3^y.$ From Lemma 10, we know that for any integer A and non-negative integer $y.$

$$(12A+3)^y \equiv \begin{cases} 3 \ (\text{mod } 12), & \text{if y is odd}, \\ 9 \ (\text{mod } 12), & \text{if y} \geq 2 \text{ and even}. \end{cases}$$

Since $n \equiv 5(\text{mod } 12),$ by Lemma 4, we obtain $n^x \equiv 5(\text{mod } 12).$

Therefore, $z^2 = n^x + 3^y \equiv 5 + 3 = 8(\text{mod } 12)$ or $z^2 = n^x + 3^y \equiv 5 + 9 = 14 \equiv 2(\text{mod } 12).$ But from Lemma 2, a square modulo 12 must be in the set $\{0, 1, 4, 9\}.$ So $z^2 \equiv 8 \text{ or } 2(\text{mod } 12)$ is a contradiction.

Therefore, the equation admits no non-negative integer solution under these conditions.

Case 4.2 x is even and $y \geq 1$

Case 4.2.1 Let $p = 2,$

from equation (1), we have $n^x + 3^y = z^2.$ Suppose $x = 2f,$ where f is a non-negative integer.

Then $n^x = (n^f)^2.$ We consider the Diophantine $(n^f)^2 + 3^y = z^2.$ We can rewrite this as a difference of squares

$$3^y = z^2 - (n^f)^2 = (z + n^f)(z - n^f).$$

Let $3^{y-h} = z + n^f$ and $3^h = z - n^f$, where $y > h$ and y, h are non-negative integers.

This yields $3^h[3^{y-2h} - 1] = 2 \cdot n^f.$ Since $y > 2h,$ it follows that $3^{y-2h} - 1$ is a positive integer.

We now consider three cases for h:

Case (i): $h = 1.$

Therefore, 3 must divide $2n^f,$ i.e., $3|2n^f.$

However, sine $\gcd(n, 3) = 1$ and 1 and 3 does not divide 2 (because 2 is a prime number not divisible by 3), it follows that $3 \nmid 2\text{n}^\text{f}.$ Which is a contradiction and cannot hold.

Case (ii): $h \geq 2.$

In this case, the equation becomes $3^h(3^{y-2h} - 1) = 2n^f.$

Since $h \geq 2,$ it follows that $3^2 = 9|3^h,$ and hence $3|2n^f.$

This implies that $3|2n^f,$ which is impossible because $\gcd(3, 2n^f) = 1$ under the assumption $\gcd(n, 3) = 1.$

Therefore, this case leads to a contradiction.

Case (iii): $h = 0.$

The equation simplifies to $3^y - 1 = 2n^f.$

We analyze this congruence modulo 3. Note that:

$3^y \equiv 0(\bmod 3)$, it follows that $3^y - 1 \equiv -1(\bmod 3)$.

Thus, $2n^f \equiv -1(\bmod 3)$, Which implies that $3|2n^f + 1$.

However, since $\gcd(n,3) = 1$, it follows that $3 \nmid 2\mathrm{n}^{\mathrm{f}}$, leading to a contradiction.

Alternatively, if y is even, then $3^y \equiv 1(\bmod 4)$, so that

$3^y - 1 \equiv 0(\bmod 4)$, which implies that $2n^f \equiv 0(\bmod 4)$.

This implies $n^f \equiv 0(\bmod 2)$, i.e., n is even.

But this contradicts the assumption that n is odd and $\gcd(n,2) = 1$.

Hence, no solution exists in the case either.

Case 4.2.2 Let $p \geq 3$,

from Lemma 9, $(2p-1)^y \equiv 1(\bmod 12)$ or $(2p-1)^y \equiv 9(\bmod 12)$, from Lemma 4, we have $n^x \equiv 1(\bmod 12)$. Thus, $z^2 = n^x + (2p-1)^y \equiv 6(\bmod 12)$ or $z^2 \equiv 2(\bmod 12)$, which contradicts Lemma 2. Therefore, in this case, there is no solution.

This completes the proof.

