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The Performance of MCUSUM Control Charts when the Multivariate Normality Assumption Is Violated

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Abstract

A multivariate cumulative sum (MCUSUM) control chart is one type of multivariate control chart for monitoring the mean vector. A multivariate normal distribution is an important assumption that is used to describe a behavior of a set of quality characteristics of interest. This research explores the sensitivity of ARLs and SDRLs when the MVN assumption is incorrect. ARLs and SDRLs for data from multivariate *t,* uniform, beta, and lognormal distributions are estimated and compared to ARLs and SDRLs under the MVN assumption. The ratios of SDRL/ARL are also computed to consider a relationship between ARL and SDRL.

Keywords: MCUSUM control chart, average run length, standard deviation of run length, multivariate distributions.

1. Introduction

In statistical process control, a control chart is one of the graphical techniques used to monitor a manufacturing process for detecting any change in a process that may affect a product's quality. A quality characteristic of a manufactured product that is measured on a numerical scale is called a variable. There are many situations in which the simultaneous monitoring of two or more related variables is necessary. Monitoring these variables independently can lead to inaccurate conclusions. Process monitoring in which several related quality characteristics are of interest is known as multivariate statistical process control. A useful tool of multivariate statistical process control is the multivariate control chart. Various types of multivariate control charts have been widely applied, including the multivariate Shewhart, multivariate exponentially weighted moving average (MEWMA), and multivariate cumulative sum (MCUSUM) control charts. Generally the multivariate Shewhart control chart uses information only from a current sample to calculate the test statistic and it is insensitive to small and moderate shifts in a mean vector. In addition a practical problem with multivariate Shewhart control charts is their lack of robustness. They are sensitive to multivariate outliers. A multivariate outlier is an observation vector $\boldsymbol{X}$ that has a large T^2 statistic. The MEWMA control chart uses weighted averages of previously observed random vectors to monitor the mean vector of the process. The MCUSUM control chart proposed by Crosier (1988) is derived by replacing the scalar quantities of a univariate CUSUM control chart by vectors and offers the advantage of a CUSUM chart over a Shewhart chart such as an ability to design the control chart to detect a specific shift in the mean vector. Therefore the MCUSUM control chart is an alternative method which has been developed to overcome the disadvantages of multivariate Shewhart control charts.

For most multivariate control charts, a multivariate normal distribution is an important assumption that is used to describe the behavior of a quality characteristic of interest. In many real situations, however, that assumption is not always met. The violations of the multivariate normality assumption affect the performance of multivariate Shewhart and MEWMA control charts in different ways (Nidsunkid et al. 2017). In this research, the performance of the MCUSUM control chart for multivariate non-normal distributions is investigated for different shifts in the mean vector. The average run length (ARL) and the standard deviation of run length (SDRL) of the MCUSUM are estimated and reported to consider how robust or sensitive a MCUSUM control chart is to violations of the multivariate normal assumption.

2. The Multivariate Cumulative Sum Control Charts

Woodall and Ncube (1985) studied the performance of a bivariate CUSUM procedure that consists of two one-sided univariate CUSUM procedures. In this case they found that the bivariate CUSUM procedure can detect small and moderate shifts in the means more quickly than the Shewhart procedure. In addition they also compared a bivariate cumulative score procedure based on the principle components to a Shewhart chart. For this case they concluded that the bivariate cumulative score procedure with the principle components detects small shifts more quickly than the Shewhart chart.

To study p response variables $(p \geq 2)$ consider the following multivariate assumptions. Let $\boldsymbol{X}_i$, $i=1,2,\ldots$, be a $p\times 1$ random vector that follows p-variate normal distribution with the $p\times 1$ in-control mean vector $\boldsymbol{\mu}_0$ and a $p\times p$ covariance matrix Σ. Let $\boldsymbol{\mu}_1$ be a $p\times 1$ out-of-control mean vector. For the following multivariate CUSUM procedures, an out-of-control signal occurs when the CUSUM statistic exceed a (scalar) decision interval H. The choice of a value for H can be found from simulations. Healy (1987) discussed the CUSUM procedures assuming the multivariate normal distribution and proposed two cases; the CUSUM for detecting a shift in a mean vector and the CUSUM for detecting a shift in a covariance matrix. The CUSUM at sample i for detecting a shift in the mean vector $\boldsymbol{\mu}_0$ towards $\boldsymbol{\mu}_1$ can be written as

$$S_i = \max\left[\left(S_{i-1} + \boldsymbol{a}'(\boldsymbol{X}_i - \boldsymbol{\mu}_0) - 0.5D\right), 0\right], \tag{1}$$

where $\boldsymbol{a}' = \left[(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)'\Sigma^{-1}\right]/D$ and D is the noncentrality parameter defined as

$$D = \sqrt{(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)'\Sigma^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)}\,. \tag{2}$$

This CUSUM procedure signals when $S_i > H$. Since S_i for this multivariate CUSUM procedure reduces to a univariate CUSUM procedure, Healy (1987) suggested that all of the available theory for calculating ARL, H and S_0 for the univariate CUSUM procedure can be used for this multivariate CUSUM.

