Application of the Simple Equation Method with Jumarie's Modified Riemann–Liouville Derivative to Space–Time Fractional Nonlinear mBBM and ZKBBM

Orapan Janngam[1], Supinan Janma[1] and Jiraporn Sanjun[2*]

[1]Faculty of Sciences and Agricultural Technology, Rajamangala University of Technology Lanna, Lampang 52000

[2]Department of Mathematics, Faculty of Science and Technology, Suratthani Rajabhat University, Suratthani 84100

[*]**Corresponding author:** Jiraporn.san@sru.ac.th

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Abstract

This paper employs the Simple Equation (SE) method in conjunction with Jumarie's modified Riemann–Liouville fractional derivative to derive exact solutions for the space–time fractional modified Benjamin–Bona–Mahony (mBBM) and Zakharov–Kuznetsov Benjamin–Bona–Mahony (ZKBBM) equations. The obtained exponential-type solutions describe kink-shaped traveling waves, which are illustrated through 2D, 3D, and contour plots using suitable parameters. The results confirm the efficiency and reliability of the proposed method as a robust analytical technique for deriving traveling wave solutions in nonlinear fractional models encountered in science and engineering. This study extends the SE method within the framework of Jumarie's modified Riemann–Liouville fractional derivative, broadening its applicability to space–time fractional systems. The approach establishes a new analytical framework for fractional traveling wave solutions, and the derived results uncover additional dynamical behaviors of the mBBM and ZKBBM equations, demonstrating both the originality and effectiveness of the proposed method.

Keywords: Space-time fractional mBBM equation, Space-time fractional ZKBBM equation, Simple equation method, Jumarie's modified Riemann-Liouville fractional derivative, Kink wave

Introduction

In applied and engineering mathematics, nonlinear fractional partial differential equations (PDEs) have gained significant attention as effective tools for modeling complex real-world phenomena. These equations are widely used in fields such as fluid mechanics, chemical kinetics, solid-state physics, and plasma wave dynamics, as well as in air pollutant dispersion and chemical physics. In recent years, various powerful analytical methods have been developed to solve nonlinear fractional PDEs, such as the Kudryashov method [1], the G'/G-expansion method [2-3], the simple equation method [4-5], the modified simple equation method [6], the Riccati sub-equation method [7-8], the modified extended tanh-function method [9], the improved modified Sardar sub-equation method [10], the improved Riccati equation method [11-12], Riccati-Bernoulli sub-ODE [13], the first integral method [14], the unified method [15], etc.

In this work, we employ Jumarie's modified Riemann–Liouville fractional derivative (JMRL derivative) in combination with the simple equation method to obtain analytic solutions for the space-time fractional modified Benjamin-Bona-Mahony (mBBM) equation and the space-time fractional Zakharov-Kuznetsov Benjamin-Bona-Mahony (ZKBBM) equation. Some fundamental properties of Jumarie's fractional derivative used in this study are presented below.

The definition of the JMRL derivative [16] of order ϕ is expressed as:

$$f^{(\phi)}(x)=\lim_{h\to 0}\left(\frac{\sum_{k=0}^{\infty}(-1)^k\binom{\phi}{k}f\left(x+(\phi-k)h\right)}{h^{\phi}}\right),\quad \phi\in\mathbb{R},0<\phi\le 1, \tag{1.1}$$

which can be written as

$$D_x^{\phi}f(x)=\begin{cases}\dfrac{1}{\Gamma(-\phi)}\displaystyle\int_0^x (x-\beta)^{(-\phi-1)}[f(\beta)-f(0)]d\beta, & \text{if } \phi<0\\ \dfrac{1}{\Gamma(1-\phi)}\dfrac{d}{dx}\displaystyle\int_0^x (x-\beta)^{(-\phi)}[f(\beta)-f(0)]d\beta, & \text{if } 0<\phi<1\\ [f^{(\phi-n)}(x)]^{(n)} \quad \text{if } n\le\phi<n+1, n\ge 1, & \end{cases} \tag{1.2}$$

where β denotes the dummy variable of integration.

The definition satisfies the properties of the modified Riemann-Liouville derivative [17] as follows:

$$\begin{aligned} D_x^{\phi}x^{\gamma}&=\frac{\Gamma(\gamma+1)x^{\gamma-\phi}}{\Gamma(\gamma+1-\phi)},\quad \gamma>0,\quad x>0,\\ D_x^{\phi}(f(x)g(x))&=g(x)D_x^{\phi}f(x)+f(x)D_x^{\phi}g(x),\\ D_x^{\phi}f(g(x))&=f_g'[g(x)]D_x^{\phi}g(x)=D_g^{\phi}f[g(x)](g'(x))^{\phi}. \end{aligned} \tag{1.3}$$

The simple equation (SE) method, introduced by Nikolai Kudryashov [18], is based on two main ideas: using general solutions of simple nonlinear differential equations and accounting for all possible singularities in the equations. The SE method has been successfully used to find exact solutions for various nonlinear equations. These include the (2+1)-dimensional breaking soliton equation and the modified generalized Vakhnenko equation (2016) [19], the nonlinear space-time fractional Estevez-Mansfield-Clarkson (EMC) and Ablowitz-Kaup-Newell-Segur (AKNS) equations [20], and the (1+1)-dimensional dispersive modified Benjamin-Bona-Mahony (DMBBM) equation as well as the (2+1)-dimensional cubic Klein-Gordon (cKG) equation (2022) [4].

The SE method is particularly advantageous for solving fractional space-time PDEs because it provides exact and straightforward solutions, even for complex nonlinear equations where other analytical or numerical techniques may struggle. Unlike some methods that cannot handle fractional operators effectively or produce cumbersome results, the SE method directly yields clear kink- or exponential-type solutions, demonstrating its suitability for addressing challenges in the mBBM and ZKBBM equations.

In particular, combining the SE method with Jumarie's modified Riemann–Liouville fractional derivative allows for a more efficient handling of fractional-order derivatives, which are otherwise challenging to treat analytically. This integration reduces limitations associated with traditional SE method applications by enabling the derivation of exact analytical solutions in a more systematic and compact form.

