Thailand Statistician
July 2018; 16(2): 156-172
http://statassoc.or.th
Contributed paper

Transmuted Generalized Power Weibull Distribution

Muhammad Shuaib Khan*
School of Mathematical and Physical Sciences, The University of Newcastle, Callaghan, NSW 2308, Australia.
*Corresponding author; e-mail: Shuaib.stat@gmail.com

Received: 20 July 2017
Accepted: 21 January 2018

Abstract

This paper introduces the transmuted generalized power Weibull distribution by using the quadratic rank transmutation map technique (QRTM) and derive some of its structural properties. This paper formulates the expressions for the moments and probability weighted moments. The method of maximum likelihood is used for estimating the model parameters and evaluate the performance of MLE using simulation. An application to data on anxiety is discussed to illustrate the flexibility of the new distribution.

Keywords: Reliability models, generalized power Weibull distribution, moment estimation, simulation, maximum likelihood estimation.

1. Introduction

The Weibull distribution is a very versatile lifetime distribution in reliability experiments and survival studies. Several extensions of the Weibull family of distributions have been proposed in the literature. Because the family of Weibull distributions are very fashionable and has flexibility that encourages its empirical use in a wide range of applications. This paper offers a new extension of the generalized power Weibull distribution by using quadratic rank transmutation map technique studied by Shaw and Buckley (2007). The aim of this study is to investigate the potential usefulness of the transmuted generalized power Weibull distribution for modelling lifetime data. A random variable X is said to have generalized power Weibull (GPW) distribution introduced by Nikulin and Haghighi (2006, 2009), its CDF is given by

$$G(x) = 1 - exp\left(1 - \left(1 + x^{\alpha}\right)^{\beta}\right). \tag{1}$$

The corresponding probability density function is given by

$$g(x) = \alpha\beta x^{\alpha-1}\left(1 + x^{\alpha}\right)^{\beta-1} exp\left(1 - \left(1 + x^{\alpha}\right)^{\beta}\right), \tag{2}$$

respectively, where $\alpha, \beta > 0$ and $x > 0$. This paper investigates the potential usefulness of this model and discuss some of its mathematical properties.

A significant amount of work has been attributed towards developing a transmuted families of lifetime distribution and discussing the effectiveness of this theory for modelling real world scenarios, where the baseline distribution does not provide better fit. Aryal and Tsokos (2009, 2011) proposed the transmuted Weibull and transmuted extreme value distributions with applications. Khan and King (2013a, 2013b) developed the transmuted modified Weibull distribution, the transmuted generalized inverse Weibull distribution and studied its mathematical properties for modelling reliability data sets. Recently, Khan et al. (2014a, 2014b) studied the transmuted Inverse Weibull distribution and studied various structural properties for modelling lifetime data. Khan et al. (2017) studied the transmuted Weibull distribution with covariates regressing modelling to analyse survival data. Ashour and Eltehiwy (2013), Elbatal and Aryal (2013), Merovci (2013) and Yuzhu et al. (2014) proposed the transmuted Lomax distribution, transmuted additive Weibull distribution, transmuted Rayleigh distribution and the transmuted linear exponential distribution with a discussion on theoretical properties of this family with applications. A random variable X is said to have transmuted distribution if its cumulative distribution function (cdf) is given by

$$F(x)=(1+\lambda)G(x)-\lambda G(x)^2, |\lambda|\le 1 \tag{3}$$

and

$$f(x)=g(x)\{(1+\lambda)-2\lambda G(x)\}, \tag{4}$$

where $G(x)$ is the cdf of the baseline model. It is important to note that at $\lambda=0$ we have the distribution of the base random variable (Shaw and Buckley (2007)).

This paper is organized as follows, In Section 2, we define the transmuted generalized power Weibull distribution by using quadratic rank transmutation map technique. This section also presented the density and hazard functions with graphical illustration. Section 3 formulates the moments and probability weighted moments. Estimation of the model parameters are performed by using the method of maximum likelihood and presented in Section 4. The results of the simulation study are performed to evaluate the asymptotic properties of the proposed model in Section 5. The new distribution is illustrated in an application to the anxiety in normal women's data in Section 6. Ultimate remarks are addressed in Section 7.

2. Transmuted Generalized Power Weibull Distribution

A random variable X is said to have transmuted generalized power Weibull distribution with parameters $\alpha,\beta>0$ and $|\lambda|\le 1,\ x>0.$ The failure probability density function defined through the QRTM technique is given by

$$f(x)=\alpha\beta x^{\alpha-1}\left(1+x^{\alpha}\right)^{\beta-1}\exp\left(1-\left(1+x^{\alpha}\right)^{\beta}\right)\left\{1-\lambda+2\lambda\exp\left(1-\left(1+x^{\alpha}\right)^{\beta}\right)\right\}. \tag{5}$$

The reliability and hazard functions corresponding to (5) is given by

$$R(x)=1-\left\{1-\exp\left(1-\left(1+x^{\alpha}\right)^{\beta}\right)\right\}\left[1+\lambda\exp\left(1-\left(1+x^{\alpha}\right)^{\beta}\right)\right], \tag{6}$$

and

$$h(x)=\frac{\alpha\beta x^{\alpha-1}\left(1+x^{\alpha}\right)^{\beta-1}\exp\left(1-\left(1+x^{\alpha}\right)^{\beta}\right)\left\{1-\lambda+2\lambda\exp\left(1-\left(1+x^{\alpha}\right)^{\beta}\right)\right\}}{1-\left\{1-\exp\left(1-\left(1+x^{\alpha}\right)^{\beta}\right)\right\}\left[1+\lambda\exp\left(1-\left(1+x^{\alpha}\right)^{\beta}\right)\right]}. \tag{7}$$