For detecting a shift in a covariance matrix Σ. Healy (1987) proposed a CUSUM for detecting a change from Σ to $C\Sigma$, where C is a real constant. The CUSUM procedure statistic for sample i is

$$S_i = \max\left[\left(S_{i-1} + \boldsymbol{a}'(\boldsymbol{X}_i - \boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{X}_i - \boldsymbol{\mu}) - \boldsymbol{K}\right), 0\right], \tag{3}$$

where $\boldsymbol{\mu}$ is a mean vector, $\boldsymbol{K} = p\log(C)\left[C/(C-1)\right]$. This CUSUM procedure signals when $S_i > H.$

Crosier (1988) proposed two schemes for a multivariate CUSUM. The first is a CUSUM of T (COT) statistics scheme which is based on the square root of the multivariate Shewhart statistic. The COT scheme is given by

$$S_i = \max\left[\left(S_{i-1} + T_i - k\right), 0\right], \tag{4}$$

where $S_0 \geq 0,$ $k > 0,$ and $T_i = +\left[(\boldsymbol{X}_i - \boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{X}_i - \boldsymbol{\mu})\right]^{1/2}$. The COT scheme signals when $S_i > H.$

The second CUSUM scheme proposed by Crosier (1988) is derived by replacing the scalar quantities of a univariate CUSUM scheme by vectors. Let

$$C_i = \left[\left(\boldsymbol{S}_{i-1} + \boldsymbol{X}_i - \boldsymbol{\mu}\right)' \Sigma^{-1} \left(\boldsymbol{S}_{i-1} + \boldsymbol{X}_i - \boldsymbol{\mu}\right)\right]^{1/2}, \tag{5}$$

and

$$\boldsymbol{S}_i = \begin{cases} \boldsymbol{0} & \text{if } C_i \leq k \\ \left(\boldsymbol{S}_{i-1} + \boldsymbol{X}_i - \boldsymbol{\mu}\right)\left(1 - k/C_i\right) & \text{if } C_i > k \end{cases}, \tag{6}$$

where $\boldsymbol{\mu}$ is the in-control mean vector, Σ is the covariance matrix, $\boldsymbol{S}_0 = \boldsymbol{0},$ and reference value $k > 0.$ Crosier (1988) stated that the choice of $k = D/2$ where D is defined in (2), appears to minimize the ARL at D for a given in-control ARL. The multivariate CUSUM is given by

$$Y_i = \left[\boldsymbol{S}_i' \Sigma^{-1} \boldsymbol{S}_i\right]^{1/2}. \tag{7}$$

The scheme signals when $Y_i > H.$ Moreover Crosier (1988) compared these two schemes of multivariate CUSUM with the multivariate Shewhart chart and concluded that the CUSUM vector-valued scheme has a better ARL performance than the COT scheme.

Pignatiello and Runger (1990) proposed two other multivariate CUSUM charts, CUSUM #1 ($MC1$) and CUSUM #2 ($MC2$). The $MC1$ chart can be defined for sample i as

$$MC1_i = \max\left[\left(\boldsymbol{C}_i' \Sigma^{-1} \boldsymbol{C}_i\right)^{1/2} - kn_i, 0\right], \tag{8}$$

where $\boldsymbol{C}_i = \sum_{l=i-n_i+1}^{i} (\boldsymbol{X}_l - \boldsymbol{\mu}_0),$ the reference value $k > 0,$ and n_i is the number of subgroup defined as

$$n_i = \begin{cases} n_{i-1} + 1 & , MC1_{i-1} > 0 \\ 1 & , \text{otherwise} \end{cases}. \tag{9}$$

The ARL performance of the $MC1$ chart depends only on the noncentrality parameter. The second scheme, $MC2$ control chart, is defined as

$$MC2_i = \max\left[0, \left(MC2_{i-1} + D_i^2 - k\right)\right], \tag{10}$$

where $MC2_0 = 0,$ $D_i^2 = (\boldsymbol{X}_i - \boldsymbol{\mu}_0)'\Sigma^{-1}(\boldsymbol{X}_i - \boldsymbol{\mu}_0).$ The upper control limits for the $MC1$ and $MC2$ charts are investigated by simulation, and they also provide a discussion about choice of the reference value k.

Pignatiello and Runger (1990) used the Markov chain approach and Monte Carlo simulation to compare the ARL performance of several schemes such as $MC1$, $MC2$, several multiple univariate CUSUM charts as developed by Woodall and Ncube (1985), and a multivariate Shewhart chart for monitoring a multivariate normal process. The ARL performance of these charts showed that $MC1$ is a good control chart for detecting a variety of shifts in the mean of a multivariate normal process. For detecting large shifts in the process mean, a multivariate Shewhart control chart appears to be a good procedure. Both $MC1$ and several multiple univariate CUSUM charts perform well compared to the multivariate Shewhart chart for detecting a small shift in the mean.

Mahmoud and Maravelakis (2013) studied the effect of estimating the vector of means and the covariance matrix on the ARL performance of the multivariate CUSUM chart proposed by Crosier (1988) and the $MC1$ chart proposed by Pignatiello and Runger (1990). They found that when parameters are unknown and replaced by parameter estimates, the $MC1$ chart has better in-control and out-of-control ARL performance than the multivariate CUSUM chart, especially when the number of Phase I samples m is small.