Algorithm of SE method

In this section, we present a systematic approach for obtaining traveling wave solutions of nonlinear fractional partial differential equations (PDEs) using the simple equation (SE) method in conjunction with the Bernoulli equation [4, 20]. The general form of a fractional PDE considered in this study is given by:

$$G\left(u, D_x^{\phi}u, D_t^{\phi}u, D_x^{2\phi}u, D_t^{2\phi}u, D_t^{\phi}D_x^{\phi}u, \dots\right) = 0,\ t > 0,\ 0 < \phi \le 1, \tag{2.1}$$

Here, $u(x,t)$ is an unknown function and G is a polynomial of $u(x,t)$. The main steps of the SE method [4, 20] are as follows:

Step 1. Wave transformation

We suppose that

$$u(x,t) = u(\beta),\ \beta = \frac{kx^{\phi}}{\Gamma(\phi+1)} - \frac{\omega t^{\phi}}{\Gamma(\phi+1)}, \tag{2.2}$$

Here, β represents a general traveling wave function, and ω is the wave velocity constant. When $\omega = 0$, the solution corresponds to a stationary wave. If $\omega > 0$, the wave propagates in the positive direction, whereas for $\omega < 0$, it travels in the negative direction.

To transform the fractional partial differential equation into an ordinary differential equation, we apply the chain rule for fractional derivatives to calculate the derivatives of $u(x,t)$ with respect to x and t. First, we compute the fractional derivatives of β with respect to x and t:

$$D_t^{\phi}\beta = D_t^{\phi}\left(\frac{kx^{\phi}}{\Gamma(\phi+1)} - \frac{\omega t^{\phi}}{\Gamma(\phi+1)}\right) = -\frac{\omega}{\Gamma(\phi+1)}D_t^{\phi}\left(t^{\phi}\right) = -\omega,$$

$$D_x^{\phi}\beta = D_x^{\phi}\left(\frac{kx^{\phi}}{\Gamma(\phi+1)} - \frac{\omega t^{\phi}}{\Gamma(\phi+1)}\right) = \frac{k}{\Gamma(\phi+1)}D_x^{\phi}\left(x^{\phi}\right) = k,$$

using the chain rule, the fractional derivatives of $u(x,t)$ are then given by:

$$D_t^{\phi}u = D_t^{\phi}u(\beta) = \frac{du}{d\beta}D_t^{\phi}\beta = -\omega\frac{du}{d\beta},$$

$$D_x^{\phi}u = D_x^{\phi}u(\beta) = \frac{du}{d\beta}D_x^{\phi}\beta = k\frac{du}{d\beta}.$$

By applying the traveling wave transformation described above, we obtain the following ordinary differential equation

$$Q(u, ku', -\omega u', k^2 u'', \ldots) = 0, \quad (2.3)$$

where Q represents a polynomial involving $u(\beta)$ and its derivatives, the prime symbol $(')$ indicates differentiation with respect to β.

Applying this wave transformation, the partial differential equation (2.3) is reduced to an ordinary differential equation (ODE) in $u(\beta)$. By performing algebraic simplifications (and integrating once if necessary), the resulting ODE can be expressed in the form of a well-known nonlinear Bernoulli equation:

$$u' + pu = qu^n,$$

Where p, q and n are constants, and the Bernoulli equation can be solved using standard methods [21].

Step 2. Reduced ODE Form

Assume that Eq. (2.3) admits the following formal solution:

$$u(\beta) = \sum_{j=0}^{M} a_j F^j(\beta), \quad (2.4)$$

where $a_j (j = 0, 1, 2, \ldots, M)$ is a constant parameter to be determined later. The functions involved satisfy simple ordinary differential equations (ODEs). In this study, we use the well-known nonlinear Bernoulli equation, whose solutions can be expressed in terms of elementary functions. Specifically, for the Bernoulli equation:

$$F'(\beta) = pF(\beta) + qF^2(\beta). \quad (2.5)$$

Step 3. Balance Principle

The balance number M in Eq. (2.4) is determined by equating the highest-order derivative term with the nonlinear terms present in the equation.

Step 4. Solution attainment

The general solutions of the simple Eq. (2.4) are given as follows [4]:

Case I: if $p > 0$ and $q < 0$, we get

$$F(\beta) = \frac{pe^{p(\beta+\beta_0)}}{1 - qe^{p(\beta+\beta_0)}}, \quad (2.6)$$

where β_0 denotes a constant arising from the integration process.

Case II: if $p < 0$ and $q > 0$, we get

$$F(\beta) = -\frac{pe^{p(\beta+\beta_0)}}{1 + qe^{p(\beta+\beta_0)}}, \quad (2.7)$$

where β_0 denotes a constant arising from the integration process.

Application of the method

We investigate the traveling wave solutions of the nonlinear space-time fractional modified Benjamin-Bona-Mahony (mBBM) equation and the space-time fractional Zakharov-Kuznetsov Benjamin-Bona-Mahony (ZKBBM) equation.

The space-time fractional mBBM equation

The nonlinear space-time fractional mBBM equation [22] is

$$D_t^{\phi}u + D_x^{\phi}u - \delta u^2 D_x^{\phi}u + D_x^{3\phi}u = 0,\ \ t>0,\ \ 0<\phi\le 1, \tag{3.1.1}$$

where δ is a nonzero constant. Using the traveling wave variable $\beta = \frac{kx^{\phi}}{\Gamma(\phi+1)} - \frac{\omega t^{\phi}}{\Gamma(\phi+1)}$.

By applying the transformation to Eq. (3.1.1), the equation is reduced to an ordinary differential equation (ODE):

$$(k-\omega)u' - \delta k u^2 u' + k^3 u''' = 0. \tag{3.1.2}$$

Integrating Eq. (3.1.2) with zero constants, we get

$$(k-\omega)u - \frac{\delta k u^3}{3} + k^3 u'' = 0. \tag{3.1.3}$$

Using the SE method, the solution is assumed in the form of Eq. (2.4). Next, we balance the highest-order derivative terms with the leading nonlinear terms in Eq. (3.1.3). Consequently, the solution of Eq. (3.1.3) is obtained as follows:

$$u(\beta) = a_0 + a_1 F, \tag{3.1.4}$$

where F satisfies Eq. (2.5), Therefore, the following is an expression for u'' and u^3:

$$\begin{aligned} u'' &= a_1 p^2 F + 3a_1 pq F^2 + 2a_1 q^2 F^3, \\ u^3 &= a_0^3 + 3a_0^2 a_1 F + 3a_0 a_1^2 F + a_1^3 F^3. \end{aligned} \tag{3.1.5}$$