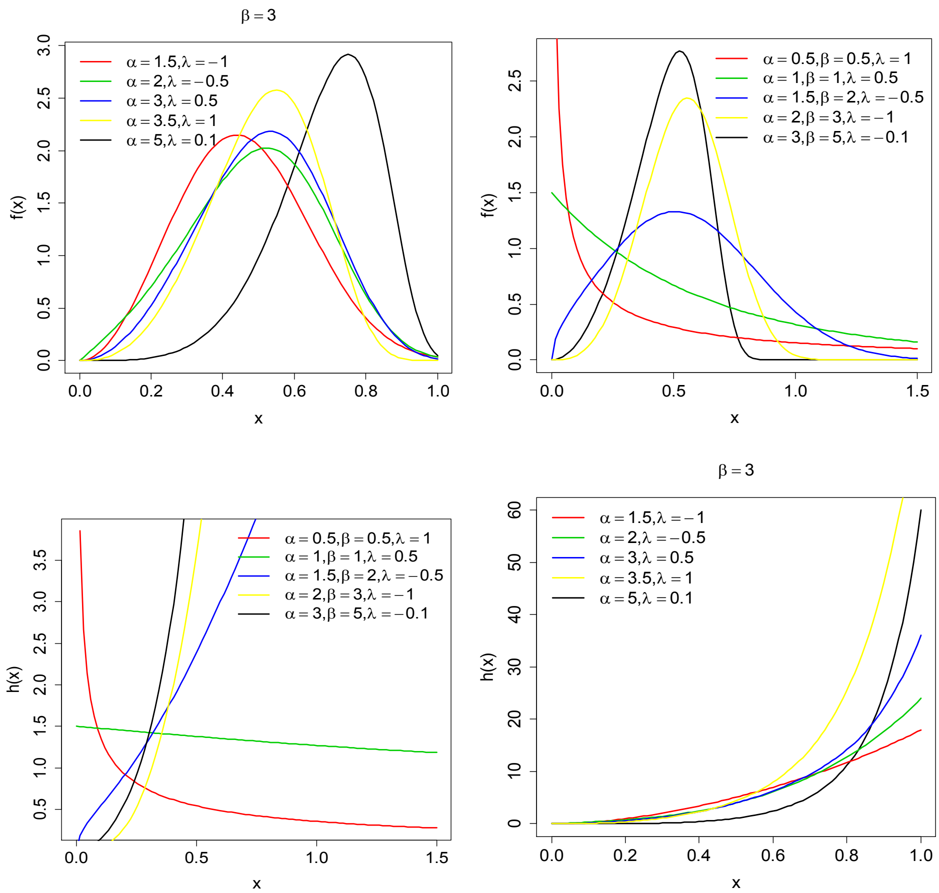


Figure 1 Plots of the TGPW PDF and HF for some selected values of parameters

$$x_q = \left\{ \left(1 - ln\left(1 - \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda} \right) \right)^{\frac{1}{\beta}} - 1 \right\}^{\frac{1}{\alpha}}, \quad (8)$$

where α and β are the shape parameters and λ is the transmuted parameter representing the different patterns of the transmuted generalized power Weibull distribution. If X is a random variable with pdf (5), we write $X \sim \text{TGPW}(x;\alpha,\beta,\lambda)$. The quantile function of the TGPW distribution is denoted as $x = Q(q)$ is the real solution given by

Therefore, for any random variable q resulting the uniform $U(0,1)$ distribution and the transmuting parameter $|\lambda| \le 1$ in (8). Figure 1 shows the shape of the transmuted generalized power Weibull PDF and hazard function (HF) for some selected choice of parameters. The transmuted generalized power Weibull HF with some selected choice of parameters has decreasing and increasing behavior of instantaneous failure rates.

3. Moments and Quantiles

This section presents the k^{th} moment, probability weighted moments and quantile functions of the transmuted generalized power Weibull distribution.

Theorem 1 *If* X *has the* $\text{TGPW}(x;\alpha,\beta,\lambda)$ *with* $|\lambda|\le 1,$ *for an integer value of* k/α *then the* k^{th} *moment of* X *say* μ'_k *is given as follows*

$$\mu'_k=(1-\lambda)\sum_{i=0}^{k/\alpha}V_{i,k,\alpha}\Gamma\left(\frac{i}{\beta}+1,1\right)+\lambda e\sum_{i=0}^{k/\alpha}\frac{V_{i,k,\alpha}}{2^{\frac{i}{\beta}}}\Gamma\left(\frac{i}{\beta}+1,2\right),$$

where

$$V_{i,k,\alpha}=e\binom{\frac{k}{\alpha}}{i}(-1)^{\frac{k}{\alpha}-i}.$$

Proof: The k^{th} moment of the TGPW distribution given by

$$\mu'_k=\int_0^\infty \alpha\beta x^{k+\alpha-1}\left(1+x^\alpha\right)^{\beta-1}\exp\left(1-\left(1+x^\alpha\right)^\beta\right)\left\{1-\lambda+2\lambda\exp\left(1-\left(1+x^\alpha\right)^\beta\right)\right\}dx.$$

