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THE EFFECTIVENESS OF THE NEW MODIFIED EULER METHOD

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บทคัดย่อ

งานวิจัยนี้มีวัตถุประสงค์เพื่อค้นหาวิธีการของออยเลอร์ปรับปรุงรูปแบบใหม่ เพื่อแก้สมการเชิงอนุพันธ์สามัญให้มีประสิทธิภาพดีที่สุดเมื่อเทียบกับวิธีออยเลอร์ และวิธีออยเลอร์ปรับปรุงดั้งเดิม โดยวิธีออยเลอร์ปรับปรุงรูปแบบใหม่นั้นสร้างเพื่อประมาณผลเฉลยค่าถัดไปโดยใช้ความชันสามค่าในช่วงปัจจุบัน นอกจากนี้ในงานวิจัยนี้ได้เปรียบเทียบค่าผิดพลาดของวิธีออยเลอร์ วิธีออยเลอร์ปรับปรุง วิธีออยเลอร์ปรับปรุงรูปแบบใหม่และวิธีรุงเงอ-คุททาอันดับสาม และยังได้มีการหาเสถียรภาพและความคงตัวของวิธีวิธีออยเลอร์ปรับปรุงรูปแบบใหม่ เมื่อนำวิธีออยเลอร์ปรับปรุงรูปแบบใหม่มาเปรียบเทียบกับวิธีออยเลอร์ วิธีออยเลอร์ปรับปรุงและวิธีรุงเงอ-คุททาอันดับสาม ผลปรากฏว่าวิธีออยเลอร์ปรับปรุงรูปแบบใหม่นั้นมีประสิทธิภาพดีกว่าวิธีออยเลอร์ และวิธีออยเลอร์ปรับปรุง และมีความผิดพลาดน้อยกว่าวิธีออยเลอร์และวิธีออยเลอร์ปรับปรุง แต่มีค่าผิดพลาดมากกว่าวิธีรุงเงอ-คุททาอันดับสามซึ่งเป็นวิธีที่มีประสิทธิภาพดีสำหรับการประมาณค่าผลเฉลยของสมการเชิงอนุพันธ์สามัญ

คำสำคัญ: ความคงตัว; ความผิดพลาด; วิธีออยเลอร์ปรับปรุง; สมการเชิงอนุพันธ์สามัญ; เสถียรภาพ

Abstract

The purpose of this paper was to discover a new modified Euler method for solving ordinary differential equations that was the most efficient compared to the Euler's method and Modified Euler methods, which were the classical methods. The new modified Euler method was developed by approximating solutions in the next intervals using three slopes in the present intervals. In addition, errors of Euler's method, modified Euler method, the new modified Euler method and the third-order Runge-Kutta method were compared. The stability and consistency of the new modified Euler method were proposed. When the new modified Euler method was compared to Euler's method, the Modified Euler method and the third-order Runge-Kutta method, the results revealed that the new modified Euler method exhibited superior effectiveness and lower error rates. However, in comparison with third-order Runge-Kutta method, the new modified Euler method exhibited higher errors due to the fact that the third-order Runge-Kutta method is highly efficient choice for approximating solutions of ordinary differential equation.

Keywords: Consistency; Errors; Modified Euler's Method; Ordinary Differential Equations; Stability

Introduction

Differential equation is an equation composed of one or more unknown functions and their derivatives, the solution is function without derivatives. Differential equation can be classified into 2 types, Ordinary Differential Equation (ODE) and Partial Differential Equation (PDE). Ordinary differential equation is differential equation with one independent variable. Whereas, partial differential equation is differential equation with more than one independent variable.

Differential equations serve as a mathematical framework for describing problems in the physical sciences. The problems can be effectively represented through mathematical models, such as models for limited growth rates, logistic growth, and predator-prey interactions. Due to the inherent complexity of these models, analytical solutions are often elusive. Consequently, numerical approximations provide a valuable means to estimate solutions.