By substituting Eqs. (3.1.4) and (3.1.5) into Eq. (3.1.3), all terms with the same power of $F(\beta)$ were collected, and their coefficients were then set to zero, where $j \ge 0,$ yields yields

$$\begin{aligned} F^0(\beta):&\quad (k-\omega)a_0 - \frac{\delta k}{3}a_0^3 = 0, \\ F^1(\beta):&\quad (k-\omega)a_1 - \delta k a_0^2 a_1 + k^3 a_1 p^2 = 0, \\ F^2(\beta):&\quad 3k^3 a_1 pq - \delta k a_0 a_1^2 = 0, \\ F^3(\beta):&\quad 2k^3 a_1 q^2 - \frac{\delta k}{3}a_1^3 = 0. \end{aligned} \tag{3.1.6}$$

Solving this system of algebraic equations (Eq. 3.1.6) gives:

$$a_0 = \sqrt{\frac{3(k-\omega)}{\delta k}},\ a_1 = \sqrt{\frac{6k^2q^2}{\delta}} \text{ and } \omega = \frac{2k-k^3p^2}{2}, \tag{3.1.7}$$

or

$$a_0 = -\sqrt{\frac{3(k-\omega)}{\delta k}},\ a_1 = -\sqrt{\frac{6k^2q^2}{\delta}} \text{ and } \omega = \frac{2k-k^3p^2}{2}, \tag{3.1.8}$$

By substituting Eqs. (3.1.7) and (3.1.8) into Eq. (3.1.4) and using the general solutions of the Bernoulli equations (2.6) and (2.7), we obtain the exact traveling wave solutions of the nonlinear space-time fractional modified mBBM equation, expressed as follows:

Case I: if $p>0$ and $q<0$, we get

$$u_1(x,t) = k\sqrt{\frac{6}{\delta}}\left(\frac{p}{2}+q\left(\frac{pe^{p(\beta+\beta_0)}}{1-qe^{p(\beta+\beta_0)}}\right)\right), \tag{3.1.9}$$

$$u_2(x,t) = -k\sqrt{\frac{6}{\delta}}\left(\frac{p}{2}+q\left(\frac{pe^{p(\beta+\beta_0)}}{1-qe^{p(\beta+\beta_0)}}\right)\right). \tag{3.1.10}$$

Case II: if $p<0$ and $q>0$, we get

$$u_3(x,t) = k\sqrt{\frac{6}{\delta}}\left(\frac{p}{2}-q\left(\frac{pe^{p(\beta+\beta_0)}}{1+qe^{p(\beta+\beta_0)}}\right)\right), \tag{3.1.11}$$

$$u_4(x,t) = -k\sqrt{\frac{6}{\delta}}\left(\frac{p}{2}-q\left(\frac{pe^{p(\beta+\beta_0)}}{1+qe^{p(\beta+\beta_0)}}\right)\right), \tag{3.1.12}$$

where $\beta = \frac{kx^{\phi}}{\Gamma(\phi+1)} - \phi\frac{\omega t^{\phi}}{\Gamma(\alpha+1)}$ and β_0 denotes a constant arising from the integration process.

The space-time fractional ZKBBM equation

The nonlinear space-time fractional Zakharov-Kuznetsov Benjamin-Bona-Mahony (ZKBBM) equation is given by [23]:

$$D_t^{\phi}u + D_x^{\phi}u - 2auD_x^{\phi}u - bD_t^{\phi}\left(D_x^{2\phi}u\right) = 0,\ t>0,\ 0<\phi\le 1, \tag{3.2.1}$$

where a and b are arbitrary constants. By employing the traveling wave variable

$\beta = \frac{kx^{\phi}}{\Gamma(\phi+1)} - \frac{\omega t^{\phi}}{\Gamma(\phi+1)}$, the equation is reduced to an ordinary differential equation (ODE).

Substituting the transformation into Eq. (3.2.1) yields the following result:

$$(k-\omega)u' - 2akuu' + b\omega k^2 u''' = 0. \tag{3.2.2}$$

Integrating Eq. (3.2.2) with zero constants gives:

$$(k-\omega)u - aku^2 + b\omega k^2 u'' = 0. \tag{3.2.3}$$

By balancing the highest-order derivatives and nonlinear terms, we obtain $M = 2.$ Thus, Eq. (2.4) becomes:

$$u(\beta) = a_0 + a_1 F + a_2 F^2, \tag{3.2.4}$$

Here, F satisfies Eq. (2.5). Therefore, the expressions for u'' and u^2 are:

$$\begin{aligned} u'' &= a_1 p^2 F + 3a_1 pqF^2 + 2a_1 q^2 F^3, \\ u^2 &= a_0^2 + 2a_0 a_1 F + 2a_0 a_2 F^2 + a_1^2 F^2 + 2a_1 a_2 F^3 + a_2^2 F^4. \end{aligned} \tag{3.2.5}$$

Substituting Eq. (3.2.4) and (3.2.5) into Eq. (3.2.3), all terms with the same power of $F(\beta)$ were collected, and their coefficients were then set to zero, where $j \geq 0,$ yields

$$\begin{aligned} F^0(\beta): &\quad (k-\omega)a_0 - aa_0^2 k = 0, \\ F^1(\beta): &\quad (k-\omega)a_1 - 2aa_0 a_1 k + a_1 bp^2 \omega k^2 = 0, \\ F^2(\beta): &\quad (k-\omega)a_2 - 2aa_0 a_2 k - aa_1^2 k + 3a_1 bpq\omega k^2 + 4a_2 bp^2 \omega k^2 = 0, \\ F^3(\beta): &\quad 2a_1 bq^2 \omega k^2 - 2aa_1 a_2 k + 10a_2 bpq\omega k^2 = 0, \\ F^4(\beta): &\quad 6a_2 bq^2 \omega k^2 - aa_2^2 k = 0. \end{aligned} \tag{3.2.6}$$

Solving this system of algebraic equations (Eq. 3.2.6) gives:

$$a_0 = 0,\ a_1 = \frac{6bpq\omega k}{a},\ a_2 = \frac{6bq^2\omega k}{a} \text{ and } \omega = \frac{k}{1 - bp^2 k^2}, \tag{3.2.7}$$

or

$$a_0 = \frac{k-\omega}{ak},\ a_1 = \frac{6bpq\omega k}{a},\ a_2 = \frac{6bq^2\omega k}{a} \text{ and } \omega = \frac{k}{1 + bp^2 k^2}, \tag{3.2.8}$$

By substituting Eqs. (3.2.7) and (3.2.8) into Eq. (3.2.4) and using the general solutions of the Bernoulli equations (2.6) and (2.7), we obtain the exact traveling wave solutions of the nonlinear space-time fractional ZKBBM equation, expressed as follows:

Case I: if $p > 0$ and $q < 0$, we get

$$u_5(x,t) = \frac{6bqk^2}{a(1 - bp^2 k^2)} \left(p \left(\frac{pe^{p(\beta+\beta_0)}}{1 - qe^{p(\beta+\beta_0)}} \right)^2 + q \left(\frac{pe^{p(\beta+\beta_0)}}{1 - qe^{p(\beta+\beta_0)}} \right) \right), \tag{3.2.9}$$

$$u_6(x,t) = \frac{bp^2 k^2}{a(1 + bp^2 k^2)} + \frac{6bqk^2}{a(1 + bp^2 k^2)} \left(p \left(\frac{pe^{p(\beta+\beta_0)}}{1 - qe^{p(\beta+\beta_0)}} \right)^2 + q \left(\frac{pe^{p(\beta+\beta_0)}}{1 - qe^{p(\beta+\beta_0)}} \right) \right). \tag{3.2.10}$$

Case II: if $p < 0$ and $q > 0$, we get

$$u_7(x,t) = \frac{6bqk^2}{a(1 - bc^2 k^2)} \left(p \left(\frac{pe^{p(\beta+\beta_0)}}{1 + qe^{p(\beta+\beta_0)}} \right)^2 - q \left(\frac{pe^{p(\beta+\beta_0)}}{1 + qe^{p(\beta+\beta_0)}} \right) \right), \tag{3.2.11}$$

$$u_8(x,t)=\frac{bp^2k^2}{a(1+bp^2k^2)}+\frac{6bqk^2}{a(1+bp^2k^2)}\left(p\left(\frac{pe^{p(\beta+\beta_0)}}{1+qe^{p(\beta+\beta_0)}}\right)^2-q\left(\frac{pe^{p(\beta+\beta_0)}}{1+qe^{p(\beta+\beta_0)}}\right)\right), \quad (3.2.12)$$

where $\beta=\frac{kx^{\phi}}{\Gamma(\phi+1)}-\frac{\omega t^{\phi}}{\Gamma(\phi+1)}$ and β_0 denotes a constant arising from the integration process.

Graphical representation of the solution

By applying the SE method, the resulting wave solutions are represented as ratios of exponential functions. Physical interpretations and graphical representations are also provided by assigning specific values to the constants p and q.

Graphical representation of the space-time fractional mBBM equation

We applied the SE method to obtain traveling wave solutions of the nonlinear space-time fractional mBBM equation in the form of exponential functions. Using the parameters listed in Table 1, the solutions were visualized graphically. Table 1 shows the effects of these parameters on Eqs. (3.1.9)–(3.1.12), illustrating the wave behaviors as kink waves, which are depicted in Figures 1–4.

Table 1. Parameter values of Eqs. (3.1.9)–(3.1.12)

Eqs.	Parameters	Figures	Wave effects
(3.1.9)	$p=1, q=-1, \phi=0.5, \delta=2, k=1,$ $\omega=\frac{2k-k^3p^2}{2}, 0\le x,t\le 100$	1	Kink
(3.1.10)	$p=1, q=-1, \phi=0.5, \delta=2, k=1,$ $\omega=\frac{2k-k^3p^2}{2}, 0\le x,t\le 100$	2	Kink
(3.1.11)	$p=-1, q=1, \phi=0.5, \delta=2, k=1,$ $\omega=\frac{2k-k^3p^2}{2}, 0\le x,t\le 100$	3	Kink
(3.1.12)	$p=-1, q=1, \phi=0.5, \delta=2, k=1,$ $\omega=\frac{2k-k^3p^2}{2}, 0\le x,t\le 100$	4	Kink

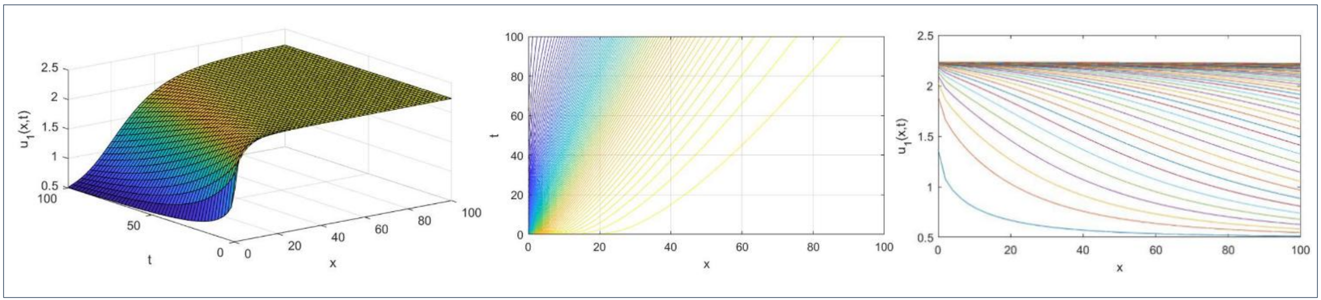

Figure 1. The kink wave solution 3D, contour, and 2D plots of (3.1.9) for the fractional mBBM equation.

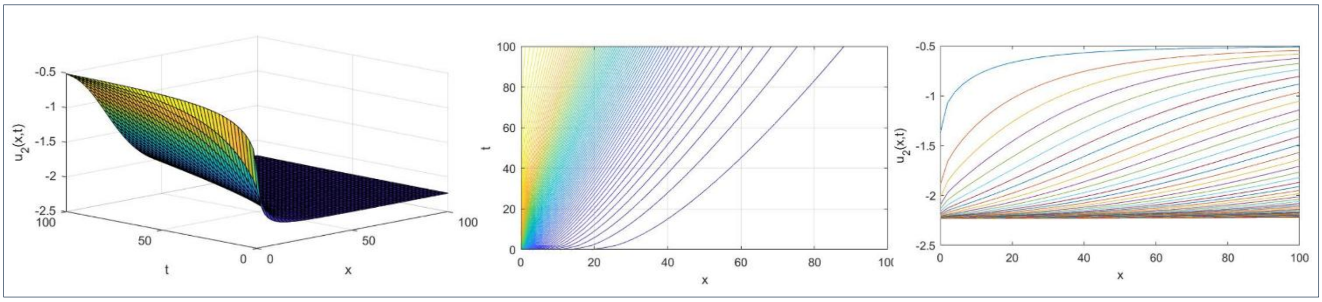

Figure 2. The kink wave solution 3D, contour, and 2D plots of (3.1.10) for the fractional mBBM equation.