3. Multivariate Distributions

Suppose we have p random variables, given by $X_1, X_2, ..., X_p$, and write these random variables in terms of random vector $\boldsymbol{X} = (X_1, X_2, ..., X_p)'$. If $X_1, X_2, ..., X_p$ are independent normal random variables with mean μ_i and variance σ^2, then the multivariate probability density function of $\boldsymbol{X}$ is given by

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{p/2}\left|\sigma^2\boldsymbol{I}\right|^{1/2}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\left(\sigma^2\boldsymbol{I}\right)^{-1}(\boldsymbol{x}-\boldsymbol{\mu})} \quad , \boldsymbol{x} \in \mathbb{R}^p. \tag{11}$$

The vector x has an independent multivariate normal (MVN) distribution (Rencher 2002), and is denoted $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \sigma^2\boldsymbol{I})$.

In general, if X_i are not independent, random vector x has a multivariate normal (MVN) distribution with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_p)'$ and $p \times p$ covariance matrix Σ, and the multivariate probability density function of $\boldsymbol{X}$ is given by

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{p/2}\left|\Sigma\right|^{1/2}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})} \quad , \boldsymbol{x} \in \mathbb{R}^p. \tag{12}$$

The MVN distribution can be defined in various ways, one is with its stochastic representation (Hofert 2013)

$$\boldsymbol{X} = \boldsymbol{\mu} + A\boldsymbol{Z}, \tag{13}$$

where $\boldsymbol{Z} = (Z_1, Z_2, ..., Z_k)$ is a k-dimensional random vector with Z_i, $i \in \{1, ..., k\}$, being independent standard normal random variables, $A \in \mathbb{R}^{p \times k}$ is a (p, k)-matrix, and $\boldsymbol{\mu} \in \mathbb{R}^p$ is the mean vector. The covariance matrix of $\boldsymbol{X}$ is $\Sigma = AA'$. We assume $k = p$, if Σ is positive definite (thus has full rank and is therefore invertible).

A random vector $\boldsymbol{X}$ is said to follow a p-variate t distribution with degrees of freedom ν, mean vector $\boldsymbol{\mu}$, and correlation matrix $\boldsymbol{R}$ (and with Σ denoting the corresponding covariance matrix) if its probability density function is given by

$$f(\boldsymbol{x}) = \frac{\Gamma\left(\frac{\nu+p}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{p/2}\left|\boldsymbol{R}\right|^{1/2}}\left(1+\frac{(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{R}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}{\nu}\right)^{-\frac{\nu+p}{2}}, \boldsymbol{x}\in\mathbb{R}^p. \tag{14}$$

The degrees of freedom ν is referred to as the shape parameter (Kotz and Naradajah 2004), and we can denote $\boldsymbol{X} \sim t_\nu(\boldsymbol{\mu},\Sigma)$.

Hofert (2013) stated that the multivariate t (MVT) distribution with degrees of freedom ν can be defined by the stochastic representation

$$\boldsymbol{X} = \boldsymbol{\mu} + \sqrt{W}A\boldsymbol{Z}, \tag{15}$$

where $W = \nu/\chi_\nu^2$, and χ_ν^2 is a random variable following a chi-squared distribution with degrees of freedom $\nu > 0$. W is independent of $\boldsymbol{Z}$, and all other quantities are as in (13). By introducing the additional random factor $\sqrt{W}$, the MVT distribution with ν degrees of freedom is more flexible than the MVN distribution (which can be recovered by taking the limit $\nu \to \infty$) especially in the tails which are heavier for $t_\nu(\boldsymbol{\mu},\Sigma)$ than for $N_p(\boldsymbol{\mu},\Sigma)$. If W and $\boldsymbol{Z}$ are independent and $\nu > 1$, the mean vector of $\boldsymbol{X}$ is $E(\boldsymbol{X}) = \boldsymbol{\mu}$, and the covariance matrix of $\boldsymbol{X}$ is $\text{Cov}(\boldsymbol{X}) = \frac{\nu}{\nu-2}\Sigma$.

The structure of dependence between two or more related variables can be defined in terms of their joint distribution. One way to obtain a multivariate distribution is joining the univariate distribution through copulation, which is one of the most useful tools when the marginal distributions are known or given. The use of copula functions enables the representation of various types of dependence between variables. Nelsen (2006) defined a copula function as a joint distribution function

$$C(u_1,u_2,...,u_p) = P(U_1 \le u_1, U_2 \le u_2, ..., U_p \le u_p), \tag{16}$$

where $0 \le u_j \le 1$ and U_j for $j = 1,2,...,p$ are marginally uniformly distributed in the interval (0,1). Sklar's Theorem shows how to obtain a joint distribution using marginal distribution functions F_i and the copula C.

Sklar's Theorem *Let H be a p-dimensional distribution function with marginal distribution functions $F_1, F_2, ..., F_p$. Then there exists a p-dimensional copula C such that for all $(y_1, y_2, ..., y_p) \in [-\infty, \infty]^p$,*

$$H(y_1,y_2,...,y_p) = C[F_1(y_1), F_2(y_2), ..., F_p(y_p)]. \tag{17}$$

If $F_1, F_2, ..., F_p$ are all continuous, then C is unique; otherwise C is uniquely determined on $\text{Range}\,F_1 \times \text{Range}\,F_2 \times ... \times \text{Range}\,F_p$, *the cartesian product of the ranges of the marginal cdf's. Conversely, if C is a p-copula and $F_1, F_2, ..., F_p$ are distribution functions, then the function H defined by* (17) *is a p-dimensional distribution function with marginal distribution functions $F_1, F_2, ..., F_p$.*

There are several families of copulas such as elliptical copulas (Gaussian and t copulas are known as elliptical copulas) and Archimedean copulas (e.g., Clayton, Frank, and Gumbel copulas) (Trivedi and Zimmer 2005). Sukparungsee et al. (2015) studied the performance of MCUSUM control chart when the observations are drawn from an exponential distribution by using Gaussian, Clayton and Frank copulas. The level of dependence between random variables is measured by Kendall's tau. Their results implied that, for small and large dependencies, it is necessary to detect the observation dependence to indicate the copula and use that detection to fit the observation on the MCUSUM chart.