In this paper, we concentrate on first order ordinary differential equation. There are many numerical methods for solving first order differential equations, including Euler's method, modified Euler's method, Runge-Kutta method, and Predictor-Corrector method. Euler's method is the simplest, but this produces a lot of errors. Leonhard Euler discovered Euler's method, which was later modified to improve accuracy by Karl Heun and named Huen's method [1]. Many studies on Euler's method have been conducted, for example, Ochoche [2] improved modified Euler's method by inserting forward Euler's method in y_n of the formula for y_{n+1} of the modified Euler's method formula. In 2016, Sampoornam [3] compared the improved Euler's method and the Runge-Kutta method. In 2017, Mohd Yusop et al. [4] investigated a modified Euler's method based on the Harmonic-Polygon approach. In 2020, Ram [5] studied a hybrid numerical method with greater efficiency for solving initial valued problems and compared the consistency, stability and accuracy to the modified Euler's method, improved Euler's method. Din Ide [6] modified Euler's method by alteration the method of Abushet H. W. In this paper, we improve Euler's method by using three slopes in each interval, and investigate the consistency and stability of each method. Furthermore, numerical experiments are provided to investigate the efficacy of the new method.

Objectives

In this paper, we modified Euler method by using three slopes in the interval of approximations. The slopes were the slope at the beginning, the slope at the middle, and the slope at the end of each interval. Moreover, we investigated stability and consistency condition for the method.

Methods

There are various methods to approximate solution of differential equations. In this paper, we compared results when using Euler's method, Modified Euler method, new Modified Euler Method and the third-order Runge-Kutta method.

The fundamental idea driving the application of the Euler's method is to utilize a simple iterative numerical approach to approximate solutions to differential equations. On the other hand, the Modified Euler method seeks to enhance the precision of solution approximations when compared to the basic Euler's method. The Modified Euler method use a two-step process within each iteration: predicting the next value and then correcting the value by using weighted average of the slopes at the beginning and end of the interval. In contrast, the third-order Runge-Kutta method involves using a third-order accurate numerical technique to approximate solutions to differential equations. This method employs three intermediate steps within each iteration, using weighted averages of the slopes at different points in the interval to estimate the next value. The third-order Runge-Kutta method offers increased accuracy over simpler methods like the Euler's method, this method is useful for solving differential equations with more precision while still maintaining computational efficiency.

1. Euler's method

Euler's method is the simplest method to solve first order ordinary differential equation. This method uses slopes to formulate equation of approximating solution. From Figure 1, the slope of the graph is $\frac{dy}{dx} = f(x, y)$, distance between and x_i and x_{i+1} is h and the difference between the exact solution and numerical solution is error. Thus, $x_{i+1} = x_i + h$, and the equation of the approximation by Euler's method is $y_{i+1} = y_i + f(x_i, y_i)h$.

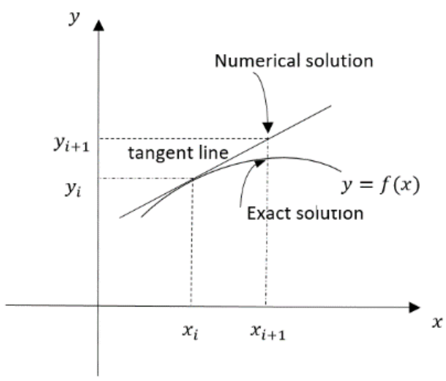

Figure 1 Exact solution and numerical solution.

In order to illustrate how the Euler's method is applied, the calculations are carried out manually.
Example1. Use Euler's method to solve the boundary-valued problem

$$\frac{dy}{dx} = 1 + (x - y)^2,\ 2 \le x \le 2.75.$$

h =0.25, $y(2)=1$ which has exact solution $y(x)=x+\dfrac{1}{1-x}$.