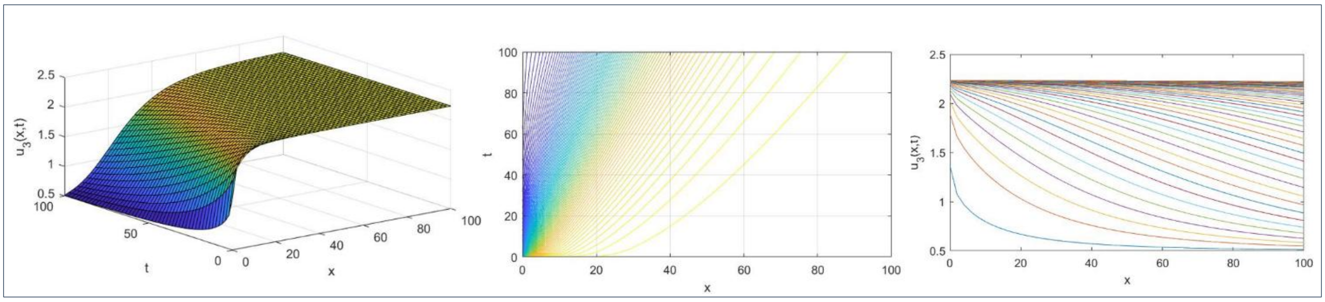

Figure 3. The kink wave solution 3D, contour, and 2D plots of (3.1.11) for the fractional mBBM equation.

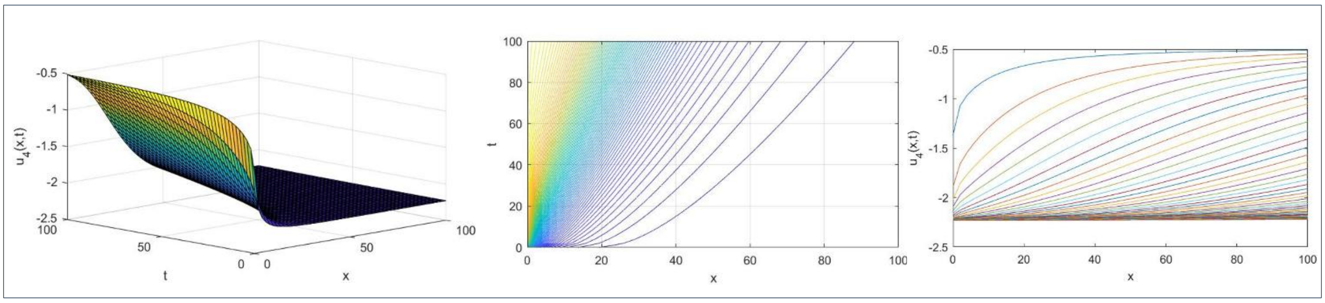

Figure 4. The kink wave solution 3D, contour, and 2D plots of (3.1.12) for the fractional mBBM equation.

Table 2. Parameter values of Eqs. (3.2.9)-(3.2.12)

Eqs.	Parameters	Figures	Wave effects
(3.2.9)	$p=1,\ q=-1,\ \phi=0.5,\ a=6,\ b=-0.5,$ $k=1,\ \omega=\dfrac{k}{1-bp^2k^2}, 0\le x,t\le 100$	5	Kink
(3.2.10)	$p=1,\ q=-1,\ \phi=0.5,\ a=6,\ b=-0.5,$ $k=1,\ \omega=\omega=\dfrac{k}{1+bp^2k^2}, 0\le x,t\le 100$	6	Kink
(3.2.11)	$p=-1,\ q=1,\ \phi=0.5,\ a=6,\ b=0.5,$ $k=1,\ \omega=\dfrac{k}{1+bp^2k^2}, 0\le x,t\le 100$	7	Kink
(3.2.12)	$p=-1,\ q=1,\ \phi=0.5,\ a=6,\ b=0.5,$ $k=1,\ \omega=\dfrac{k}{1-bp^2k^2}, 0\le x,t\le 100$	8	Kink

Graphical representation of the space-time fractional ZKBBM equation

The traveling wave solutions of the nonlinear space-time fractional ZKBBM equation, given by Eqs. (3.2.9)–(3.2.12), were analyzed using the SE method. Table 2 summarizes the effects of the chosen parameters on these solutions, illustrating the corresponding kink wave behaviors in Figures 5–8.

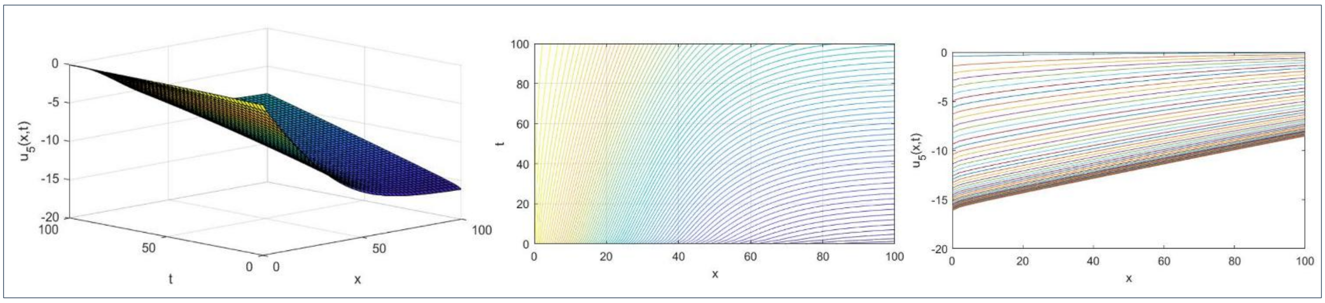

Figure 5. The kink wave solution 3D, contour, and 2D plots of (3.2.9) for the fractional ZKBBM equation.