In this research, we use the Gaussian copula to construct the multivariate distributions. Let Φ_R is the joint cumulative distribution function of a multivariate normal distribution with mean vector zero and covariance matrix equal to the correlation matrix R. Then C_R, the Gaussian copula corresponding to Φ_R, is given by

$$C_R(u_1, u_2, ..., u_p) = \Phi_R(\Phi^{-1}(u_1), \Phi^{-1}(u_2), ..., \Phi^{-1}(u_p)), \tag{18}$$

where Φ^{-1} is the inverse cumulative distribution function of a standard normal.

If $U_1, U_2, ..., U_p$ are independent, $C(u_1, u_2, ..., u_p) = u_1 \times u_2 \times \cdots \times u_p$. Therefore, for independent case, we can construct many multivariate distributions, such as a multivariate lognormal (MVL), a multivariate beta (MVB), and a multivariate uniform (MVU) distribution, by using a copula function and their marginal univariate distributions. The details of these three univariate distributions are now presented.

If $Z = \log(X)$ is normally distributed, the distribution of X is said to be lognormal (Johnson et al. 1994). Thus, X is a positive random variable, and the probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\log x-\mu)^2}{2\sigma^2}} & , x > 0 \\ 0 & , \text{otherwise,} \end{cases} \tag{19}$$

where $-\infty < \mu < \infty$, and $\sigma > 0$.

The standard beta distribution is a distribution defined on the interval [0,1]. The probability density function of the standard beta with parameters a and b is given by

$$f(x) = \begin{cases} \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} & , 0 \le x \le 1 \\ 0 & , \text{otherwise,} \end{cases} \tag{20}$$

where $a > 0$, and $b > 0$ (Johnson et al. 1995).

The continuous uniform distribution is a member of the family of symmetric probability distributions, and is defined by the two parameters a and b, which are its minimum and maximum values. The probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & , a \le x \le b \\ 0 & , \text{otherwise}. \end{cases} \tag{21}$$

The case where $a = 0$ and $b = 1$ is called the standard uniform distribution which is a special case of standard beta distribution when the parameters a and b are both equal to one (Johnson et al. 1995).

4. Run Length Properties

For a control chart, the run length is a random variable and is defined to be the number of subgroups, which must be collected (or equivalently, the number of charting statistics that must be plotted) until the first signal is observed suggesting a change from the in-control process.

A control chart is useful in detecting if a process is or is not in a state of statistical control. In practice, the performance of a control chart is considered in terms of certain measures associated with its run length distribution, including the average run length (ARL) and standard deviation of the run length (SDRL). The ARL and the SDRL are an average and the standard deviation of the number of subgroups sampled (or number of charting statistics needed to be plotted) before an out-of-control signal is detected, respectively.

The ARL and SDRL are widely used to show the performance of control charts. Somran et al. (2015) presented ARL results for a negative CUSUM chart for a lower-sided case when the observations are from exponential distribution. Mahmoud and Maravelakis (2013) used ARLs and SDRLs to study the effect of estimating the vector of means and the variance-covariance matrix on the performance of two of the MCUSUM chart proposed by Crosier (1988) and the $MC1$ chart proposed by Pignatiello and Runger (1990)

5. Research Methodology and Simulation Study

For studying the performance of MCUSUM control chart when the multivariate normality assumption is violated, we consider MCUSUM control charts is proposed by Crosier (1988) for individual observations which monitor the mean vector of the process where 2, 3 and 5 quality characteristics are controlled jointly in which mean vector and covariance matrix are known. The reference values, k = 0.5, 1.0, and 1.5 are specified and the values of decision interval (h) are obtained by simulation which are shown in Table 1. Without loss of generality, the in–control mean vectors are (5, 5), (5, 5) and (5, 5, 5, 5, 5), and the covariance are diag (5, 5), diag (5, 5, 5) and (5, 5, 5, 5, 5) for 2, 3 and 5 variables, respectively. Thus, only uncorrelated random variables are consider in this study. The ARLs and SDRLs were then estimated for various shifts in the noncentrality parameter (δ) = 0.00, 0.45, 1.01, 1.58, 2.06 and 3.00 for 2 variables, δ = 0.00, 0.45, 1.03, 1.52, 2.10 and 3.10 for 3 variables and δ = 0.00, 0.45, 1.03, 1.55, 2.02 and 3.00 for 5 variables.