From the initial condition, we have the first point is (2,1). Employing the formula

$$y_{n+1}=y_n+hf(x_n,y_n).$$

Therefore, $y_1=y_0+hf(x_0,y_0)$, we have

$$y_1=1+0.25f(2,1)=1.5.$$

Next, find y_2 by using $y_2=y_1+hf(x_1,y_1)=1.5+0.25f(2.25,1.5)=1.8906.$

Then, find y_3 by using $y_3=y_2+hf(x_2,y_2)=1.8906+0.25f(2.5,1.8906)=2.2334.$

The errors are shown in Table 1.

Table 1 Results and errors of example 1.

x_i	y (actual solution)	y_i (approximating solution)	Absolute error
2	0	0	0
2.25	1.45	1.5	0.05
2.5	1.833	1.833	0.0576
2.75	2.1786	2.1786	0.05487

2. Modified Euler method

In this section, the Euler's method is modified by using the average of slopes at the end points. The slope of the left end point of the interval is $\dfrac{dy}{dx}=f(x,y)$. Then estimate y_{i+1} by using the slope at (x_i,y_i), we have

$$y_{i+1}^{Eu}=y_i+f(x_i,y_i)h. \quad (1)$$

The term y_{i+1}^{Eu} is the approximated solution of y_{i+1} by using Euler's method. Then, use the right end point (x_{i+1},y_{i+1}^{Eu}) to obtain $\dfrac{dy}{dx}$ as $\dfrac{dy}{dx}=f(x_{i+1},y_{i+1}^{Eu})$. Therefore, the formula of Modified Euler method is

$$y_{i+1}=y_i+\frac{h}{2}[f(x_i,y_i)+f(x_{i+1},y_{i+1}^{Eu})]. \quad (2)$$

In order to illustrate how the Modified Euler method is applied, the calculations are carried out manually.

Example 2. Use Modified Euler method to solve the boundary-valued problem

$$\frac{dy}{dx}=1+(x-y)^2,\ 2\le x\le 2.75.$$

h =0.25, $y(2)=1$ which has exact solution $y(x)=x+\dfrac{1}{1-x}$.

From the initial condition, we have the first point is (2,1). Employing the formula

$$y_{i+1}=y_i+\frac{h}{2}[f(x_i,y_i)+f(x_{i+1},{y_{i+1}}^{Eu})].$$

We have, $y_1^{Eu} = y_0 + hf(x_0, y_0) = 1 + 0.25 f(2,1) = 1.5$, thus

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{Eu})] = 1 + \frac{0.25}{2}[f(2,1) + f(2.25,1.5)] = 1.4453125.$$

Next, find y_2^{Eu} by using $y_2^{Eu} = y_1 + hf(x_1, y_1)$, thus

$$y_2^{Eu} = 1.4453125 + 0.25 f(2.25,1.4453125) = 1.85719299.$$

Substitute y_2^{Eu} into $y_2 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{Eu})]$.

Therefore,

$$y_2 = 1.4453125 + \frac{0.25}{2}[f(2.25,1.4453125) + f(2.5,1.85719299)] = 1.82790285.$$

Then, find y_3^{Eu} by using $y_3^{Eu} = y_2 + hf(x_2, y_2)$, thus

$$y_3^{Eu} = 1.82790285 + 0.25 f(2.5,1.82790285) = 2.1908315.$$

Substitute y_3^{Eu} into $y_3 = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3^{Eu})]$.

Therefore, $y_3 = 1.82790285 + \frac{0.25}{2}[1.451715 + f(2.75, 2.1908315)] = 2.17345085.$

The errors are shown in Table 2.

Table 2 Results and errors of example 2.

x_i	y (actual solution)	y_i (approximating solution)	Absolute error
2	1	1	0
2.25	1.45	1.4453125	0.004687
2.5	1.833	1.82790285	0.005097
2.75	2.1786	2.17345085	0.005149

3. Third-order- Runge-Kutta method

Runge-Kutta methods are a family of single step, explicit method for finding solution of first-order ordinary differential equation [7]. There are second-order Runge-Kutta, third-order Runge-Kutta and fourth-order Runge-Kutta methods. In this paper we interested on third-order Runge-Kutta due to this method use three point in the subinterval for the approximations. Solutions by Runge-Kutta methods are more accurate solutions than the Euler's method. Moreover, the Runge-Kutta methods are widely used for approximating solution of ordinary differential equation. The accuracy increases when the order of Runge-Kutta increase.