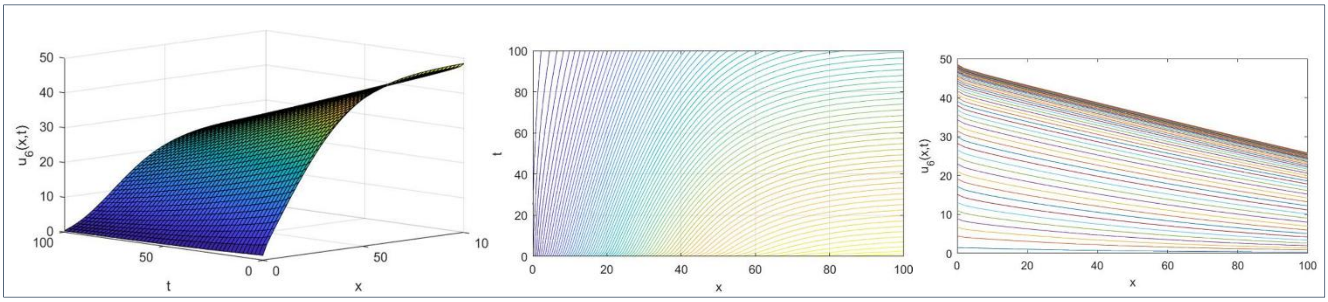

Figure 6. The kink wave solution 3D, contour, and 2D plots of (3.2.10) for the fractional ZKBBM equation.

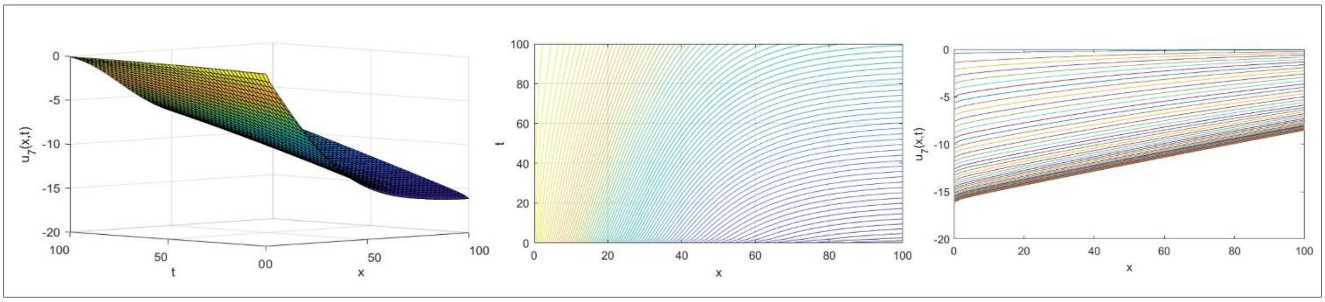

Figure 7. The kink wave solution 3D, contour, and 2D plots of (3.2.11) for the fractional ZKBBM equation.

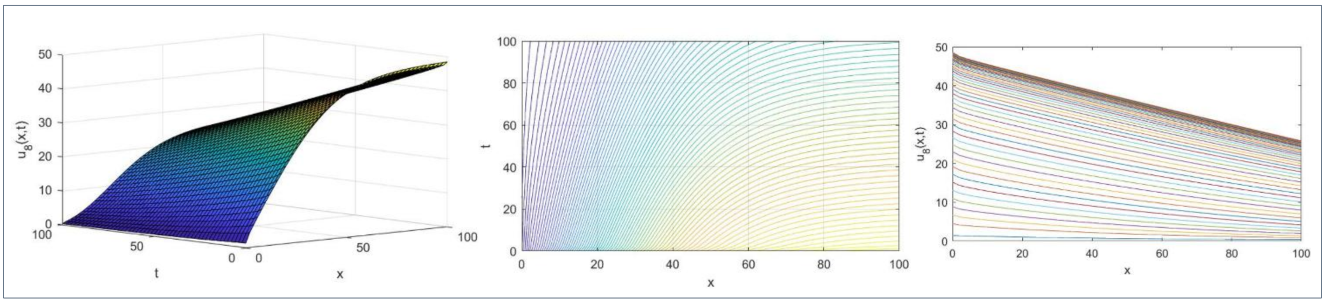

Figure 8. The kink wave solution 3D, contour, and 2D plots of (3.2.12) for the fractional ZKBBM equation.

Figures 1–8 show the solutions of the space-time fractional mBBM and ZKBBM equations, which exhibit kink-type wave behavior. A kink wave is a smooth, monotonic traveling wave that connects two different constant states, rather than forming a localized peak. The graphs clearly illustrate these stable transitions, confirming the presence of kink waves in the studied models. This behavior provides insight into the dynamics of the fractional equations and can be applied to model nonlinear wave propagation in various physical systems, including fluid dynamics, plasma physics, and optical fiber systems, highlighting the practical relevance of these solutions in science and technology.

Solutions Comparison

This section compares the solutions of the space–time fractional mBBM equation obtained by the SE method with those derived using the first integral method [24], as shown in Table 3. It also compares the SE method results for the space–time fractional ZKBBM equation with those obtained by the complete discriminant system [25], as presented in Table 4.

Table 3 shows that the solutions of the space–time fractional mBBM equation obtained by the SE method possess simpler and more compact analytical forms compared with those derived using the first integral method. The SE method expresses the solutions in exponential form, which makes them easier to manipulate and analyze. In contrast, the solutions obtained by the first integral method involve trigonometric and fractional power terms, leading to more complicated structures. Overall, these results demonstrate that the SE method offers a simpler analytical structure and greater computational efficiency compared with the first integral method.

Table 4 shows that the solutions of the space–time fractional ZKBBM equation obtained by the SE method are clearly simpler and more compact. These solutions appear in a direct exponential form, which makes them easier to interpret and analyze. In contrast, the complete discriminant system produces significantly more complex solutions involving layered trigonometric and hyperbolic functions, making the physical interpretation of the wave structures considerably more difficult.

