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The Third Order Approximation for the Coverage Probability of a Confidence Set Centered at the Positive Part James-Stein Estimator

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Abstract

In this paper, we continue the work of Ahmed et al. (2006, 2009, 2015) by investigating the asymptotic expansion approximation for the coverage probability of a confidence set centered at the positive-part James-Stein estimator. The third order Taylor expansion is the main tool here. The theoretical part provides a formula of the approximation for the coverage probability in the case of a noncentrality parameter $\tau \to 0$, where $\tau^2 = n\|\boldsymbol{\theta}\|^2$, n is the sample size and $\boldsymbol{\theta}$ is the mean vector of the $p-$ variate normal distribution with independent components and equal unit variances. In the computational part, we compare the first, second and third orders of the asymptotic expansion with the exact values of the coverage probabilities in order to obtain the accuracy of estimation. The results show that all of these approximations are reliable. However, the first order of the asymptotic expansion gives the best result, especially when the noncentrality parameter τ is far from 0.

Keywords: Confidence sets, positive part James-Stein estimator, multivariate normal distribution, coverage probability, asymptotic expansions, third order asymptotic.

1. Introduction

The problem of estimating the mean vector, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)$ of a $p-$dimensional multivariate normal distribution with independent components and equal unit variances has begun to receive much attention after Stein (1962) showed that the usual sample mean estimator $\bar{\mathbf{X}}$ is inadmissible for $p \geq 3$. The shrinkage estimator defined by

$$\delta(\bar{\mathbf{X}}) = \left(1 - \frac{a}{\|\bar{\mathbf{X}}\|^2}\right)\bar{\mathbf{X}},$$

where a is a constant, has smaller risk than $\bar{\mathbf{X}}$. The value $a = p-2$ gives the uniformly best estimator in the class of $0 < a < 2(p-2)$. This optimal estimator is called the James-Stein estimator and is defined by

$$\delta_{JS}(\bar{\mathbf{X}}) = \left(1 - \frac{p-2}{\|\bar{\mathbf{X}}\|^2}\right)\bar{\mathbf{X}}.$$

Later, Baranchik (1970) proposed an improved estimator which has uniformly smaller risk than $\delta_{JS}(\bar{\mathbf{X}})$. The positive part James-Stein estimator is defined as

$$\delta^+(\bar{\mathbf{X}}) = \left(1 - \frac{p-2}{\|\bar{\mathbf{X}}\|^2}\right)^+ \bar{\mathbf{X}},$$

where $x^+ = \max\{x, 0\}$.

We are interested in the coverage probability of the set centered at the positive part James-Stein estimator

$$D_{\delta^+} = \{\boldsymbol{\theta} : n \|\boldsymbol{\theta} - \delta^+(\bar{\mathbf{X}})\|^2 \le c^2\},$$

where n is the sample size. An advantage of Stein-type estimators is that it can be used to construct a new confidence set with a smaller volume and a higher coverage probability. This fact has been first proved in Hwang and Casella (1982, 1984) but they did not provide any exact values for the coverage probability. Therefore, the calculation of this coverage probability becomes a relevant problem.

Ahmed et al. (2006) investigated the asymptotic expansion of the coverage probability for the James-Stein estimator. The main tool of their work combines a geometrical and analytical methodology and based on Taylor expansion. They established that the coverage probability of the confidence sets centered at the James-Stein estimator and its positive part depends on the noncentrality parameter $\tau^2 = n\|\boldsymbol{\theta}\|^2$. The same parameter appears in the calculation of the risks of these estimators. The main result of their work is the simple formulae of the first order approximation of the coverage probabilities for the cases when $\tau \to 0$ and $\tau \to \infty$.

Later, Ahmed et al. (2009) continue their work by considering only the confidence set centered at the positive part James-Stein estimator. They investigated the asymptotic behavior of the coverage probability by using second order Taylor expansion. They found that this second order approximation has small influence on the accuracy of the estimation.

Thus, the goal of the present work is to estimate the coverage probability of the confidence set centered at the positive part James-Stein estimator by investigating the asymptotic expansion approximation using the third order Taylor expansion for the case $\tau \to 0$. Moreover, to show the accuracy of estimation, we provide numerical illustrations that compare the first, second and third orders of asymptotic estimation with the exact values of the coverage probabilities.