The equation of approximation by third-order Runge-Kutta is

$$y_{i+1} = y_i + \frac{h}{6}[(K_1 + 4K_2 + K_3)]$$, where

$$K_1 = f(x_i, y_i)$$

$$K_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1 h)$$

$$K_3 = f(x_i + h, y_i - K_1 h + 2K_2 h).$$

Example 3. Use third-order Runge-Kutta method to solve the boundary-valued problem

$$\frac{dy}{dx} = 1+(x-y)^2, \; 2 \le x \le 2.75.$$

h =0.25, $y(2)=1$ which has exact solution $y(x) = x + \frac{1}{1-x}$.

From the initial condition, we have the first point is (2,1). We have

$y_{i+1} = y_i + \frac{h}{6}[(K_1 + 4K_2 + K_3)]$. Thus $y_1 = y_0 + \frac{h}{6}[(K_1 + 4K_2 + K_3)]$, where

$$K_1 = f(x_0, y_0) = f(2,1) = 2,$$

$$K_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1 h) = f(2.125, 1.25) = 1.765625,$$

$$K_3 = f(x_i + h, y_i - K_1 h + 2K_2 h) = f(2.25, 1.3828125) = 1.75201416.$$

Therefore, $y_1 = 1.45060476$. Next, find y_2 by using $y_2 = y_1 + \frac{h}{6}[(K_1 + 4K_2 + K_3)]$, where

$$K_1 = f(x_1, y_1) = f(2.25, 1.45060476) = 1.63903276,$$

$$K_2 = f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}K_1 h) = f(2.375, 1.65548385) = 1.517703489,$$

$$K_3 = f(x_1 + h, y_1 - K_1 h + 2K_2 h) = f(2.5, 1.79969831) = 1.49042245.$$

Therefore, $y_2 = 1.83394931.$ The results and errors are shown in Table 3.

Table 3 Results and errors of example 3.

x_i	y (actual solution)	y_i (approximating solution)	Absolute error
2	1	1	0
2.25	1.45	1.45060476	0.00060476
2.5	1.833	1.83394931	0.00061597
2.75	2.1786	2.1791027	0.00053128

Results

In this section we improved the efficiencies of Euler's method and Modified Euler method by using the average of slopes. The method was called New Modified Euler Method (NME).

1. New Modified Euler Method

To approximate the solution, the method takes the average of three slopes in the interval. The average of slopes in each interval uses the slopes at the left end point, the slopes at the middle, and the slope at the right end point to form a slope of the approximation.

We denote slope at the starting of the interval as $\frac{dy}{dx} = f(x_i, y_i)$, then use this slope to approximate $y_{i+\frac{1}{2}}$ by using Euler's method, we denote this by $y^{Eu}_{i+\frac{1}{2}}$, thus we obtain

$$y^{Eu}_{i+\frac{1}{2}} = y_i + f(x_i, y_i)\frac{h}{2}. \quad (3)$$

Then approximate the slope at $(x_i + \frac{h}{2}, y^{Eu}_{i+\frac{1}{2}})$, we obtain a slope at the point $(x_i + \frac{h}{2}, y^{Eu}_{i+\frac{1}{2}})$ as

$$\frac{dy}{dx} = f(x_i + \frac{h}{2}, y^{Eu}_{i+\frac{1}{2}}). \tag{4}$$

Then approximate y_{i+1} by using Euler's method (denoting by y^{Eu}_{i+1}), we obtain

$$y^{Eu}_{i+1} = y^{Eu}_{i+\frac{1}{2}} + f(x_i + \frac{h}{2}, y^{Eu}_{i+\frac{1}{2}})\frac{h}{2}. \tag{5}$$