Table 1 The decision interval (h) for MCUSUM to achieve an in-control ARL of 200

	k		
	0.5	1.0	1.5
2 variables	5.49	3.01	1.93
3 variables	6.88	3.77	2.42
5 variables	9.38	5.20	3.36

The random vectors $\boldsymbol{X}$ considered in the simulation come from multivariate normal (MVN), multivariate t (MVT) with 3, 6, 12, 24, and 48 degrees of freedom, multivariate uniform (MVU), multivariate beta (MVB) with shape parameter vector (4, 2), and multivariate lognormal (MVL) distributions. Thus, the comparisons will include several symmetric (MVN, MVT, MVU) distributions, a left-skewed distribution (MVB) and a right-skewed distribution (MVL) with the MVN, MVL, MVU and MVB having the same in-control means and variances. Each simulation is replicated 50,000 times to provide accurate results.

6. Results

The estimated ARLs and SDRLs of the MCUSUM control charts for monitoring the mean vector for each distribution and noncentrality parameters when $k = 0.5$, 1.0, and 1.5 are shown in Tables 2-7. Note that $\delta = 0.00$ corresponds to the in-control ARLs for all 2, 3 and 5 process variables. For MVN distributions, the in-control ARLs are approximately the same value, 200, and the out-of-control ARLs vary inversely as a size of noncentrality parameter, because the out-of-control ARLs depend on the noncentrality parameter.

From Tables 2-4, we can see that when $k = 0.5$ in 2, 3 and 5 variables, the in-control ARLs for the MVU and MVB are greater than the in-control ARLs of MVN, while the out-of-control ARLs for the MVU and MVB close to the out-of-control ARLs of MVN. For MVT, the in-control ARLs are shorter than the in-control ARLs of MVN, moreover, as the degrees of freedom increase, the distribution approaches normality, and the in-control ARLs approach 200. However, the out-of-control ARLs for MVT are greater than MVN out-of-control ARLs, and when the degrees of freedom increase, the out-of-control ARLs for MVT approach MVN out-of-control ARLs. For MVL, the in-control ARLs are smaller than the in-control ARLs of MVN, while the out-of-control ARLs are close to the out-of-control ARLs of MVN.

When $k = 1.0$, the in-control and the first out-of-control ARLs for the MVU and MVB are greater than the in-control ARLs of MVN, and the other out-of-control ARLs are close to MVN out-of-control ARLs. And for MVU, in some cases ARLS were very large, for instance, the estimated in-control ARLs for MVU are 839.25 and 29,045.34. The in-control and the first out-of-control MVT ARLs are smaller than MVN ARLs, while the other out-of-control ARLs are slightly greater than MVN ARLs. For MVL, the in-control and the first out-of-control ARLs are smaller than the ARLs of MVN, and the other out-of-control ARLs are close to MVN ARLs.

When $k = 1.5$, the ARLs are similar to $k = 1.0$ for all multivariate distributions.

Table 2 The ARLs of the MCUSUM control charts for monitoring the mean vector in 2 variables

k	distributions	δ					
		0.00	0.45	1.01	1.58	2.06	3.00
0.5	MVN	200.11	35.60	9.75	5.41	3.99	2.69
	MVT(3)	132.03	90.48	26.70	11.85	7.93	4.85
	MVT(6)	150.78	50.20	13.37	7.01	5.03	3.30
	MVT(12)	176.84	41.65	11.17	6.05	4.42	2.95
	MVT(24)	188.99	38.52	10.39	5.71	4.18	2.80
	MVT(48)	195.35	36.93	10.07	5.55	4.08	2.74
	MVU	241.67	35.06	9.68	5.40	3.99	2.68
	MVB	209.11	35.34	9.60	5.38	3.97	2.66
	MVL	142.70	35.94	10.07	5.50	4.02	2.70
1.0	MVN	200.48	60.77	11.80	4.91	3.25	2.02
	MVT(3)	61.88	58.65	43.23	19.71	9.75	4.36
	MVT(6)	79.59	57.19	19.27	7.26	4.48	2.57
	MVT(12)	119.02	60.25	14.48	5.79	3.72	2.24
	MVT(24)	152.85	61.50	13.00	5.30	3.45	2.12
	MVT(48)	175.21	61.44	12.33	5.08	3.34	2.07
	MVU	839.25	71.46	11.30	4.81	3.23	2.02
	MVB	225.90	82.39	11.29	4.78	3.20	2.01
	MVL	65.37	37.07	12.34	5.15	3.34	2.04
1.5	MVN	199.84	92.07	19.14	6.08	3.40	1.81
	MVT(3)	44.43	43.22	38.19	29.28	18.81	5.88
	MVT(6)	54.99	48.08	27.29	11.15	5.59	2.50
	MVT(12)	86.00	63.17	23.31	7.84	4.14	2.07
	MVT(24)	123.46	75.74	21.16	6.82	3.73	1.92
	MVT(48)	153.32	84.09	20.18	6.43	3.56	1.86
	MVU	29045.34	353.82	19.31	5.76	3.32	1.76
	MVB	278.92	261.21	21.84	5.91	3.28	1.75
	MVL	45.54	33.17	16.31	6.47	3.63	1.86

In addition to the ARL performance, we consider the SDRL for all scenarios. From Tables 5-7, the SDRLs have the same behavior as the ARLs for both in-control and out-of-control processes. That is, the ARLs and SDRLs are highly correlated. Although the ARL and SDRL values appear very similar, their differences can be seen by examining the ratios of SDRL/ARL. These ratios are summarized in Table 8. For 2 variables cases, the ratios of SDRL/ARL≈ 1 for the noncentrality parameter value is 0.00, the ratios decrease as the noncentrality parameter increases but the rate of change decreases as k increases. The patterns of the ratios for 3 and 5 variables are very similar to those in Table 8.