Substituting $u=\left(1+x^\alpha\right)^\beta$, the above expression reduces to

$$\mu'_k=\int_1^\infty\left(u^{\frac{1}{\beta}}-1\right)^{\frac{k}{\alpha}}\exp(-u)\left\{1-\lambda+2\lambda\exp(1-u)\right\}du.$$

By using the Binomial expansion, the above integral reduces to

$$\mu'_k=(1-\lambda)e\sum_{i=0}^{k/\alpha}\binom{\frac{k}{\alpha}}{i}(-1)^{\frac{k}{\alpha}-i}\int_1^\infty u^{\frac{i}{\beta}}\exp(-u)\,du$$
$$+2\lambda e\sum_{i=0}^{k/\alpha}\binom{\frac{k}{\alpha}}{i}(-1)^{\frac{k}{\alpha}-i}\int_1^\infty u^{\frac{i}{\beta}}\exp(-2u)\,du.$$

Hence, it follows that

$$\mu'_k=(1-\lambda)e\sum_{i=0}^{k/\alpha}\binom{\frac{k}{\alpha}}{i}(-1)^{\frac{k}{\alpha}-i}\Gamma\left(\frac{i}{\beta}+1,1\right)+2\lambda e\sum_{i=0}^{k/\alpha}\binom{\frac{k}{\alpha}}{i}\frac{(-1)^{\frac{k}{\alpha}-i}}{2^{\frac{i}{\beta}+1}}\Gamma\left(\frac{i}{\beta}+1,2\right). \tag{9}$$

A parameterization of the incomplete gamma function in terms of generalized Laguerre polynomials is defined by, Olver et al. (2010)

$$\Gamma(x,y)=y^x\exp(-y)\sum_{n=0}^{\infty}\frac{L_n^{(x)}(y)}{n+1},$$

which converges for $\text{Re}(x)>-1$ and $x>0$.

Here in (9), the abbreviation of $e=\exp$ and k/α is an integer value. The important features and characteristics of the TGPW distribution can be studied through moments. The skewness and kurtosis measures can now be calculated from the ordinary moments using renowned relationships. The computations of the moments are performed using the R language for some selected choices of parameters are listed in Table 1. These important features and characteristics of the TGPW distribution are obtained through the ordinary moments such as mean, variance, coefficient of variation, coefficient of skewness and coefficient of kurtosis are showed in Table 2.

Theorem 2 *If* X *has the* $\text{TGPW}(x;\alpha,\beta,\lambda)$ *with* $|\lambda|\le 1,$ *for an integer value of* k/α *then the probability weighted moment (PWM) is given as follows*

$$\xi_{(k,m)} = \sum_{i,j=0}^{\infty}\sum_{p=0}^{k/\alpha}\binom{\frac{k}{\alpha}}{p}\frac{\mathcal{V}_{i,j,m}}{(-1)^{p-\frac{k}{\alpha}}}\left\{\frac{(1-\lambda)\Gamma\left(\frac{p}{\beta}+1,i+j+1\right)}{(i+j+1)^{\frac{p}{\beta}+1}}+\frac{2\lambda e\Gamma\left(\frac{p}{\beta}+1,i+j+2\right)}{(i+j+2)^{\frac{p}{\beta}+1}}\right\}.$$

Proof: The PWM of the TGPW distribution as follows

$$\xi_{(k,m)} = \int_0^{\infty} x^k F(x)^m f(x)\,dx.$$

From the above integral the expansion of cdf in terms of infinite weighted sum as

$$F(x)^m = \sum_{i,j=0}^{\infty}\binom{m}{i}\binom{m}{j}(-1)^i\lambda^j\exp\left\{(i+j)\left(1-\left(1+x^{\alpha}\right)^{\beta}\right)\right\}.$$

The expression for the PWM reduces to

$$\xi_{(k,m)} = \sum_{i,j=0}^{\infty}\mathcal{V}_{i,j,m}\int_1^{\infty}\left(u^{\frac{1}{\beta}}-1\right)^{\frac{k}{\alpha}}\exp\left(-(i+j+1)u\right)\left\{1-\lambda+2\lambda\exp(1-u)\right\},$$

where

$$\mathcal{V}_{i,j,m} = \binom{m}{i}\binom{m}{j}(-1)^i\lambda^j exp(i+j+1).$$

By using the Binomial expansion, the above integral reduces to

$$\xi_{(k,m)} = (1-\lambda)\sum_{i,j=0}^{\infty}\sum_{p=0}^{k/\alpha}\binom{\frac{k}{\alpha}}{p}\mathcal{V}_{i,j,m}(-1)^{\frac{k}{\alpha}-p}\int_1^{\infty}u^{\frac{p}{\beta}}\exp\left(-(i+j+1)u\right)du$$
$$+2\lambda\sum_{i,j=0}^{\infty}\sum_{p=0}^{k/\alpha}\binom{\frac{k}{\alpha}}{p}\mathcal{V}_{i,j,m}(-1)^{\frac{k}{\alpha}-p}\int_1^{\infty}u^{\frac{p}{\beta}}\exp\left(-(i+j+2)u\right)du.$$