Therefore at (x_{i+1}, y^{Eu}_{i+1}), we obtain $\frac{dy}{dx} = f(x_{i+1}, y^{Eu}_{i+1})$. Thus, the formula of New Modified Euler Method is

$$y_{i+1} = y_i + \frac{h}{3}[f(x_i, y_i) + f(x_i + \frac{h}{2}, y^{Eu}_{i+\frac{1}{2}}) + f(x_{i+1}, y^{Eu}_{i+1})]. \tag{6}$$

The New Modified Euler Method is illustrated in Figure 2 – Figure 5. From Figure 2, uses the slopes of the beginning of interval approximates $y^{Eu}_{i+\frac{1}{2}}$. Figure 3 uses $y^{Eu}_{i+\frac{1}{2}}$ approximates $f(x_i + \frac{h}{2}, y^{Eu}_{i+\frac{1}{2}})$. Figure 4 uses $f(x_i + \frac{h}{2}, y^{Eu}_{i+\frac{1}{2}})$ approximates y^{Eu}_{i+1}. Thus, the average of these slopes is illustrated in Figure 5.

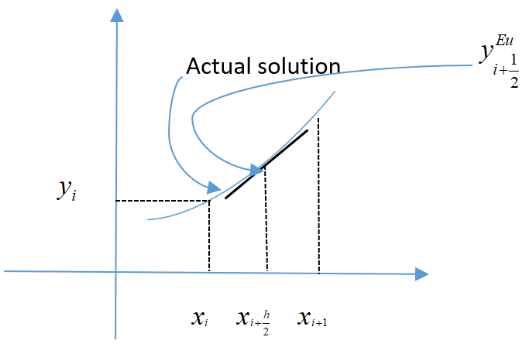

Figure 2 Slope approximates for $y^{Eu}_{i+\frac{1}{2}}$.

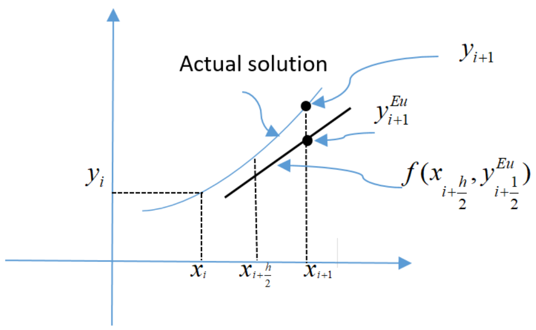

Figure 3 Slope obtains from $y^{Eu}_{i+\frac{1}{2}}$.

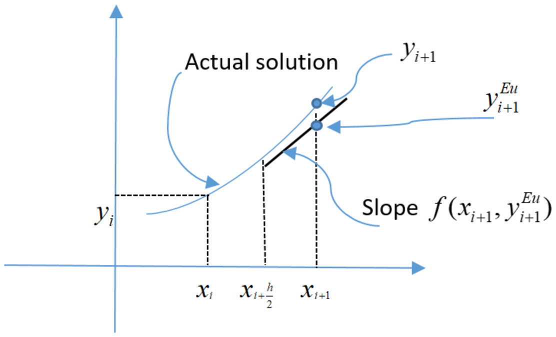

Figure 4 Slope approximates for y_{i+1}^{Eu}.