2. Asymptotic Expansion of the Coverage Probability

In Ahmed et al. (2006), the approximation of the coverage probability by a confidence set centered at the positive part James-Stein estimator was established that

$$Q_p^+(\tau) = P(D_{\delta^+}) = K_p(w) + R_P(\tau),$$

where

$$w = a + \frac{c^2 - \tau^2}{2} + \sqrt{\frac{\left(c^2 - \tau^2\right)^2}{4} + c^2\tau^2 - a(\tau^2 - c^2)}. \tag{1}$$

The term $K_p(w)$ is the chi-square distribution with p degrees of freedom and $R_P(\tau)$ is represented as a double integral depended on the relation between the radius of the confidence set c and τ.

The coverage probability can be rewritten as

$$Q_p^+(\tau) = K_p(w) + \int_w^{v_2}\int_{-\sqrt{x}}^{h(x)} f(x,y)dydx - \int_{v_1}^{w}\int_{h(x)}^{\sqrt{x}} f(x,y)dydx, \qquad (2)$$

where (cf. also Budsaba and Suraphee (2012))

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\, 2^{\frac{p-1}{2}} \Gamma\left(\dfrac{p-1}{2}\right)} \left(x-y^2\right)^{\frac{p-1}{2}} \cdot e^{-x/2} & ; x - y^2 > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

and the limits of integration v_1 and v_2 are defined as the points of intersection of the right branch of the hyperbola

$$h(x) = \frac{(x-a)^2 - c^2\tau^2 + x(\tau^2 - c^2)}{2\tau(c^2 + a - x)}$$

and parabola $\sqrt{x}$. Therefore we need to solve the equation

$$(x-a)^2 - c^2\tau^2 + x(\tau^2 - c^2) = 2\tau\sqrt{x}(c^2 + a - x). \qquad (3)$$

This is a polynomial equation of fourth order which has four roots. We need only points of intersection v_1 and v_2 of the right branch of the hyperbola $h(x)$ with parabola $\sqrt{x}$. In order to find them, it is sufficient to consider only the (3) by making the substitution $x = w + z$.

For the solution of (3) for v_1 and v_2 we make use of Lemma 1 from the articles Ahmed et al. (2006, 2009). With minor corrections of typos in these articles and taking into consideration that we not using function f_2, we present this lemma in the following form.

Lemma *If $\tau \to 0$ then the following asymptotic expansion are true for the roots of v_1 and v_2*

$$v_{1,2} = w + \sum_{k=1}^{\infty} \lambda_k (\pm\tau)^k,$$

where $\lambda_k = \dfrac{1}{k!}\left[\left(\dfrac{d^{k-1} f_1^k(z)}{dz^{k-1}}\right)\right]_{z=0}$, $k \ge 1$, $f_1(z) = \dfrac{2\sqrt{w+z}(c^2 + a - w - z)}{2(w-a) + z + (\tau^2 - c^2)}$ *and w is given in (1).*

Note that
$$\lambda_1 = \frac{2\sqrt{w}(c^2 + a - w)}{2(w-a) - c^2}$$

and
$$\lambda_2 = \frac{2\left(c^2 + a - w\right)\left(2(w-a) - c^2\right)\left(c^2 + a - 3w\right) - 4w\left(c^2 + a - w\right)^2}{\left(2(w-a) - c^2\right)^3}.$$

In previous publications of Ahmed et al. (2006, 2009, 2015), the first and second orders of approximation for the coverage probabilities for $\tau \to 0$ have been investigated. By using the first order of asymptotic expansion, the term $R_P(\tau)$ as represented in double integrals in (2) was established that $R_p(\tau) = O(\tau^2)$. For the second order approximation in Ahmed et al. (2009), it has been established that

$$R_p(\tau) = \tau^2 \left(\frac{4\lambda_2 + \left((a-w)w^{-1}\right)\lambda_1^2}{2^{(p+4)/2}\Gamma(p/2)} \right) e^{-w/2} w^{(p-2)/2} + O(\tau^3).$$

2.1. The third order approximation for the coverage probability

To provide the third order of asymptotic expansion of the coverage probability for $\tau \to 0$, we note that by Lemma the roots of equations $h(x) = \pm\sqrt{x}$ can be represented as

$$v_1 = w + \Delta_1 + O\left(\tau^4\right) \text{ and } v_2 = w + \Delta_2 + O\left(\tau^4\right),$$

where

$$\Delta_1 = \lambda_1\tau + \lambda_2\tau^2 + \lambda_3\tau^3 \tag{4}$$

and

$$\Delta_2 = -\lambda_1\tau + \lambda_2\tau^2 - \lambda_3\tau^3. \tag{5}$$

With the preliminaries accounted for, we are now ready to present and prove the main result of the article.