Finally, we obtain

$$\begin{aligned}\xi_{(k,m)} &= (1-\lambda)\sum_{i,j=0}^{\infty}\sum_{p=0}^{k/\alpha}\binom{\frac{k}{\alpha}}{p}\frac{\mathcal{V}_{i,j,m}}{(i+j+1)^{\frac{p}{\beta}+1}}(-1)^{\frac{k}{\alpha}-p}\,\Gamma\left(\frac{p}{\beta}+1,\ i+j+1\right)\\ &\quad+2\lambda\sum_{i,j=0}^{\infty}\sum_{p=0}^{k/\alpha}\binom{\frac{k}{\alpha}}{p}\frac{\mathcal{V}_{i,j,m}}{(i+j+2)^{\frac{p}{\beta}+1}}(-1)^{\frac{k}{\alpha}-p}\,\Gamma\left(\frac{p}{\beta}+1,\ i+j+2\right).\end{aligned} \tag{10}$$

The PWMs of the TGPW distribution can be very useful to determine the L-moments estimators and advantageous for the moments of order statistics.

Table 1 Moments for selected parameter values

α	β	λ	Moments			
			μ_1'	μ_2'	μ_3'	μ_4'
1	1	-1	1.4990	3.4889	11.1259	45.0959
		-0.50	1.2492	2.7416	8.5319	34.1969
		0.50	0.7497	1.2472	3.3439	12.3989
		1	0.5000	0.4999	0.7499	1.4999
1	2	-1	0.5471	0.4056	0.3703	0.3960
		-0.50	0.4630	0.3238	0.2877	0.3033
		0.50	0.2948	0.1603	0.1226	0.1179
		1	0.2106	0.0786	0.0401	0.0252

Table 2 Moments based measures for selected parameter values

α	β	λ	Moments measures				
			Mean	Var	CV	CS	CK
1	1	-1	1.4990	1.2418	0.7434	1.5699	6.6623
		-0.5	1.2492	1.1811	0.8699	1.6798	7.1174
		0.5	0.7497	0.6851	1.1041	2.4361	11.9921
		1	0.5000	0.2499	0.9998	2.0016	9.0064
1	2	-1	0.5471	0.1062	0.5958	0.9265	4.0087
		-0.5	0.4630	0.1094	0.7144	1.0068	4.0996
		0.5	0.2948	0.0733	0.9189	1.6129	6.3601
		1	0.2106	0.0342	0.8787	1.4392	5.4863

Table 3 PWMs for selected parameter values

α	β	λ	Probability weighted moments			
			$\xi_{(1,1)}$	$\xi_{(2,2)}$	$\xi_{(3,3)}$	$\xi_{(4,4)}$
1	1	-1	1.0406	2.4868	8.6060	37.6212
		-0.50	0.9055	2.0558	6.8917	29.4925
		0.50	0.5726	1.0170	2.9369	11.4602
		1	0.3750	0.3935	0.6342	1.3441
1	2	-1	0.3632	0.2626	0.2527	0.2892
		-0.50	0.3221	0.2230	0.2085	0.2335
		0.50	0.2175	0.1222	0.0995	0.1012
		1	0.1541	0.0587	0.0317	0.0211

Table 4 The Quartiles of the TGPWD for selected parameter values

α	β	λ	Quantiles					
			Q_1	Q_2	Q_3	*I.Q.R*	*Q.D*	*C.Q.D*
1	1	-1	0.6931	1.2279	2.0101	1.3169	0.6584	0.4871
		-0.5	0.4557	0.9624	1.7309	1.2751	0.6375	0.5831
		0.5	0.1949	0.4812	1.0050	0.8101	0.4050	0.6751
		1	0.1438	0.3465	0.6931	0.5493	0.2746	0.6562
1	2	-1	0.3012	0.4926	0.7349	0.4337	0.2168	0.4186
		-0.5	0.2065	0.4008	0.6525	0.4460	0.2230	0.5191
		0.5	0.0931	0.2170	0.4159	0.3228	0.1614	0.6341
		1	0.0695	0.1604	0.3012	0.2317	0.1158	0.6250
2	2	-1	0.5488	0.7018	0.8573	0.3084	0.1542	0.2193
		-0.5	0.4544	0.6331	0.8078	0.3533	0.1766	0.2799
		0.5	0.3051	0.4658	0.6449	0.3397	0.1698	0.3576
		1	0.2636	0.4005	0.5488	0.2851	0.1425	0.3510
2	3	-1	0.4380	0.5532	0.6662	0.2282	0.1140	0.2066
		-0.5	0.3651	0.5019	0.6306	0.2655	0.1327	0.2666
		0.5	0.2473	0.3740	0.5108	0.2635	0.1317	0.3476
		1	0.2140	0.3229	0.4380	0.2239	0.1119	0.3435

I.Q.R: Inter Quartile Range; Q.D: Quartile Deviation; C.Q.D: Coefficient of Quartile Deviation