Table 3 The ARLs of the MCUSUM control charts for monitoring the mean vector in 3 variables

k	distributions	δ					
		0.00	0.45	1.03	1.52	2.10	3.10
0.5	MVN	198.60	37.76	10.78	6.61	4.58	3.08
	MVT(3)	134.32	93.04	27.40	14.27	8.98	5.53
	MVT(6)	153.75	52.03	14.41	8.52	5.76	3.77
	MVT(12)	178.36	43.76	12.25	7.37	5.07	3.37
	MVT(24)	191.54	40.56	11.43	6.98	4.81	3.21
	MVT(48)	196.02	39.16	11.10	6.78	4.70	3.15
	MVU	227.83	37.51	10.72	6.59	4.60	3.08
	MVB	205.09	37.47	10.64	6.57	4.56	3.08
	MVL	151.89	38.21	11.04	6.71	4.64	3.09
1.0	MVN	198.87	61.52	11.70	5.73	3.57	2.24
	MVT(3)	58.40	55.65	40.44	21.78	9.94	4.65
	MVT(6)	77.43	56.02	18.34	8.38	4.81	2.81
	MVT(12)	117.00	60.85	14.16	6.73	4.03	2.46
	MVT(24)	154.12	62.27	12.79	6.17	3.77	2.34
	MVT(48)	175.98	61.77	12.11	5.94	3.66	2.28
	MVU	523.52	71.00	11.31	5.63	3.54	2.25
	MVB	222.54	74.75	11.44	5.61	3.50	2.25
	MVL	69.31	40.22	11.94	5.96	3.66	2.25
1.5	MVN	200.44	92.55	17.90	6.85	3.52	1.92
	MVT(3)	37.84	37.35	33.75	28.10	17.33	5.68
	MVT(6)	47.71	42.35	24.94	11.95	5.52	2.60
	MVT(12)	77.70	58.50	21.70	8.71	4.23	2.18
	MVT(24)	115.35	73.67	19.53	7.63	3.82	2.04
	MVT(48)	149.27	82.35	18.91	7.20	3.65	1.97
	MVU	4819.11	260.10	18.53	6.65	3.47	1.90
	MVB	262.72	183.60	20.65	6.68	3.42	1.88
	MVL	42.72	32.62	14.83	7.14	3.71	1.97

7. Discussion

The results in this simulation study highlight the sensitivity (or lack of robustness) of ARLs and SDRLs to violations of the multivariate normality assumptions.

If the violation of the MVN assumption occurs when sampling from a more heavy-tailed MVT distributions then there is a significant reduction in the in-control ARLs (and in the first out-of-control when k =1.0 and 1.5). Thus, the in-control ARL for the MVN case is an overestimate of the true ARLs for the MVT distributions with the bias decreasing as the degrees of freedom increase. However, ARLs for the MVN is an underestimate for all out-of-control cases for k =0.5, and for all cases for k =1.0 and 1.5 when the noncentrality parameter is greater than 1.

If the violation of the MVN assumption occurs when data are sampled from the MVU and MVB distributions, there are now a significant increase in the in–control ARLs. Thus, the ARLs for the MVN are an underestimate of the ARLs for MVU and MVB when the noncentrality parameter is 0.00 but the ARLs for MVU and MVB are very close to MVN ARLs for the cases with the noncentrality parameter is greater than 1. And because of the symmetry and finite support for the

MVU distributions, if the noncentrality parameter was not sufficiently large, then the in–control ARLs are very large, especially when k =1.0 and 1.5.

Table 4 The ARLs of the MCUSUM control charts for monitoring the mean vector in 5 variables

k	distributions	δ					
		0.00	0.45	1.03	1.55	2.02	3.00
0.5	MVN	199.04	41.41	13.09	8.08	6.02	4.03
	MVT(3)	140.95	96.09	30.69	16.70	11.65	7.26
	MVT(6)	158.07	55.70	17.12	10.25	7.52	4.93
	MVT(12)	181.76	47.13	14.71	9.00	6.65	4.41
	MVT(24)	191.77	44.10	13.84	8.51	6.30	4.20
	MVT(48)	195.63	42.59	13.42	8.27	6.16	4.12
	MVU	215.50	41.26	13.02	8.06	6.00	4.02
	MVB	203.78	41.08	12.98	8.04	5.99	4.02
	MVL	167.41	41.73	13.30	8.18	6.06	4.05
1.0	MVN	200.00	62.27	12.68	6.43	4.47	2.81
	MVT(3)	58.41	55.90	40.81	21.40	12.07	5.91
	MVT(6)	78.02	56.65	19.11	9.09	5.96	3.55
	MVT(12)	122.45	61.49	15.09	7.45	5.06	3.11
	MVT(24)	157.47	63.05	13.72	6.95	4.73	2.94
	MVT(48)	178.81	62.14	13.20	6.66	4.60	2.88
	MVU	352.51	68.81	12.51	6.41	4.45	2.80
	MVB	218.92	69.04	12.52	6.39	4.43	2.78
	MVL	84.01	45.21	12.94	6.64	4.56	2.85
1.5	MVN	199.92	90.61	17.50	6.86	4.18	2.37
	MVT(3)	34.16	33.46	30.89	25.91	18.27	6.75
	MVT(6)	43.42	39.22	23.38	11.17	6.31	3.15
	MVT(12)	73.40	55.47	20.51	8.51	4.93	2.66
	MVT(24)	113.30	71.39	18.99	7.58	4.53	2.50
	MVT(48)	149.09	80.74	18.14	7.22	4.34	2.43
	MVU	1181.51	176.32	17.78	6.78	4.14	2.38
	MVB	242.63	133.38	18.56	6.75	4.10	2.35
	MVL	45.99	34.81	15.30	7.13	4.34	2.42

If the violation of the MVN assumption occurs when data are sampled from skewed-right MVL distributions, then once again there is a significant reduction in the in-control ARLs. Thus, the process engineer may have a very biased overestimates of the in-control ARLs while the out-of-control ARLs are very close to MVN ARLs.