Theorem *If $\tau \to 0$, then for $p \geq 4$ the following asymptotic expansion of the probability for the coverage of the true value by the confidence set D_{δ^+} centered at the positive-part James-Stein estimation, is true*

$$Q_p^+(\tau) = K_p(w) + R_p(\tau) + O(\tau^4),$$

where w is defined in (1) and

$$R_p(\tau) = \tau^2 \left(\frac{4\lambda_2 + \left((a-w)w^{-1}\right)\lambda_1^2}{2^{(p+4)/2}\Gamma(p/2)} - \tau g(w,\tau) \right) e^{-w/2} w^{(p-2)/2},$$

where

$$g(w,\tau) = \frac{\lambda_1\lambda_2 a w^{-\frac{1}{2}}}{\left(c^2+a-w\right)^2 \sqrt{2\pi}\, 2^{(p-1)/2}\Gamma\left(\frac{p-1}{2}\right)} + \frac{\lambda_1^3 a w^{-\frac{3}{2}}\left(p-3-w\right)}{6\left(c^2+a-w\right)^2 \sqrt{2\pi}\, 2^{(p-1)/2}\Gamma\left(\frac{p-1}{2}\right)} + \frac{\lambda_1^3 a w^{-\frac{1}{2}}}{3\left(c^2+a-w\right)^3 \sqrt{2\pi}\, 2^{(p-1)/2}\Gamma\left(\frac{p-1}{2}\right)}$$

and $a = p-2$.

Proof: Let

$$S_1(t) = \int_w^{w+t}\int_{h(x)}^{\sqrt{x}} f\left(x,y\right)dydx,$$
$$S_2(t) = \int_{w+t}^{w}\int_{-\sqrt{x}}^{h(x)} f(x,y)dydx.$$

By using the Mean Value theorem for the outer integral by dx, we can show that

$$R_p\left(\tau\right) = S_1\left(\Delta_1\right) - S_2\left(\Delta_2\right) + O\left(\tau^4\right).$$

Taylor series expansion for $S_1(\Delta_1)$ and $S_2(\Delta_2)$ gives:

$$S_i(\Delta_i) = S_i(0) + S_i'(0)\Delta_i + S_i''(0)\frac{\Delta_i^2}{2!} + S_i'''(0)\frac{\Delta_i^3}{3!} + O(\Delta_i^4), i = 1,2.$$

Substituting values of Δ_1 and Δ_2 from (4) and (5) and since $S_1(0) = S_2(0) = 0$, we obtain that

$$\begin{aligned} S_1(\Delta_1) - S_2(\Delta_2) &= 2S_1'(0)\lambda_2\tau^2 + \frac{\lambda_1\tau^2}{2}(S_1''(0) - S_2''(0)) + \lambda_1\lambda_2\tau^3(S_1''(0) + S_2''(0)) \\ &+ \frac{\lambda_1^3\tau^3}{6}(S_1'''(0) + S_2'''(0)) + \lambda_3\tau^3(S_1'(0) - S_2'(0)) + O(\tau^4). \end{aligned} \tag{6}$$

Our next step is to find $S_1'(0), S_1''(0), S_1'''(0), S_2'(0), S_2''(0),$ and $S_2'''(0)$.