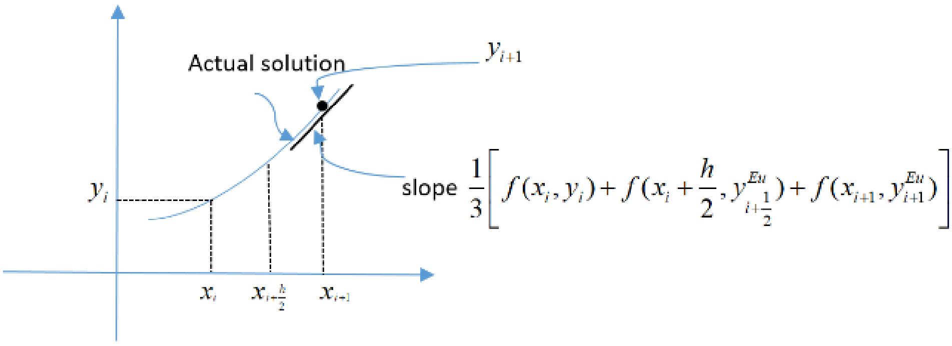

Figure 5 Averaging of slopes.

2. Algorithm of New Modified Euler Method

Initial and boundary value problem: $y' = f(x, y), a \le x \le b, y(a) = \alpha$.

i) Input points a, b, N, α.

ii) Output approximation value of y at N+1 point.

Step 1 Let $h = \dfrac{b-a}{N}, x_0 = a, y_0 = \alpha$.

Step 2 Compute $f(x_i, y_i)$.

Step 3 for $i = 1, 2, 3, ..., N+1$ do step 4-7.

Step 4 Approximate y_{i+1} by

$$y_{i+\frac{1}{2}}^{Eu} = y_i + f(x_i, y_i)\frac{h}{2}.$$

$$y_{i+1}^{Eu} = y_{i+\frac{1}{2}}^{Eu} + f(x_i + \frac{h}{2}, y_{i+\frac{1}{2}}^{Eu})\frac{h}{2}.$$

Step 5 Compute $f(x_{i+1}, y_{i+1}^{Eu})$.

Step 6 Compute the approximation value of y_{i+1},

$$y_{i+1} = y_i + \frac{h}{3}[f(x_i, y_i) + f(x_i + \frac{h}{2}, y_{i+\frac{1}{2}}^{Eu}) + f(x_{i+1}, y_{i+1}^{Eu})].$$

Step 7 Stop.

3. Consistency of New Modified Euler Method

For testing the consistency of New Modified Euler Method, we use Ochoche [2], Let

$$y_{i+1} = y_i + h\phi(x_i, y_i, h), \tag{7}$$

where

$$\phi(x_i, y_i, h) = \frac{1}{3}[f(x_i, y_i) + f(x_i + \frac{h}{2}, y^{Eu}_{i+\frac{1}{2}}) + f(x_{i+1}, y^{Eu}_{i+1})]. \tag{8}$$

Therefore,

$$\lim_{h\to 0} \phi(x_i, y_i, h) = f(x_i, y_i). \tag{9}$$

Hence, NME is consistency.

4. Stability of New Modified Euler Method

To investigate the stability of New Modified Euler Method, we employ Ochoche [2] and let

$$\frac{dy}{dx} = \lambda y_i,\ y(x_0) = y_0,\ \lambda \in C. \tag{10}$$

From the formula of NME as

$$y_{i+1} = y_i + \frac{h}{3}[f(x_i, y_i) + f(x_i + \frac{h}{2}, y^{Eu}_{i+\frac{1}{2}}) + f(x_{i+1}, y^{Eu}_{i+1}). \tag{11}$$

Thus,

$$y^{Eu}_{i+\frac{1}{2}} = y_i + f(x_i, y_i)\frac{h}{2} = y_i + \lambda y_i \frac{h}{2} = y_i(1 + \frac{\lambda h}{2}), \tag{12}$$

and,

$$y^{Eu}_{i+1} = y_i + f(x_i, y_i)h = y_i + \lambda y_i h = y_i(1 + \lambda h). \tag{13}$$

Therefore,

$$y_{i+1} = y_i + \frac{h}{3}[\lambda y_i + \lambda y^{Eu}_{i+\frac{1}{2}} + \lambda y^{Eu}_{i+1}] = y_i + [1 + \lambda h + \frac{\lambda h^2}{2}]. \tag{14}$$

Hence, NME is stable when $\left|1 + \lambda h + \frac{\lambda^2 h^2}{2}\right| \le 1.$

5. Numerical Experiments

To assess the effectiveness of this approach, numerical illustrations are provided in this section. These examples involve a comparison of errors across Euler's method, the modified Euler method, the New modified Euler method, and the third-order Runge-Kutta method. It's noteworthy that the third-order Runge-Kutta method is employed here as the method also employs three slopes, similar to the new modified Euler method.