4. Parameter Estimation

Consider the random samples $x_1, x_2, \ldots, x_n$ consisting of n observations from the $\text{TGPW}(x;\alpha,\beta,\lambda)$ distribution. Let $\Theta = (\alpha,\beta,\lambda)^T$ be the parameter vector. The log-likelihood function of the density (5) is given by

$$\mathcal{L} = n\ln\alpha + n\ln\beta + (\alpha-1)\sum_{i=1}^{n}\ln x_i + (\beta-1)\sum_{i=1}^{n}\ln\left(1+x_i^{\alpha}\right) - n - \sum_{i=1}^{n}\left(1+x_i^{\alpha}\right)^{\beta} + \sum_{i=1}^{n}\ln\left(1-\lambda+2\lambda\exp\left(1-\left(1+x_i^{\alpha}\right)^{\beta}\right)\right). \tag{11}$$

The associated score function is given by $U_n(\Theta) = \left(\partial\mathcal{L}/\partial\alpha, \partial\mathcal{L}/\partial\beta, \partial\mathcal{L}/\partial\lambda\right)^{\mathrm{T}}$, where

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln x_i + (\beta-1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\ln(x_i)}{\left(1+x_i^{\alpha}\right)} - \sum_{i=1}^{n}\beta\left(1+x_i^{\alpha}\right)^{\beta-1}x_i^{\alpha}\ln(x_i)$$
$$-\sum_{i=1}^{n}\frac{2\lambda\exp\left(1-\left(1+x_i^{\alpha}\right)^{\beta}\right)\beta\left(1+x_i^{\alpha}\right)^{\beta-1}x_i^{\alpha}\ln(x_i)}{\left(1-\lambda+2\lambda\exp\left(1-\left(1+x_i^{\alpha}\right)^{\beta}\right)\right)},$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln\left(1 + x_i^{\alpha}\right) - \sum_{i=1}^{n} \left(1 + x_i^{\alpha}\right)^{\beta} \ln\left(1 + x_i^{\alpha}\right)$$

$$-\sum_{i=1}^{n} \frac{2\lambda \exp\left(1 - \left(1 + x_i^{\alpha}\right)^{\beta}\right)\left(1 + x_i^{\alpha}\right)^{\beta} \ln\left(1 + x_i^{\alpha}\right)}{\left(1 - \lambda + 2\lambda \exp\left(1 - \left(1 + x_i^{\alpha}\right)^{\beta}\right)\right)},$$

and

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{i=1}^{n} \frac{2\exp\left(1 - \left(1 + x_i^{\alpha}\right)^{\beta}\right) - 1}{\left(1 - \lambda + 2\lambda \exp\left(1 - \left(1 + x_i^{\alpha}\right)^{\beta}\right)\right)}.$$

These nonlinear systems of equations cannot be solved analytically, and statistical software can be used to solve them numerically by using iterative procedures such as Newton Raphson method, BFGS, BHHH and L-BFGS-B through R-package (Adequacy Model). For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The 3×3 observed information matrix is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\lambda} \end{pmatrix} \sim N\left[\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} \end{pmatrix}\right], \tag{12}$$

$$V_n\left(\Theta\right)^{-1} = -E\begin{bmatrix} \frac{\partial^2 \mathcal{L}}{\partial \alpha^2} & \frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \beta} & \frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \beta} & \frac{\partial^2 \mathcal{L}}{\partial \beta^2} & \frac{\partial^2 \mathcal{L}}{\partial \beta \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \lambda} & \frac{\partial^2 \mathcal{L}}{\partial \beta \partial \lambda} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{bmatrix}. \tag{13}$$

The asymptotic multivariate normal $N_3\left(0,\ V_n\left(\Theta\right)^{-1}\right)$ distribution can be used to construct the approximate confidence intervals and confidence region of individual parameters for the transmuted generalized power Weibull distribution. We can compute the maximum values of log-likelihood to obtain the likelihood ratio (LR) statistics for testing the sub-models of the TGPW distribution. For testing hypothesis, we formulated the null hypothesis $H_0 : \Theta = \Theta_0$ versus $H_A : \Theta \neq \Theta_0$ for the LR statistics to compare the TGPW distribution with other lifetime distributions to see the superiority of the model for a given data set. The likelihood ratio (LR) statistics can be defined as $\Lambda = 2\left\{l\left(\hat{\alpha}, \hat{\beta},\ \hat{\lambda}\right) - l\left(\tilde{\alpha}, \tilde{\beta},\ \tilde{\lambda}\right)\right\}$, where $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ are the MLEs under H_A and $\tilde{\alpha}, \tilde{\beta}, \tilde{\lambda}$ are the estimates under H_0. By using the observed information matrix an approximately $100\left(1-\gamma\right)\%$ confidence intervals for α, β and λ can be determined as

$$\hat{\alpha} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{V}_{11}}, \qquad \hat{\beta} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{V}_{22}}, \qquad \hat{\lambda} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{V}_{33}},$$

where $Z_{\frac{\gamma}{2}}$ is the upper γ^{th} percentile of the standard normal distribution.

5. Simulation

In order to evaluate the performance of MLEs of the TGPW distribution, this section shows the results of Monte Carlo experiment on finite samples. The inversion method was used to generate samples from the TGPW distribution for different sample sizes $n = 50, 100, 200, 400$ and for fixed choice of parameters using (8). The simulation results were obtained from 1000 Monte Carlo replications using the BFGS optimization method in R and results were displayed in Tables 5-9. The results of the bias estimates and mean square error (MSE) for the parameters α, β, λ are decreasing as the sample size n increases.