We have

$$S_1'(t) = \frac{d}{dt}\left[\int_w^{w+t}\int_{h(x)}^{\sqrt{x}} f\left(x,y\right)dydx\right] = \int_{h(w+t)}^{\sqrt{w+t}} f(w+t,y)dy$$

and since $h(w) = 0$,

$$S_1'(0) = \int_0^{\sqrt{w}} f(w,y)dy = \frac{e^{-w/2}w^{(p-2)/2}}{2^{(p+2)/2}\Gamma(p/2)}.$$

From

$$S_1''(t) = \frac{d}{dt}\left[\int_{h(w+t)}^{\sqrt{w+t}} f(w+t)dy\right],$$

by Leibniz theorem and the fact that $f(w+t,\sqrt{w+t}) = 0$, we get

$$S_1''(t) = -h'(w+t)\cdot f(w+t,h(w+t)) + \int_{h(w+t)}^{\sqrt{w+t}} \frac{\partial}{\partial(w+t)} f(w+t,y)dy.$$

Therefore, since $h(w) = 0$,

$$S_1''(0) = -h'(w)f(w,0) + \int_0^{\sqrt{w_2}} \frac{\partial}{\partial w} f(w,y)dy = -h'(w)f(w,0) + \frac{(a-w)e^{-w/2}w^{(p-4)/2}}{2^{(p+4)/2}\Gamma(p/2)}.$$

From

$$S_1'''(t) = \frac{d}{dt}\left[\int_{h(w+t)}^{\sqrt{w+t}} \frac{\partial}{\partial(w+t)} f(w+t,y)dy - h'(w+t)f(w+t,h(w+t))\right],$$

by Leibniz theorem, we get

$$S_1'''(0) = \int_{h(w)}^{\sqrt{w}} \frac{\partial^2}{\partial w^2} f(w,y)dy - h'(w)\frac{\partial}{\partial w} f(w,h(w))$$
$$- h''(w) f(w,h(w)) - h'(w)\frac{\partial}{\partial w} f(w,h(w))$$
$$-\left(h'(w)\right)^2 \frac{\partial}{\partial h(w)} f\left(w,h(w)\right).$$

In exactly the same way as for derivatives of $S_1(t)$, we obtain

$$S_2'(0) = \int_0^{-\sqrt{w}} f(w,y)dy = -h'(w) f(w,0) - \frac{(a-w)e^{-w/2} w^{(p-4)/2}}{2^{(p+4)/2}\Gamma(p/2)},$$

$$S_2''(0) = -h'(w) f(w,0) + \int_0^{-\sqrt{w}} \frac{\partial}{\partial w} f(w,y)dy = -h'(w) f(w,0) - \frac{(a-w)e^{-w/2} w^{(p-4)/2}}{2^{(p+4)/2}\Gamma(p/2)},$$

$$S_2'''(0) = -\frac{\partial}{\partial w} f(w,h(w))h'(w_2) + \int_{h(w)}^{-\sqrt{w}} \frac{\partial^2}{\partial w^2} f(w,y)dy - h''(w) f(w,h(w))$$
$$- h'(w)\frac{\partial}{\partial w} f(w,h(w)) - (h'(w))^2 \frac{\partial}{\partial h(w)} f(w,h(w)).$$

Note that from the equations above we have

$$S_1'(0) + S_2'(0) = 0,$$

$$S_1''(0) - S_2''(0) = \frac{2(a-w)e^{-w/2} w^{(p-4)/2}}{2^{(p+4)/2}\Gamma(p/2)},$$

$$S_1''(0) + S_2''(0) = -2h'(w) f(w,0),$$

$$S_1'''(0) + S_2'''(0) = -4h'(w)\frac{\partial}{\partial w} f(w,0) - 2h''(w) f(w,0) - 2h'(w)^2 \frac{\partial}{\partial h(w)} f(w,h(w)).$$

Note also that

$$f(w,0) = \frac{e^{-w/2} w^{\frac{p-1}{2}-1}}{\sqrt{2\pi}\, 2^{(p-1)/2}\Gamma\left(\frac{p-1}{2}\right)},$$

$$\frac{\partial}{\partial w} f(w,0) = \frac{e^{-w/2} w^{(p-5)/2}\left(p-3-w\right)}{\sqrt{\pi}\, 2^{(p+2)/2}\Gamma\left(\frac{p-1}{2}\right)},$$

$$h'(w) = \frac{\tau}{2}\left(-1 + \frac{a\left(\frac{1}{\tau^2} - c^2\right)}{\left(c^2 + a - w\right)^2}\right),$$

$$h''(w) = \tau\left(\frac{a\left(\frac{1}{\tau}\right)^2 - c^2}{\left(c^2 + a - w\right)^3}\right).$$

Substituting these formulae to (6), we obtain the result.