Table 3. Solutions comparison of the space–time fractional mBBM equation

The SE method	The first integral method
$u_1(x,t)=k\sqrt{\frac{6}{\delta}}\left(\frac{p}{2}+q\left(\frac{pe^{p(\beta+\beta_0)}}{1-qe^{p(\beta+\beta_0)}}\right)\right),$	$u_1(x,t)=\sqrt{-\frac{3(k+c)}{vk}}\tan\left(\sqrt{-\frac{(k+c)}{2k^3}}\left(\frac{kx^\alpha+ct^\alpha}{\Gamma(1+\alpha)}\right)+\sqrt{-\frac{(k+c)}{2k^3}}\gamma\right),$
$u_2(x,t)=-k\sqrt{\frac{6}{\delta}}\left(\frac{p}{2}+q\left(\frac{pe^{p(\beta+\beta_0)}}{1-qe^{p(\beta+\beta_0)}}\right)\right),$	$u_2(x,t)=-\sqrt{-\frac{3(k+c)}{vk}}\tan\left(\sqrt{-\frac{(k+c)}{2k^3}}\left(\frac{kx^\alpha+ct^\alpha}{\Gamma(1+\alpha)}\right)+\sqrt{-\frac{(k+c)}{2k^3}}\gamma\right),$
$u_3(x,t)=k\sqrt{\frac{6}{\delta}}\left(\frac{p}{2}-q\left(\frac{pe^{p(\beta+\beta_0)}}{1+qe^{p(\beta+\beta_0)}}\right)\right),$	$u_3(x,t)=\sqrt{-\frac{3(k+c)}{vk}}\tan\left(\sqrt{-\frac{(k+c)}{2k^3}}\left(\frac{kx^\alpha}{\alpha}+\frac{ct^\alpha}{\alpha}\right)+\sqrt{-\frac{(k+c)}{2k^3}}\gamma\right),$
$u_4(x,t)=-k\sqrt{\frac{6}{\delta}}\left(\frac{p}{2}-q\left(\frac{pe^{p(\beta+\beta_0)}}{1+qe^{p(\beta+\beta_0)}}\right)\right).$	$u_4(x,t)=-\sqrt{-\frac{3(k+c)}{vk}}\tan\left(\sqrt{-\frac{(k+c)}{2k^3}}\left(\frac{kx^\alpha}{\alpha}+\frac{ct^\alpha}{\alpha}\right)+\sqrt{-\frac{(k+c)}{2k^3}}\gamma\right).$

Table 4. Solutions comparison of the space–time fractional ZKBBM equation

The SE method	The complete discriminant system
$u_5(x,t)=\frac{6bqk^2}{a(1-bp^2k^2)}\times\left(p\left(\frac{pe^{p(\beta+\beta_0)}}{1-qe^{p(\beta+\beta_0)}}\right)^2+q\left(\frac{pe^{p(\beta+\beta_0)}}{1-qe^{p(\beta+\beta_0)}}\right)\right)$	$\Theta_1(x,t)=-\left(\frac{3bpq}{2a}\right)^{\frac{1}{3}}\left\{R_1\tanh^2\left[\frac{R_1^2}{2}\left(-\frac{2a}{3bpq}\right)^{\frac{1}{3}}\left(\frac{pt^\alpha}{\alpha}+\frac{qx^\alpha}{\beta}-\zeta_0\right)\right]+R_2\right\},$ $\Theta_2(x,t)=-\left(\frac{3bpq}{2a}\right)^{\frac{1}{3}}\left\{R_1\coth^2\left[\frac{R_1^2}{2}\left(\frac{2a}{3bpq}\right)^{\frac{1}{3}}\left(\frac{pt^\alpha}{\alpha}+\frac{qx^\alpha}{\beta}-\zeta_0\right)\right]+R_2\right\},$
$u_6(x,t)=\frac{bp^2k^2}{a(1+bp^2k^2)}+\frac{6bqk^2}{a(1+bp^2k^2)}\times\left(p\left(\frac{pe^{p(\beta+\beta_0)}}{1-qe^{p(\beta+\beta_0)}}\right)^2+q\left(\frac{pe^{p(\beta+\beta_0)}}{1-qe^{p(\beta+\beta_0)}}\right)\right),$	$\Theta_3(x,t)=-\left(\frac{3bpq}{2a}\right)^{\frac{1}{3}}\left\{-R_1\tanh^2\left[\frac{(-R)_1^2}{2}\left(-\frac{2a}{3bpq}\right)^{\frac{1}{3}}\left(\frac{pt^\alpha}{\alpha}+\frac{qx^\alpha}{\beta}-\zeta_0\right)\right]+R_2\right\},$
$u_7(x,t)=\frac{6bqk^2}{a(1-bc^2k^2)}\times\left(p\left(\frac{pe^{p(\beta+\beta_0)}}{1+qe^{p(\beta+\beta_0)}}\right)^2-q\left(\frac{pe^{p(\beta+\beta_0)}}{1+qe^{p(\beta+\beta_0)}}\right)\right),$	$\Theta_4(x,t)=4\left(\frac{3bpq}{2a}\right)^{\frac{2}{3}}\left(\frac{pt^\alpha}{\alpha}+\frac{qx^\beta}{\beta}-\zeta_0\right)^{-2}-\frac{p+q}{\left(12a^2bpq^4\right)^{\frac{1}{3}}},$ $\Theta_5(x,t)=-\left(\frac{3bpq}{2a}\right)^{\frac{1}{3}}\left[\frac{-p-q}{\left(12a^2bpq^4\right)^{\frac{1}{3}}}-\frac{2}{3}R_1\cos\frac{\theta}{3}+\frac{\sqrt[3]{3}R_1}{3}\cos\left(\frac{\pi+2\theta}{6}\right)\times sn^2\left(-\frac{a^{\frac{1}{3}}\cos^{\frac{1}{2}}\left(\frac{\pi-2\theta}{6}\right)}{(12bpq)^{\frac{1}{3}}}\left(\frac{pt^\alpha}{\alpha}+\frac{qx^\beta}{\beta}-\zeta_0\right),m\right)\right],$
$u_8(x,t)=\frac{bp^2k^2}{a(1+bp^2k^2)}+\frac{6bqk^2}{a(1+bp^2k^2)}\times\left(p\left(\frac{pe^{p(\beta+\beta_0)}}{1+qe^{p(\beta+\beta_0)}}\right)^2-q\left(\frac{pe^{p(\beta+\beta_0)}}{1+qe^{p(\beta+\beta_0)}}\right)\right).$	$\Theta_6(x,t)=-\left(\frac{3bpq}{2a}\right)^{\frac{1}{3}}\left[\frac{\vartheta-\mu sn^2\left(\frac{-a^{\frac{1}{3}}\cos^{\frac{1}{2}}\left(\frac{\pi-2\theta}{6}\right)}{(12bpq)^{\frac{1}{3}}}\left(\frac{pt^\alpha}{\alpha}+\frac{qx^\beta}{\beta}-\zeta_0\right),m\right)}{cn^2\left(\frac{-a^{\frac{1}{3}}\cos^{\frac{1}{2}}\left(\frac{\pi-2\theta}{6}\right)}{(12bpq)^{\frac{1}{3}}}\left(\frac{pt^\alpha}{\alpha}+\frac{qx^\beta}{\beta}-\zeta_0\right),\right)}\right].$

Overall, the integration of the SE method with Jumarie's modified Riemann–Liouville fractional derivative plays a key role in achieving these simplified and compact solutions. This combination allows fractional-order derivatives to be handled more effectively than in previous approaches [24-25], enabling the derivation of exact analytical solutions in a systematic and concise form. In contrast, solutions obtained in other studies typically take more complex forms involving layered trigonometric and hyperbolic functions, as illustrated in Tables 3 and 4. This highlights the advantage of the current approach in producing clearer and more manageable analytical results.