Tables 2-7 each highlight the potential for extreme differences for the in-control ARLs and SDRLs when MVN is assumed. Typically, the process engineer has a desired in-control ARL when determining the upper control limit and an ARL for a desired shift to be detected quickly. Thus, is can be very risky to use these upper control limits when samples are based in individual measurements.

Table 5 The SDRLs of the MCUSUM control charts for monitoring the mean vector in 2 variables

k	distributions	δ					
		0.00	0.45	1.01	1.58	2.06	3.00
0.5	MVN	192.00	28.41	4.71	1.91	1.20	0.67
	MVT(3)	129.87	86.46	18.84	5.73	3.12	1.45
	MVT(6)	146.41	43.02	7.45	2.81	1.67	0.87
	MVT(12)	172.83	34.03	5.75	2.25	1.39	0.75
	MVT(24)	182.27	30.96	5.17	2.07	1.28	0.70
	MVT(48)	190.65	29.47	4.94	1.99	1.23	0.69
	MVU	234.37	27.55	4.74	1.92	1.22	0.67
	MVB	202.96	27.52	4.63	1.87	1.17	0.67
	MVL	139.48	29.81	5.01	2.02	1.23	0.67
1.0	MVN	199.30	58.09	8.92	2.66	1.41	0.68
	MVT(3)	61.12	58.08	42.08	16.65	6.43	1.91
	MVT(6)	79.26	56.09	16.51	4.55	2.24	0.93
	MVT(12)	117.61	57.71	11.69	3.34	1.72	0.77
	MVT(24)	151.21	58.83	10.16	2.95	1.55	0.72
	MVT(48)	173.34	58.42	9.50	2.78	1.47	0.70
	MVU	836.36	67.53	8.43	2.59	1.40	0.71
	MVB	224.16	78.17	8.24	2.46	1.33	0.68
	MVL	64.58	35.52	9.99	2.98	1.51	0.69
1.5	MVN	198.89	91.19	17.38	4.53	2.03	0.80
	MVT(3)	43.90	42.51	37.35	28.47	17.57	3.88
	MVT(6)	54.38	47.12	26.35	9.64	3.97	1.21
	MVT(12)	85.30	62.30	21.97	6.23	2.68	0.96
	MVT(24)	122.29	74.17	19.70	5.24	2.30	0.87
	MVT(48)	151.14	82.80	18.75	4.87	2.17	0.83
	MVU	29159.95	352.07	17.31	4.13	1.98	0.85
	MVB	277.49	258.17	19.64	4.10	1.85	0.79
	MVL	44.77	32.61	15.20	5.14	2.30	0.82

Tables 2-7 also confirm what is expected as the noncentrality paremeter become larger. That is, the ARLs converge to zero for all distributions as the magnitude in the shift vector increases.

Table 6 The SDRLs of the MCUSUM control charts for monitoring the mean vector in 3 variables

k	distributions	δ					
		0.00	0.45	1.03	1.52	2.10	3.10
0.5	MVN	189.62	27.89	4.61	2.16	1.23	0.68
	MVT(3)	131.25	86.98	17.82	6.57	3.19	1.50
	MVT(6)	147.22	42.69	7.21	3.17	1.74	0.90
	MVT(12)	170.76	33.97	5.63	2.57	1.42	0.76
	MVT(24)	183.26	30.85	5.07	2.37	1.32	0.71
	MVT(48)	186.87	29.50	4.85	2.28	1.28	0.69
	MVU	217.12	27.63	4.63	2.17	1.23	0.69
	MVB	196.78	27.28	4.47	2.11	1.19	0.66
	MVL	145.80	29.87	5.00	2.29	1.30	0.70
1.0	MVN	195.89	57.56	8.11	2.87	1.39	0.65
	MVT(3)	57.76	54.94	38.57	18.32	6.02	1.84
	MVT(6)	75.88	54.13	14.98	5.03	2.16	0.93
	MVT(12)	114.49	57.69	10.48	3.65	1.67	0.76
	MVT(24)	151.27	58.74	9.16	3.23	1.51	0.70
	MVT(48)	172.58	57.85	8.44	3.05	1.45	0.67
	MVU	525.14	65.80	7.64	2.83	1.39	0.65
	MVB	218.69	69.71	7.53	2.70	1.29	0.60
	MVL	68.20	38.15	8.88	3.21	1.51	0.69
1.5	MVN	199.47	90.67	15.77	4.85	1.87	0.77
	MVT(3)	37.38	36.58	33.04	27.50	15.67	3.39
	MVT(6)	47.11	41.72	23.62	10.15	3.58	1.14
	MVT(12)	76.63	57.56	19.89	6.67	2.46	0.89
	MVT(24)	113.79	72.05	17.58	5.61	2.13	0.82
	MVT(48)	147.79	81.14	16.92	5.20	2.00	0.79
	MVU	4808.25	257.29	16.01	4.59	1.84	0.79
	MVB	260.67	181.98	17.80	4.46	1.70	0.74
	MVL	42.09	31.81	13.48	5.48	2.18	0.81