3. Numerical Results for Estimating the Coverage Probabilities

The estimates of the coverage probabilities will be compared with the confidence coefficient, $1-\alpha$, and the exact values, $Q_p^+(\tau)$, to judge the validity and the accuracy of them, respectively. The notations are defined by

$Q_p^+(\tau)$ is the exact coverage probability of the confidence set centered at the positive part James-Stein estimator.

$Q_1 = K_p(w_2)$, Q_2 and Q_3 are the estimates of coverage probability by the first, second and third orders of asymptotic expansions, respectively.

$\Delta_1 = Q_1 - 0.95, \Delta_2 = Q_2 - 0.95$ and $\Delta_3 = Q_3 - 0.95$ are validity of the estimates of coverage probability by the first, second and third orders of asymptotic expansions, respectively.

$\Delta_{a\_1} = Q_p^+(\tau) - Q_1, \Delta_{a\_2} = Q_p^+(\tau) - Q_2$ and $\Delta_{a\_3} = Q_p^+(\tau) - Q_3$ are the accuracy of the estimates of coverage probability by the first, second and third orders of asymptotic expansions, respectively.

Table 1 The estimates of the coverage probabilities by the first, second and third orders of asymptotic expansions and their accuracies in the case $p = 4$, $1-\alpha = 0.95$, $c = 3.0802$

τ	$Q_p^+(\tau)$	Q_1	Q_2	Q_3	Δ_1
0.0	0.9895907	0.9895907	0.9895907	0.9895907	0.0395907
0.1	0.9895812	0.9895777	0.9895917	0.9895926	0.0395777
0.2	0.9895527	0.9895388	0.9895950	0.9895989	0.0395388
0.3	0.9895053	0.9894739	0.9896011	0.9896101	0.0394739
0.4	0.9894388	0.9893830	0.9896113	0.9896276	0.0393830
0.5	0.9893531	0.9892660	0.9896269	0.9896533	0.0392660
0.6	0.9892482	0.9891230	0.9896498	0.9896894	0.0391230
0.7	0.9891240	0.9889538	0.9896820	0.9897384	0.0389538
0.8	0.9889803	0.9887585	0.9897256	0.9898033	0.0387585
0.9	0.9888169	0.9885372	0.9897828	0.9898872	0.0385372
1.0	0.9886336	0.9882897	0.9898552	0.9899927	0.0382897
2.0	0.9856227	0.9844645	0.9911019	0.9923714	0.0344645
3.0	0.9795461	0.9788823	0.9809897	0.9880586	0.0288823
4.0	0.9624427	0.9730516	0.9060755	0.9325821	0.0230516
5.0	0.9577025	0.9680648	0.6839772	0.7555455	0.0180648

Table 1 (Continued)

τ	Δ_2	Δ_3	Δ_{a_1}	Δ_{a_2}	Δ_{a_3}
0.0	0.0395907	0.0395907	0.0000000	0.0000000	0.0000000
0.1	0.0395917	0.0395926	0.0000035	-0.0000105	-0.0000114
0.2	0.0395950	0.0395989	0.0000139	-0.0000423	-0.0000462
0.3	0.0396011	0.0396101	0.0000314	-0.0000958	-0.0001048
0.4	0.0396113	0.0396276	0.0000558	-0.0001725	-0.0001888
0.5	0.0396269	0.0396533	0.0000871	-0.0002738	-0.0003002
0.6	0.0396498	0.0396894	0.0001252	-0.0004016	-0.0004412
0.7	0.0396820	0.0397384	0.0001702	-0.0005580	-0.0006144
0.8	0.0397256	0.0398033	0.0002218	-0.0007453	-0.0008230
0.9	0.0397828	0.0398872	0.0002797	-0.0009659	-0.0010703
1.0	0.0398552	0.0399927	0.0003439	-0.0012216	-0.0013591
2.0	0.0411019	0.0423714	0.0011582	-0.0054792	-0.0067487
3.0	0.0309897	0.0380586	0.0006638	-0.0014436	-0.0085125
4.0	-0.0439245	-0.0174179	-0.0106089	0.0563672	0.0298606
5.0	-0.2660228	-0.1944545	-0.0103623	0.2737253	0.2021570

Table 2 The estimates of the coverage probabilities by the first, second and third orders of asymptotic expansions and their accuracies in the case $p = 7$, $1-\alpha = 0.95$, $c = 3.7506$