Example 4. The boundary-valued problem

$$y' = -\frac{1}{1+x} + y - y^2,\ h = 0.1, \text{ on } [0,1].$$

$$y(0) = 1 \text{ which has exact solution } y(x) = \frac{1}{1+x}.$$

The errors of each method is shown in Table 4.

Table 4 Errors of each method of example 4.

x	EU	ME	NME	RK3
0	0	0	0	0
0.1	0.00909091	0.00405046	0.00001755	0.00000580
0.2	0.01524242	0.00648985	0.00004262	0.00001000
0.3	0.01959138	0.00797496	0.00006916	0.00001300
0.4	0.02280138	0.00887943	0.00009474	0.00001500
0.5	0.02527753	0.00942443	0.00011853	0.00001600
0.6	0.02727662	0.00974518	0.00014040	0.00001800
0.7	0.02896690	0.00992660	0.00016051	0.00001800
0.8	0.03046196	0.01002326	0.00017913	0.00001900
0.9	0.03184072	0.01007085	0.01007085	0.00002000
1.0	0.03315956	0.01009312	0.00021311	0.00002000
Maximum value of absolute error	0.03315956	0.01009312	0.00021311	0.00002000

Note: EU means Euler's method, ME means Modified Euler method, NME means New modified Euler method and RK3 means third-order Runge-Kutta methods.

Example 5. The boundary-valued problem

$$y' = \frac{x^2 + y}{x}, h = 0.1, \text{ on } [-2, -1].$$

$$y(-2) = -2 \text{ which has exact solution } y(x) = 3x + x^2.$$

The errors of each method is shown in Table 5.

Table 5 Errors of each method of example 5

x	EU	ME	NME	RK3
-2	0	0	0	0
-1.9	0.01000000	0.00210526	0.00012933	0.00000450
-1.8	0.01947368	0.00371391	0.00025891	0.00000927
-1.7	0.02839181	0.00479624	0.00038878	0.00001440
-1.6	0.03672171	0.00531867	0.00051901	0.00001980
-1.5	0.04442660	0.00524295	0.00064965	0.00002580
-1.4	0.05146483	0.00452504	0.00078081	0.00003220
-1.3	0.05778877	0.00311371	0.00091260	0.00003940
-1.2	0.06334348	0.00094855	0.00104517	0.00004750

-1.1	0.06806485	0.00204282	0.00117876	0.00005670
-1.0	0.07187714	0.00594875	0.00131366	0.00006740
Maximum value of absolute error	0.07187714	0.00594875	0.00131366	0.00006740

Example 6. The boundary-valued problem

$$y' = 3 - 2x - \frac{y}{2},\ h = 0.1,\ \text{on } [0,\ 2].$$

$$y(0) = 1 \text{ which has exact solution } y(x) = 14 - 4x - 13e^{-0.5x}.$$

The errors of each method is shown in Table 6.