Corollary 1. Let n be a positive number such that $n \equiv 5(\bmod 12)$. Then the Diophantine equation $n^x + (2p-1)^y = u^{2w}$ has a unique solution $(n,p,x,y,u) = (n,2,0,1,2)$ where p be prime number and x,y,u,w are non-negative integers.

Proof. Let p be an odd prime and x,y,u are non-negative integers. Suppose that $u^{2w} = z^2$,

Then Diophantine equation $n^x + (2p-1)^y = u^{2w} = z^2$ has a unique solution $(n,p,x,y,u) = (n,2,0,1,2)$.

By Theorem 2.

Corollary 2. The Diophantine equation $5^{2x} + 13^y = z^2$ has no non-negative integer solutions, where x, y and z are non-negative integers.

Proof. By Lemma 4, $5^{2x} \equiv 1(\bmod 12)$ and by Lemma 9, $(2p-1)^y \equiv 1,9(\bmod 12)$.

Therefore, $z^2 \equiv 2\,,10(\bmod 12)$, which contradicts Lemma 2. Hence, by Theorem 2, the Diophantine equation $5^{2x} + 13^y = z^2$ has no non-negative integer solution.

Corollary 3. The Diophantine equation $29^x + (2p-1)^{2m+1} = k^{2t+2}$, where p be prime, has no non-negative integer solution, where m, t, x, y and k are non-negative integers.

Proof. Let p be prime number.

Suppose that $y = 2m+1$, $z = k^{t+1}$, so $29^x + (2p-1)^y = k^{2t+2} = z^2$. From Theorem 2,

Then the Diophantine equation $29^x + (2p-1)^{2m+1} = k^{2t+2}$ has no non-negative integer solutions.

Corollary 4. The Diophantine equation $17^x + (2p-1)^y = h^{2t}$, where p be prime, has no non-negative integer solution, where t, x, y and h are non-negative integers.

Proof. Let p be prime number.

Suppose that $z = h^t$, so $17^x + (2p-1)^y = h^{2t} = z^2$. From Theorem 2,

then the Diophantine equation $17^x + (2p-1)^y = h^{2t}$ has no non-negative integer solution.

Discussion

This study provides an in-depth analysis of the exponential Diophantine equation

$$n^x + (2p-1)^y = z^2,$$

This study provides an in-depth analysis of the exponential. Diophantine equation $n^x + (2p-1)^y = z^2$, under the modular condition $n \equiv 5(\text{mod}12)$ and $\gcd(n, 2p-1) = 1$. The results indicate that the the equation admits a single non-trivial solution when $p = 2$, reflecting the rarity of non-negative integer solutions under these constraints. Moreover, it is confirmed that for other values of p, no non-negative integer solutions exist, and z cannot be a perfect square outside of this solution. These findings align with previous studies on related Diophantine equations. For instance, Kumar et al. [6] and Fernando [5] showed that certain non-linear Diophantine equations with prime parameters admit no positive integer solutions, while Viriyapong and Sroysang [9, 12] found that unique solutions exist under specific conditions. Despite these advances, several questions remain open regarding the behavior of such equations under broader conditions and more complex structures. To address these gaps, future research may focus on:

1. Extending the analysis to other modular conditions, such as $n \equiv 1(\text{mod } 4)$ or $n \equiv 3(\text{mod } 8)$, to examine whether the uniqueness of solutions persists or varies under different arithmetic constraints.

2. Exploring additional families of exponential Diophantine equations, particularly those involving multiple exponential terms or higher-degree exponents, to gain deeper insight into the general structure of solutions and to potentially formulate new theoretical conjectures.

3. Investigating computational methods using modern mathematical software such as SageMath, Python (with SymPy or NumPy), or Mathematica to empirically verify the uniqueness of solutions over broader parameter ranges, especially in cases where theoretical proofs are difficult to obtain.

Recommendations

Based on the results and the scope of this study, the researcher offers several recommendations for future research and educational applications.

1. Broaden the investigation of exponential Diophantine equations by exploring more diverse modular conditions and wider parameter ranges. This could help uncover new solution structures or unexpected behaviors. For instance, future studies may examine equations of the form $a^x + b^y \equiv c(\text{mod } m)$ across varying values of mmm, to analyze patterns of solvability and periodicity under modular constraints.