τ	$Q_p^+(\tau)$	Q_1	Q_2	Q_3	Δ_1
0.0	0.9982811	0.9982811	0.9982811	0.9982811	0.0482811
0.1	0.9982784	0.9982783	0.9982790	0.9982791	0.0482783
0.2	0.9982703	0.9982699	0.9982729	0.9982730	0.0482699
0.3	0.9982567	0.9982557	0.9982627	0.9982629	0.0482557
0.4	0.9982377	0.9982359	0.9982490	0.9982494	0.0482359
0.5	0.9982130	0.9982101	0.9982318	0.9982325	0.0482101
0.6	0.9981827	0.9981782	0.9982116	0.9982129	0.0481782
0.7	0.9981466	0.9981401	0.9981892	0.9981912	0.0481401
0.8	0.9981045	0.9980955	0.9981652	0.9981683	0.0480955
0.9	0.9980564	0.9980442	0.9981404	0.9981452	0.0480442
1.0	0.9980019	0.9979859	0.9981161	0.9981230	0.0479859
2.0	0.9970488	0.9969280	0.9982286	0.9983785	0.0469280
3.0	0.9950096	0.9946789	0.9992730	1.0008916	0.0446789
4.0	0.9873941	0.9909656	0.9829493	0.9945707	0.0409656
5.0	0.9765813	0.9861608	0.8424270	0.9004856	0.0361608

Table 2 (Continued)

τ	Δ_2	Δ_3	Δ_{a_1}	Δ_{a_2}	Δ_{a_3}
0.0	0.0482811	0.0482811	0.0000000	0.0000000	0.0000000
0.1	0.0482790	0.0482791	0.0000001	-0.0000006	-0.0000007
0.2	0.0482729	0.0482730	0.0000004	-0.0000026	-0.0000027
0.3	0.0482627	0.0482629	0.0000010	-0.0000060	-0.0000062
0.4	0.0482490	0.0482494	0.0000018	-0.0000113	-0.0000117
0.5	0.0482318	0.0482325	0.0000029	-0.0000188	-0.0000195
0.6	0.0482116	0.0482129	0.0000045	-0.0000289	-0.0000302
0.7	0.0481892	0.0481912	0.0000065	-0.0000426	-0.0000446
0.8	0.0481652	0.0481683	0.0000090	-0.0000607	-0.0000638
0.9	0.0481404	0.0481452	0.0000122	-0.0000840	-0.0000888
1.0	0.0481161	0.0481230	0.0000160	-0.0001142	-0.0001211
2.0	0.0482286	0.0483785	0.0001208	-0.0011798	-0.0013297
3.0	0.0492730	0.0508916	0.0003307	-0.0042634	-0.0058820
4.0	0.0329493	0.0445707	-0.0035715	0.0044448	-0.0071766
5.0	-0.107573	-0.0495144	-0.0095795	0.1341543	0.0760957

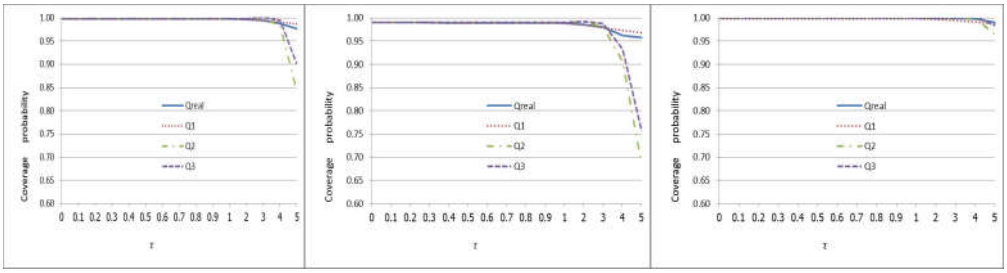
Table 3 The estimates of the coverage probabilities by the first, second and third orders of asymptotic expansions and their accuracies in the case $p=10$, $1-\alpha=0.95$, $c=4.2787$