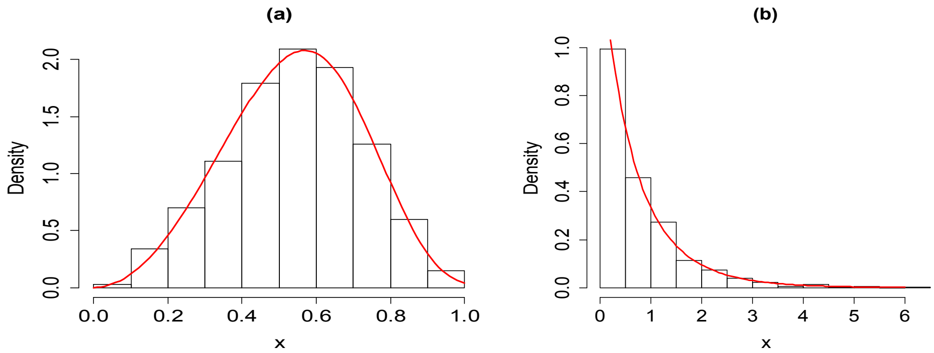


Figure 2 Plots of the TGPW densities for simulated data sets,
(a) $\alpha = 3; \beta = 2; \lambda = 1$ (b) $\alpha = 1; \beta = 1; \lambda = 0.5$

Table 5 Estimated values of $\hat{\Theta}$ based on MLE

$\Theta = (\alpha, \beta, \lambda)$	n	Estimates		
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
1, 1, 0.8	25	1.4233	0.8796	0.7003
	50	1.2173	0.8456	0.7142
	100	1.1782	0.9545	0.7587
	200	1.0364	0.9960	0.7742
	400	1.0136	1.0211	0.8011
1, 2, 0.5	25	0.8561	2.0642	0.4513
	50	1.1086	1.9244	0.4419
	100	1.1102	2.1523	0.5495
	200	0.9906	2.0548	0.5311
	400	1.0754	1.9985	0.5231
2, 2, 0.5	25	1.6434	2.1852	0.2399
	50	2.0960	1.9435	0.5540
	100	1.8641	2.0636	0.0979
	200	2.0655	2.0419	0.5766
	400	2.0049	2.0989	0.4309
2, 3, 0.8	25	2.0603	2.6243	0.6740
	50	1.9952	3.1160	0.6708
	100	2.0339	3.1256	0.6700
	200	2.0380	2.9008	0.8670
	400	2.0123	3.0910	0.6860

Table 6 Standard deviations of the estimate of $\hat{\Theta}$ based on MLE

$\Theta = (\alpha, \beta, \lambda)$	n	SD		
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
1, 1, 0.8	25	0.4125	0.1985	0.3231
	50	0.3561	0.2044	0.4536
	100	0.2011	0.1521	0.4328
	200	0.1112	0.0712	0.1576
	400	0.1022	0.0651	0.1114
1, 2, 0.5	25	0.3481	0.3269	0.6013
	50	0.2502	0.2243	0.5133
	100	0.2211	0.1360	0.5020
	200	0.1260	0.0900	0.2510
	400	0.1113	0.0615	0.2344
2, 2, 0.5	25	0.3743	0.4834	0.7615
	50	0.2463	0.4304	0.4979
	100	0.4705	0.3885	0.8811
	200	0.1410	0.3394	0.3995
	400	0.1143	0.1878	0.2612
2, 3, 0.8	25	0.2951	0.6777	0.4386
	50	0.2147	0.8273	0.5230
	100	0.1500	0.4844	0.2818
	200	0.1061	0.3509	0.1473
	400	0.0763	0.3026	0.1915

Table 7 Bias of the estimate of $\hat{\Theta}$ based on MLE

$\Theta=(\alpha,\beta,\lambda)$	n	Bias		
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
1, 1, 0.8	25	0.4233	-0.1204	-0.0997
	50	0.2173	-0.1544	-0.0858
	100	0.1782	-0.0455	-0.0413
	200	0.0364	-0.004	-0.0258
	400	0.0136	0.0211	0.0011
1, 2, 0.5	25	-0.1439	0.0642	-0.0487
	50	0.1086	-0.0756	-0.0581
	100	0.1102	0.1523	0.0495
	200	-0.0094	0.0548	0.0311
	400	0.0754	-0.0015	0.0231
2, 2, 0.5	25	-0.3566	0.1852	-0.2601
	50	0.0960	-0.0565	0.0540
	100	-0.1359	0.0636	-0.4021
	200	0.0655	0.0419	0.0766
	400	0.0049	0.0989	-0.0691
2, 3, 0.8	25	0.0603	-0.3757	-0.1260
	50	-0.0048	0.1160	-0.1292
	100	0.0339	0.1256	-0.1300
	200	0.0380	-0.0992	0.0670
	400	0.0123	0.0910	-0.1140