2. Utilize computational techniques to support theoretical work and explore related number-theoretic equations such as Pell-type equations or exponential forms involving recurrence relations. Performing exhaustive searches for small integer parameters (e.g., $a, b, x, y \leq 1000$) may reveal special or exceptional solutions that can guide the formulation of more general conjectures or proofs.

3. Incorporate these methods into mathematics education, particularly in teaching number theory and problem-solving strategies. Computational tools, such as Python, PARI/GP, or SageMath, could be used to help

students visualize and analyze Diophantine equations, thereby strengthening logical reasoning and conceptual understanding through hands-on experimentation.

Conclusions

This study examined the exponential Diophantine equation $n^x + (2p-1)^y = z^2$, where p is a prime number and n, x, y, z are non-negative integers, under the conditions $n \equiv 5 (\mathrm{mod} 12)$ and $\gcd(n, 2p-1) = 1.$ The analysis shows that the equation has a unique non-trivial solution given by

$(n, p, x, y, z) = (n, 2, 0, 1, 2).$

No other prime numbers yield non-negative integer solutions, and z^2cannot be a perfect square in any other case. These results highlight the rarity of solutions for this type of equation and provide a framework for exploring more complex cases. They also offer clear and illustrative examples for teaching number theory and Diophantine equations.

References

1. Burshtein N. On the Diophantine equation $\mathrm{p^x + (p+5)^y = z^2}$. Ann Pure Appl Math 2020;9(1):41-4.
2. Catalan E. Note extradite dune letter adressee a l'editeur. J Reine Angew Math 1844;27:192.
3. Chotchaistith S. On the Diophantine equation $\mathrm{4^x + p^y = z^2}$ where p is a prime number. Am J Math Sci. 2012;1:191–3.
4. Burton DM. Elementary Number Theory. 6th ed. Singapore: McGraw-Hill; 2007.
5. Fernando N. On the solvability of the Diophantine equation $\mathrm{p^x + (p+8)^y = z^2}$. when $\mathrm{p > 3}$ and $\mathrm{p + 8}$ are primes. Ann Pure Appl Math 2018;18(1):9–13.
6. Kumar S, Gupta S, Kishan H. On the non-linear Diophantine equation $\mathrm{p^x + (p+6)^y = z^2}$. Ann Pure Appl Math 2018;8(1):125–8.
7. Mihăilescu P. Primary cycolotomic units and a proof of Catalan's conjecture. J Reine Angew Math. 2004;27:167–95.
8. Pakapongpun T, Chattae C. On the Diophantine equation $\mathrm{p^x + 7^y = z^2}$ where p is prime and $\mathrm{x, y, z}$ are non-negative integers. Int J Math Comput Sci 2022;17(4):1535–40.
9. Sroysang B. The Diophantine equation $\mathrm{3^x + 5^y = z^2}$. Int J Pure Appl Math 2012;81(4):605–8.
10. Sroysang B. On the Diophantine equation $\mathrm{7^x + 31^y = z^2}$. Int J Pure Appl Math 2014;92(1):109–12.
11. Suvarnamani A. Solutions of the Diophantine equation $\mathrm{2^x + q^y = z^2}$. Int J Pure Appl Math 2011;1(3):1415–9.
12. Viriyapong N, Viriyapong C. On a Diophantine equation $\mathrm{n^x + 13^y = z^2}$ where $\mathrm{n \equiv 2 (mod\ 39)}$ and $\mathrm{n+1}$ is not a square number. J Appl Math 2021;29(1):33–41.
13. Tanjai W, Chubthaisong C. On the Diophantine equation $\mathrm{3^x + p^y = z^2}$ where $\mathrm{p \equiv 2\ (mod\ 3)}$. WSEAS Trans Math 2020:245–56.
14. Tadee S, Siraworakun A. Non-existence of positive integer solutions of the Diophantine equation $\mathrm{p^x + (p+2q)^y = z^2}$ where $\mathrm{p, q}$ and $\mathrm{p + 2q}$ are prime numbers. Eur J Pure Appl Math 2023;16(2): 724–35.