τ	$Q_p^+(\tau)$	Q_1	Q_2	Q_3	Δ_1
0.0	0.9996469	0.9996469	0.9996469	0.9996469	0.0496469
0.1	0.9996462	0.9996463	0.9996460	0.9996460	0.0496463
0.2	0.9996443	0.9996444	0.9996433	0.9996432	0.0496444
0.3	0.9996410	0.9996412	0.9996388	0.9996386	0.0496412
0.4	0.9996363	0.9996367	0.9996325	0.9996321	0.0496367
0.5	0.9996303	0.9996309	0.9996245	0.9996239	0.0496309
0.6	0.9996228	0.9996237	0.9996148	0.9996139	0.0496237
0.7	0.9996139	0.9996151	0.9996036	0.9996023	0.0496151
0.8	0.9996035	0.9996049	0.9995906	0.9995890	0.0496049
0.9	0.9995915	0.9995931	0.9995763	0.9995743	0.0495931
1.0	0.9995778	0.9995797	0.9995607	0.9995583	0.0495797
2.0	0.9993246	0.9993193	0.9994064	0.9994093	0.0493193
3.0	0.9987186	0.9986602	0.9998135	1.0000323	0.0486602
4.0	0.9973294	0.9972363	0.9994045	1.0019881	0.0472363
5.0	0.9902316	0.9947171	0.9641057	0.9836416	0.0447171

Table 3 (Continued)

τ	Δ_2	Δ_3	Δ_{a_1}	Δ_{a_2}	Δ_{a_3}
0.0	0.0496469	0.0496469	0.0000000	0.0000000	0.0000000
0.1	0.0496460	0.0496460	-0.0000001	0.0000002	0.0000002
0.2	0.0496433	0.0496432	-0.0000001	0.0000010	0.0000011
0.3	0.0496388	0.0496386	-0.0000002	0.0000022	0.0000024
0.4	0.0496325	0.0496321	-0.0000004	0.0000038	0.0000042
0.5	0.0496245	0.0496239	-0.0000006	0.0000058	0.0000064
0.6	0.0496148	0.0496139	-0.0000009	0.0000080	0.0000089
0.7	0.0496036	0.0496023	-0.0000012	0.0000103	0.0000116
0.8	0.0495906	0.0495890	-0.0000014	0.0000129	0.0000145
0.9	0.0495763	0.0495743	-0.0000016	0.0000152	0.0000172
1.0	0.0495607	0.0495583	-0.0000019	0.0000171	0.0000195
2.0	0.0494064	0.0494093	0.0000053	-0.0000818	-0.0000847
3.0	0.0498135	0.0500323	0.0000584	-0.0010949	-0.0013137
4.0	0.0494045	0.0519881	0.0000931	-0.0020751	-0.0046587
5.0	0.0141057	0.0336416	-0.0044855	0.0261259	0.0065900

Numerical illustrations presented in Tables 1-3 show that the estimates of the coverage probability by the first, second and third orders of the asymptotic expansions with τ close to 0 are not significantly different. All of them are close to the exact values. This means that they have high accuracy. For large τ, the first order asymptotic expansions slowly decrease to nominal coverage probability $1-\alpha = 0.95$ but the second and the third orders of asymptotic expansions show that the estimates are below $1-\alpha = 0.95$ when τ becomes close to c. In this particular case of τ close to c, the accuracy of the first order approximation is actually better than the second order and third order approximations. In the neighborhood of the point $\tau = c$, there is an irregular behavior of the second and third order approximations.



(a) $p=4, c=3.0802$ (b) $p=7, c=3.7506$ (c) $p=10, c=4.2787$

Figure 1 The comparison of the estimates of coverage probabilities, $1-\alpha = 0.95$

Figure 1 shows that the estimates of coverage probabilities by the first, second and third orders of asymptotic expansions produce reliable approximations, but the first order approximation gives the best result especially when τ is far from 0.

4. Conclusion and Discussion

The numerical illustrations presented show that all approximations provide high accuracy for the coverage probability with $\tau \to 0$. In the case large τ, the third order approximation reduces the accuracy of both the first and second order approximations. The first order approximation has the simplest formula. The accuracy depends on both p and τ. Note the accuracy is meaningless in the neighborhood of the point $\tau = c$. It looks like there is some relationship between p and τ that affects the accuracy. Namely, for larger p the asymptotic probability approaches nominal coverage probability more slowly than for smaller p. However, it can be confidently stated that all of estimations provide substantial improvements in the coverage probability than for the confidence set centered at the usual sample mean.

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