Table 6 Errors of each method of example 6.

x	EU	ME	NME	RK3
0	0	0	0	0
0.1	0.01598252	0.00535752	0.00013206	0.00000340
0.2	0.03038643	0.00974761	0.00025125	0.00000640
0.3	0.04332869	0.01325266	0.00035850	0.00000910
0.4	0.05491854	0.01594932	0.00045468	0.00001200
0.5	0.06525799	0.01790879	0.00054064	0.00001400
0.6	0.07444229	0.01919723	0.00061713	0.00001600
0.7	0.08256032	0.01987608	0.00068487	0.00001700
0.8	0.08969499	0.02000234	0.00074454	0.00001900
0.9	0.09592364	0.01962885	0.00079677	0.00002000
1.0	0.10131837	0.01880462	0.00084212	0.00002100
Maximum value of absolute error	0.10131837	0.02000234	0.00084212	0.00002100

Example 7. The boundary-valued problem

$$y' = \frac{y}{4}\left(1 - \frac{y}{20}\right),\ h = 0.1,\ \text{on } [0,\ 1].$$

$$y(0) = 1 \text{ which has exact solution } y(x) = \frac{20}{1 + 19e^{-\frac{x}{4}}}.$$

The errors of each method is shown in Table 7.

Table 7 Errors of each method of example 7.

x	EU	ME	NME	RK3
0	0	0	0	0
0.1	0.00026896	0.00024223	0.00000107	0.00000001
0.2	0.00054933	0.00049477	0.00000218	0.00000003
0.3	0.00084139	0.00075783	0.00000334	0.00000004
0.4	0.00114544	0.00103169	0.00000455	0.00000005
0.5	0.00146177	0.00131662	0.00000580	0.00000007
0.6	0.00179070	0.00161290	0.00000711	0.00000008
0.7	0.00213251	0.00192079	0.00000846	0.00000000
0.8	0.00248751	0.00224056	0.00000986	0.00000012
0.9	0.00285601	0.00257249	0.00001132	0.00000013
1.0	0.00323830	0.00291685	0.00001282	0.00000015
Maximum value of absolute error	0.00323830	0.00291685	0.00001282	0.00000015

From the problems, the percentage of maximum errors are shown in Figure 6.

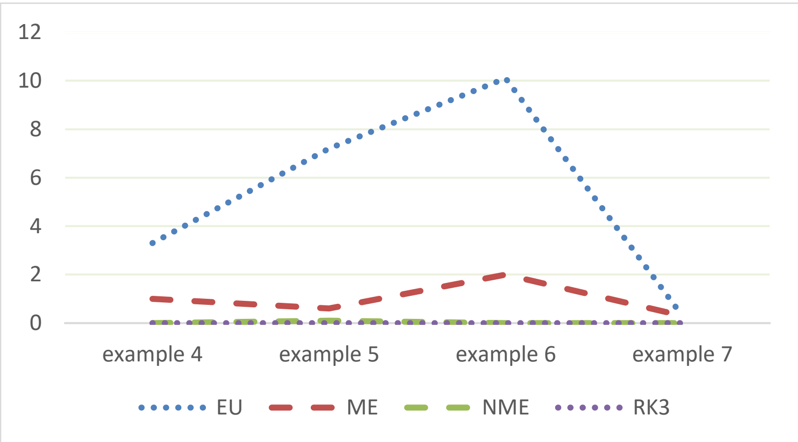

Figure 6 Percentage error of maximum absolute error.

Conclusions and Discussion

Euler's method serves as a fundamental and classical approach for solving ordinary differential equations. However, because of its simplicity and ease of use, approximation results are less applicable due to the large of error. This paper constructs the new modified Euler method as well as its stability and consistency. The results show that the new modified Euler method produces less error than the Euler's method and the modified Euler method, implying that new modified Euler method is superior to the other methods and applicable for approximations. However, when compared to the third-order Runge-Kutta method, the new modified Euler method produces more error than the third-order Runge-Kutta method, since the fact that the third-order Runge-Kutta method is one of the most efficient methods of ordinary differential equation approximation.

The results obtained from this research indicate that the new modified Euler method can be employed as a replacement for the traditional Euler method in applications involving advanced numerical solution techniques. These methods are characterized by their complexity but high accuracy. Notably, these methods incorporate the traditional Euler method within their algorithms for approximating solutions to differential equations. Examples of such methods include adaptive step-size methods, multistep methods, or other approximation techniques. This integration aims to enhance accuracy in numerical approximation process.

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