Table 8 95% confidence intervals of the estimate of $\hat{\Theta}$ based on MLE

$\Theta=(\alpha,\beta,\lambda)$	n	95% confidence intervals					
		LL (α)	UL (α)	LL (β)	LL (β)	LL (λ)	LL (λ)
1, 1, 0.8	25	1.2530	1.5936	0.7977	0.9615	0.5669	0.8337
	50	1.1161	1.3185	0.7875	0.9037	0.5853	0.8431
	100	1.1383	1.2181	0.9243	0.9847	0.6728	0.8446
	200	1.0209	1.0519	0.9861	1.0059	0.7522	0.7962
	400	1.0036	1.0236	1.0147	1.0275	0.7901	0.8121
1, 2, 0.5	25	0.7124	0.9998	1.9293	2.1991	0.2031	0.6995
	50	1.0375	1.1797	1.8607	1.9881	0.2960	0.5878
	100	1.0663	1.1541	2.1253	2.1793	0.4499	0.6491
	200	0.9730	1.0082	2.0423	2.0673	0.4961	0.5661
	400	1.0645	1.0863	1.9925	2.0045	0.5001	0.5461
2, 2, 0.5	25	1.4889	1.7979	1.9857	2.3847	-0.0744	0.5542
	50	2.0260	2.1660	1.8212	2.0658	0.4125	0.6955
	100	1.7707	1.9575	1.9865	2.1407	-0.0769	0.2727
	200	2.0458	2.0852	1.9946	2.0892	0.5209	0.6323
	400	1.9937	2.0161	2.0804	2.1174	0.4052	0.4566
2, 3, 0.8	25	1.9385	2.1821	2.3446	2.9040	0.4930	0.8550
	50	1.9342	2.0562	2.8809	3.3511	0.5222	0.8194
	100	2.0041	2.0637	3.0295	3.2217	0.6141	0.7259
	200	2.0232	2.0528	2.8519	2.9497	0.8465	0.8875
	400	2.0048	2.0198	3.0613	3.1207	0.6672	0.7048

Table 9 MSE of the estimate of $\hat{\Theta}$ based on MLE

$\Theta=(\alpha,\beta,\lambda)$	n	MSE		
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
1, 1, 0.8	25	0.3493	0.0538	0.1143
	50	0.1740	0.0656	0.2131
	100	0.0722	0.0252	0.1890
	200	0.0137	0.0051	0.0255
	400	0.0106	0.0046	0.0124
1, 2, 0.5	25	0.1418	0.1109	0.3639
	50	0.0744	0.0560	0.2668
	100	0.0610	0.0417	0.2545
	200	0.0159	0.0111	0.0639
	400	0.0181	0.0037	0.0554
2, 2, 0.5	25	0.2672	0.2679	0.6475
	50	0.0698	0.1884	0.2508
	100	0.2398	0.1549	0.9380
	200	0.0241	0.1169	0.1654
	400	0.0130	0.0450	0.0730
2, 3, 0.5	25	0.0907	0.6004	0.2082
	50	0.0461	0.6978	0.2902
	100	0.0236	0.2504	0.0963
	200	0.0127	0.1329	0.0261
	400	0.0059	0.0998	0.0496

6. Application

In this section we provide data analysis to assess the goodness-of-fit of the proposed model with the study on anxiety performed in a group of 166 “normal" women, i.e., outside of a pathological clinical picture (Townsville, Queensland, Australia), which were originally reported by Smithson and Verkuilen (2006) and recently studied by Marcelo et al. (2016). The data observations are listed below

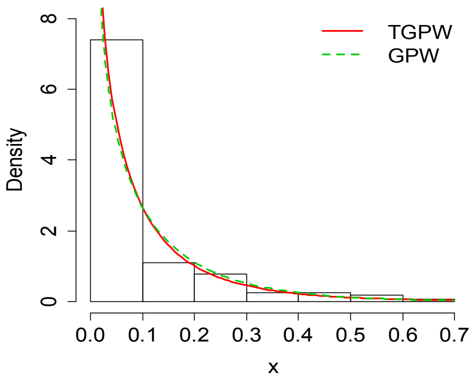
0.01, 0.17, 0.01, 0.05, 0.09, 0.41, 0.05, 0.01, 0.13, 0.01, 0.05, 0.17, 0.01, 0.09, 0.01, 0.05, 0.09, 0.09, 0.05, 0.01, 0.01, 0.01, 0.29, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.09, 0.37, 0.05, 0.01, 0.05, 0.29, 0.09, 0.01, 0.25, 0.01, 0.09, 0.01, 0.05, 0.21, 0.01, 0.01, 0.01, 0.13, 0.17, 0.37, 0.01, 0.01, 0.09, 0.57, 0.01, 0.01, 0.13, 0.05, 0.01, 0.01, 0.01, 0.01, 0.09, 0.13, 0.01, 0.01, 0.09, 0.09, 0.37, 0.01, 0.05, 0.01, 0.01, 0.13, 0.01, 0.57, 0.01, 0.01, 0.09, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.05, 0.01, 0.01, 0.01, 0.13, 0.01, 0.25, 0.01, 0.01, 0.09, 0.13, 0.01, 0.01, 0.05, 0.13, 0.01, 0.09, 0.01, 0.05, 0.01, 0.05, 0.01, 0.09, 0.01, 0.37, 0.25, 0.05, 0.05, 0.25, 0.05, 0.05, 0.01, 0.05, 0.01, 0.01, 0.01, 0.17, 0.29, 0.57, 0.01, 0.05, 0.01, 0.09, 0.01, 0.09, 0.49, 0.45, 0.01, 0.01, 0.01, 0.05, 0.01, 0.17, 0.01, 0.13, 0.01, 0.21, 0.13, 0.01, 0.01, 0.17, 0.01, 0.01, 0.21, 0.13, 0.69, 0.25, 0.01, 0.01, 0.09, 0.13, 0.01, 0.05, 0.01, 0.01, 0.29, 0.25, 0.49, 0.01, 0.01.