Conclusions

In this study, we successfully applied the simple equation (SE) method combined with JMRL derivative to obtain exact traveling wave solutions for two significant nonlinear space-time fractional PDEs: the modified Benjamin-Bona-Mahony (mBBM) equation and the Zakharov-Kuznetsov Benjamin-Bona-Mahony (ZKBBM) equation. The solutions, expressed in exponential form, exhibit kink-type wave structures and were effectively visualized using 2D, 3D, and contour plots, demonstrating the robustness and practicality of the SE method in analyzing fractional models. These results provide valuable insights for applied mathematics by offering direct and exact solutions to complex nonlinear equations. They also demonstrate clear applicability in real-world systems.

The SE method employs simple procedures that yield exact closed-form results, highlighting its analytical strength and versatility. This approach not only reinforces its usefulness in applied mathematics but also opens avenues for extending its application to other nonlinear fractional equations, including coupled systems and variable-coefficient models. Future research may focus on modifying the SE method or integrating it with other symbolic approaches to further enhance its applicability in practical modeling across physics, engineering, and applied mathematics.

References

1. Thadee W, Chankaew A, Phoosree S. Effects of wave solutions on shallow-water equation, optical-fibre equation and electric-circuit equation. Maejo Int J Sci Technol 2022;16(3):262-74.
2. Djilali M, Ali H. (G’/G)-expansion method to seek traveling wave solutions for some fractional nonlinear PDEs arising in natural sciences. Adv Theory Nonlinear Anal Appl 2013;7(2):303-18.
3. Krishnan E, Ghabshi MA, Alquran M. (G'/G)-expansion method and Weierstrass elliptic function method applied to coupled wave equation. Nonlinear Dyn Syst Theory 2019;19(4):512-22.
4. Sanjun J, Chankaew A. Wave solutions of the DMBBM equation and the cKG equation using the simple equation method. Front Appl Math Stat 2022;8:952668.
5. Janma S, Janngam O, Sanjun J. Kink wave solutions for the (1+1)-dimensional nonlinear evolution equation by the simple method with the bernoulli equation. RMUTSB Sci Technol J 2025;9(1):19-28.
6. Sheikh MAN, Taher MA, Hossain MM, Akter S, Roshid HO. Variable coefficient exact solution of Sharma–Tasso–Olver model by enhanced modified simple equation method. Partial Differ Equ Appl Math 2023;7:100527.
7. Thadee W, Phoosree S. New wave behaviors generated by simple equation method with Riccati equation of some fourth-order fractional water wave equations. J Phys Soc Japan 2024;93(1):014002.
8. Sanjun J, Promkwan K, Korkiatsakul T, Janma S. Closed form exact solutions to the combined kdv-mkdv equation and the (2+1)-dimensional gbs equation via the Riccati sub-equation method. RMUTSB Sci Technol J 2024;8(2):46-60.
9. Sanjun J, Muenduang K, Phoosree S. Wave solutions to the combined KdV-mKdV equation via two methods with the Riccati equation. J Appl Sci Emerg Technol 2024;23(2):e256328.
10. Khan MI, Marwat DNK, Sabi’u J, Inc M. Exact solutions of Shynaray-IIA equation (S-IIAE) using the improved modified Sardar sub-equation method. Opt Quantum Electron 2024;56(3):459.
11. Ibrahim IS, Sabi’u J, Gambo YY, Rezapour S, Inc M. Dynamic soliton solutions for the modified complex Korteweg-de Vries system. Opt Quantum Electron 2024;56(6):954.
12. Sabi’u J, Sirisubtawee S, Sungnul S, Inc M. Wave dynamics for the new generalized (3+1)-D Painlevé-type nonlinear evolution equation using efficient techniques. AIMS Math 2024;9(11):32366-98.

13. Sanjun J, Aphaisawat W, Korkiatsakul T. Wave solutions to the Landau-Ginzburg-Higgs equation and modified KdV-Zakharov-Kuznetsov equation by the Riccati-Bernoulli sub-ODE method. J Appl Sci Emerg Technol 2024;23(1):e253520.
14. Hossain AKMKS, Akbar MA. Traveling wave solutions of Benny Luke equation via the enhanced (G'/G)-expansion method. Ain Shams Eng J 2021;12(4):4181-7.
15. Abdel-Gawad HI, Tantawy M, Abdelwahab AM. A new technique for solving Burgers-Kadomtsev-Petviashvili equation with an external source. Suppression of wave breaking and shock wave. Alex Eng J 2023;69:167-76.
16. Jumarie G. Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results. Comp Math Appl 2006;51:1367-76.
17. Jumarie G. Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions. Appl Math Lett 2009;22(3):378-85.
18. Kudryashov NA. Simplest equation method to look for exact solutions of nonlinear differential equations. Chaos Solit Fract 2005;24(5):1217-31.
19. Nofal TA. Simple equation method for nonlinear partial differential equations and its applications. J Egypt Math Soc 2016;24(2):204-9.
20. Poosree S, Thadee W. Wave effects of the fractional shallow water equation and the fractional optical fiber equation. Front Appl Math Stat 2022;8:900369.
21. Parker AE. Who solved the Bernoulli differential equation and how did they do it?. Coll Math J 2013;44(2):89-97.
22. Arefin MA, Zaman UHM, Uddin MH, Inc M. Consistent travelling wave characteristic of space–time fractional modified Benjamin–Bona–Mahony and the space–time fractional Duffing models. Opt Quantum Electron 2024;56(4):588.
23. Islam MN, Parvin R, Pervin MR, Akbar MA. Adequate soliton solutions to the time fractional Zakharov-Kuznetsov equation and the space-time fractional Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation. Arab J Basic Appl Sci 2021;28(1):370-85.
24. Javeed S, Saif S, Waheed A, Baleanu D. Exact solutions of fractional mBBM equation and coupled system of fractional Boussinesq-Burgers. Results Phys 2018;9:1275-81.
25. Zhao S, Li Z. Bifurcation, chaotic behavior, and traveling wave solutions of the space–time fractional Zakharov–Kuznetsov–Benjamin–Bona–Mahony equation. Front Phys 2025;13:1502570.