Table 7 The SDRLs of the MCUSUM control charts for monitoring the mean vector in 5 variables

k	distributions	δ					
		0.00	0.45	1.03	1.55	2.02	3.00
0.5	MVN	185.11	27.52	4.86	2.27	1.43	0.78
	MVT(3)	137.91	87.56	17.95	6.84	3.89	1.84
	MVT(6)	147.93	42.29	7.49	3.34	2.05	1.06
	MVT(12)	168.88	33.18	5.87	2.69	1.68	0.88
	MVT(24)	178.07	30.12	5.32	2.46	1.56	0.83
	MVT(48)	181.69	28.76	5.05	2.36	1.49	0.80
	MVU	200.50	27.22	4.83	2.27	1.42	0.78
	MVB	188.95	26.70	4.73	2.22	1.38	0.76
	MVL	155.79	28.96	5.19	2.42	1.52	0.82
1.0	MVN	195.34	56.77	7.68	2.75	1.54	0.75
	MVT(3)	57.64	54.67	38.12	16.39	6.89	2.20
	MVT(6)	75.22	53.58	14.25	4.79	2.46	1.07
	MVT(12)	118.39	57.20	10.05	3.50	1.89	0.88
	MVT(24)	152.95	57.53	8.70	3.12	1.69	0.81
	MVT(48)	176.02	56.75	8.28	2.91	1.62	0.78
	MVU	345.80	61.93	7.48	2.73	1.54	0.76
	MVB	213.52	61.84	7.33	2.60	1.46	0.72
	MVL	81.90	41.80	8.57	3.10	1.71	0.82
1.5	MVN	196.89	87.46	14.48	4.21	1.99	0.81
	MVT(3)	33.30	32.78	29.94	24.68	16.29	3.85
	MVT(6)	42.32	38.40	21.48	8.60	3.80	1.26
	MVT(12)	71.56	53.93	18.00	5.80	2.59	0.98
	MVT(24)	111.60	69.16	16.07	4.89	2.26	0.88
	MVT(48)	147.66	78.63	15.10	4.57	2.12	0.85
	MVU	1175.74	171.71	14.35	4.05	1.93	0.80
	MVB	240.17	130.24	14.79	3.90	1.84	0.72
	MVL	44.85	33.66	13.35	4.87	2.35	0.91

Table 8 The ratios of SDRL/ARL of the MCUSUM control charts for monitoring the mean vector in 2 variables

k	distributions	δ					
		0.00	0.45	1.01	1.58	2.06	3.00
0.5	MVN	0.96	0.80	0.48	0.35	0.30	0.25
	MVT(3)	0.98	0.96	0.71	0.48	0.39	0.30
	MVT(6)	0.97	0.86	0.56	0.40	0.33	0.26
	MVT(12)	0.98	0.82	0.51	0.37	0.31	0.25
	MVT(24)	0.96	0.80	0.50	0.36	0.31	0.25
	MVT(48)	0.98	0.80	0.49	0.36	0.30	0.25
	MVU	0.97	0.79	0.49	0.36	0.31	0.25
	MVB	0.97	0.78	0.48	0.35	0.29	0.25
	MVL	0.98	0.83	0.50	0.37	0.31	0.25
1.0	MVN	0.99	0.96	0.76	0.54	0.43	0.34
	MVT(3)	0.99	0.99	0.97	0.84	0.66	0.44
	MVT(6)	1.00	0.98	0.86	0.63	0.50	0.36
	MVT(12)	0.99	0.96	0.81	0.58	0.46	0.34
	MVT(24)	0.99	0.96	0.78	0.56	0.45	0.34
	MVT(48)	0.99	0.95	0.77	0.55	0.44	0.34
	MVU	1.00	0.95	0.75	0.54	0.43	0.35
	MVB	0.99	0.95	0.73	0.51	0.42	0.34
	MVL	0.99	0.96	0.81	0.58	0.45	0.34
1.5	MVN	1.00	0.99	0.91	0.75	0.60	0.44
	MVT(3)	0.99	0.98	0.98	0.97	0.93	0.66
	MVT(6)	0.99	0.98	0.97	0.86	0.71	0.48
	MVT(12)	0.99	0.99	0.94	0.79	0.65	0.46
	MVT(24)	0.99	0.98	0.93	0.77	0.62	0.45
	MVT(48)	0.99	0.98	0.93	0.76	0.61	0.45
	MVU	1.00	1.00	0.90	0.72	0.60	0.48
	MVB	0.99	0.99	0.90	0.69	0.56	0.45
	MVL	0.98	0.98	0.93	0.79	0.63	0.44

8. Conclusions

The simulation results in this study are meant to provide a warning to process engineers who rely on MCUSUM control charts to monitor a process having multiple responses and assume the MVN assumption is reasonable. The type of skewness for a distribution, whether or not the distribution has a finite interval for support for continuous responses, all affect the ARLs and SDRLs in different ways when the MVN assumption has been violated.

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