The required numerical evaluations were implemented using the R language (2013). We fitted the TGPW and GPW distributions by the method of maximum likelihood. The MLEs of the parameters with their corresponding standard errors (SE) and the Akaike information criteria (AIC), BIC (Bayesian Information Criterion) and HQIC (Hannan-Quinn information criterion) for the fitted models are displayed in Table 10. The values in Table 10 indicate that the TGPW distribution provides a better fit than the GPW distribution, because the TGPW distribution has the lowest AIC, BIC and HQIC comparing with the baseline distribution. Therefore, the TGPW distribution provides better fit for modelling anxiety performed in a group of 166 normal women's data.

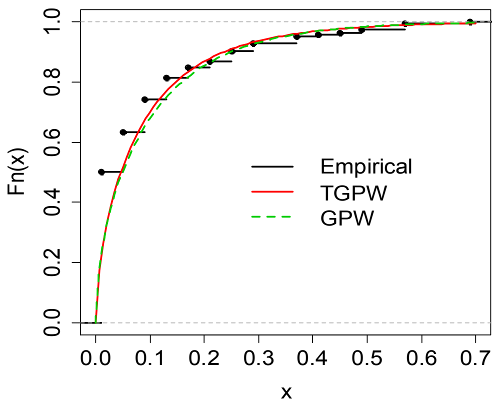
Table 10 MLEs of the Parameters for anxiety in women's data with goodness of fit measures

Distribution	Parameter Estimates			AIC	BIC	HQIC
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$			
TGPW	0.6545	2.8750	0.6537	-486.06	-476.72	-482.27
	(0.0371)	(0.3213)	(0.2006)			
GPW	0.5689	3.1906	-	-479.98	-473.76	-477.45
	(0.0324)	(0.1881)				

The likelihood ratio statistics for testing the hypothesis $H_0 : \lambda = 0,\ H_A : \lambda \neq 0$ is $\Lambda = 8.078$ (p-value = 0.0044). Therefore, we reject the null hypothesis in favour of TGPW distribution.

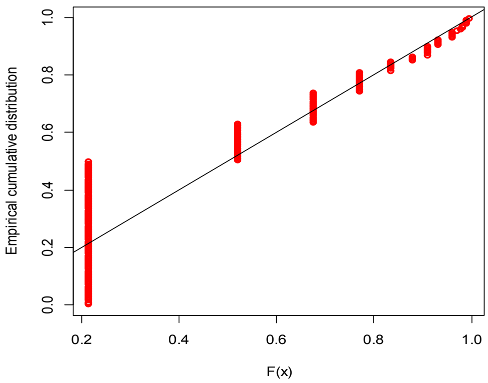


(a) Estimated pdfs

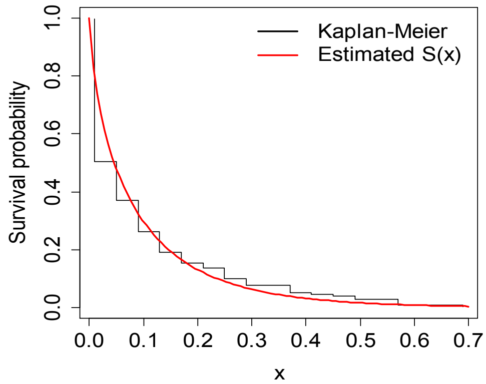


(b) Estimated cdfs

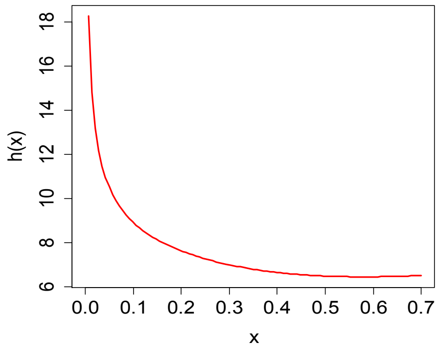
Figure 3 Plots of the fitted TGPW and GPW models for the anxiety in women's data



(a) P-P Plot



(b) Estimated Survival function



(c) Estimated hazard function

Figure 4 Estimated Fitted Models for the TGPW distribution for the anxiety in women's data

Using the maximum likelihood estimates of the unknown parameters, the approximately 95% two-sided confidence interval for the parameters α, β and λ are [0.5817, 0.7272], [2.2452, 3.5047] and [0.2605, 1.0468] respectively. Figure 3(a) illustrate the fitted TGPW and GPW distributions with histogram for the anxiety in normal women's data. Figure 3b shows the empirical non-survival function and the estimated non-survival functions for the anxiety in women's data. Figure 4 display the PP-Plot, estimated survival function and estimated hazard function for the TGPW distribution which gives a better fit for the data under analysis

7. Concluding Remarks

This research introduced a new distribution, so-called transmuted generalized power Weibull distribution and studied its theoretical properties. The research displays the analytical shapes of density and hazard functions of the TGPW distribution. The moments and probability weighted moments are formulated. The estimation of the unknown parameter is approached by using the method of maximum likelihood. The TGPW distribution has increasing and decreasing failure rate pattern for lifetime data. The usefulness of the TGPW model is illustrated by using the anxiety in normal women's data.

Acknowledgements

The author would like to thank the referees and the editor for careful reading and comments that significantly improved the article.

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