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Monitoring Mean Shift in INAR(1)s Processes based on CLSE-CUSUM Procedure

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Abstract

In this paper, we consider a new control procedure for monitoring mean shift using the conditional least squares estimator (CLSE)-based cumulative sum (CUSUM) test for the first-order seasonal integer-valued autoregressive (INAR(1)s) processes. Numerical experiments show that the proposed CLSE-CUSUM procedure outperforms conventional CUSUM charts for small to moderate up-shifts in mean of innovation processes, in terms of average run length (ARL), standard deviation (SD) and median.

Keywords: Average run length, INAR(1)s process, CLSE-based CUSUM test, CUSUM chart, small to moderate shift.

1. Introduction

There has been a growing interest in modeling autocorrelated count processes and studying their statistical properties. Models using a thinning operator have received much attention from researchers. Steutel and van Harn (1979) first designated the binomial thinning operator '$\circ$': for random variable X and $\alpha \in [0,1]$, $\alpha \circ X$ is defined as $\sum_{i=1}^{X} B_i(\alpha)$, wherein $B_i(\alpha)$ are independent and identically distributed (i.i.d.) Bernoulli random variables with success probability $P(B_i(\alpha)=1)=\alpha$, independent of X. Using this operator, Al-Osh and Alzaid (1987) proposed the first-order integer-valued autoregressive (INAR(1)) process: $X_t = \alpha \circ X_{t-1} + \epsilon_t$, $t \in \mathbf{Z}$, where ϵ_t are i.i.d. non-negative random variables, independent of X_u, with mean μ_ϵ and variance σ_ϵ^2. Since then, a number of articles have been published in the literature: see Weiß (2008) and Scotto et al. (2015) for a review. Bourguignon et al. (2016) recently consider the seasonal INAR(1) process with seasonal period $s \in \mathbf{N}$ (INAR(1)s) as follows:

$$X_t = \alpha \circ X_{t-s} + \epsilon_t,\ t \in \mathbf{Z}. \quad (1)$$

Statistical properties, such as the existence of stationary solutions, explicit form of autocorrelation function and transition probabilities can be found in Bourguignon et al. (2016).

Control chart is one of the most useful tools in statistical process control (SPC). Among control charts, the cumulative sum (CUSUM) control scheme, introduced by Page (1954), is well known as a standard tool for detecting small to moderate abnormal changes in the process of interest: see Montgomery (2012) for a general review. A conventional upper one-sided CUSUM control statistic for detecting a mean increase is expressed as:

$$C_0 = c_0, \tag{2}$$

$$C_t = \max\left(0, X_t - k + C_{t-1}\right), \quad t \in \mathbf{N}, \tag{3}$$

where $c_0 (\geq 0)$ is called the 'starting value', usually set to be 0, $k(\geq EX_t)$ is called the 'reference value', and $h(>0)$ is the 'control limit'. When $C_t \geq h$ occurs, the process of interest is regarded as 'out-of-control'. The reference value k acts as a tuning parameter for the sensitive detection of a mean increase, preventing the statistic from drifting towards the control limit h. Weiss and Testik (2009) study the upper one-sided CUSUM control charts for Poisson INAR(1) processes and Yontay et al. (2013) consider the two-sided CUSUM control chart by combining the one-sided CUSUM charts. The two-sided CUSUM control statistic is defined as:

$$\begin{aligned}
C_0^+ &= c_0^+, \\
C_t^+ &= \max\left(0, X_t - k^+ + C_{t-1}^+\right), \quad t \in \mathbf{N}, \\
C_0^- &= c_0^-, \\
C_t^- &= \max\left(0, k^- - X_t + C_{t-1}^-\right), \quad t \in \mathbf{N},
\end{aligned}$$

where c_0^+ and c_0^- are starting values, k^+ and k^- are reference values, and h^+, h^- are control limits of upper and lower one-sided CUSUM charts, respectively. When $C_t^+ \geq h^+$ or $C_t^- \geq h^-$ occurs, the process of interest is regarded as out-of-control. Conventionally, the performance evaluation of control charts is carried out based on the average run length (ARL), which is the average number of observations until the signal indicating an out-of-control state occurs. Two types of ARLs are considered: the ARL_0 (in-control ARL) is the average number of observations until a false alarm occurs when the process is in-control; the ARL_1 (out-of-control ARL) is the average number of observations until the control procedure triggers a correct signal, indicating an abnormal change from its start.

In this study, we consider the CUSUM test based on conditional least squares estimator (CLSE) from INAR(1)s processes and apply it to monitoring the mean shift. Based on this, we construct a one-sided control chart and compare its performance with the CUSUM charts. As a relevant work, we refer to Huh et al. (2017) who use the conditional maximum likelihood (MLE) in integer-valued generalized autoregressive conditional heteroskedasticity (INGARCH) models. This paper is organized as follows. Section 2 presents the difference equations as to the moments of INAR(1)s processes. Moreover, the CLSE-CUSUM test statistic is introduced. Section 3 designs a new control statistic for monitoring mean shift based on the results in Section 2. Section 4 compares the proposed control procedure with the CUSUM chart based on numerical experiments using ARL, standard deviation (SD) and median. Section 5 illustrates a real data analysis demonstrating the superiority of our proposed procedure to the CUSUM chart. Sections 6 and 7 provide technical proofs and concluding remarks.

2. Higher Moments and CLSE-based CUSUM Test

In this section, we show the existence of a 4th moment and provide related difference equations for stationary INAR(1)s processes. These are important in the construction of the control statistic in Section 3.

Proposition 2.1 *Suppose that* $\alpha \in [0,1)$ *and* $E\epsilon_t^4 < \infty$. *Then, we have* $EX^k \le C < \infty,\ k = 1,2,3,4,$ *for some* $C > 0$, *where* X *is the unique stationary limit of* X_t *in distribution.*

Difference equations for moments are obtained by using the properties of Binomial thinning operator.

Proposition 2.2 *Let* $\{X_t\}_{t\in\mathbf{Z}}$ *be a stationary INAR(1)s process with* $\alpha \in [0,1)$, $\mu_{\epsilon,k} = E\epsilon_t^k < \infty,\ k = 1,2,3,4,$ *and* $\sigma_\epsilon^2 = \mu_{\epsilon,2} - \mu_{\epsilon,1}^2$. *Then, for* $u, v, w \in \mathbf{N}$ *with* $u < v < w$,

$$(a)\mu_{\mathrm{x},1} := EX_t = \frac{\mu_{\epsilon,1}}{1-\alpha}.$$

$$(b)\mu_{\mathrm{x},2} := EX_t^2 = \frac{\alpha\mu_{\epsilon,1} + \sigma_\epsilon^2}{1-\alpha^2} + \frac{\mu_{\epsilon,1}^2}{(1-\alpha)^2}.$$

$$(c)\mu_{\mathrm{x},3} := EX_t^3 = \frac{1}{1-\alpha^3}\Big[3\alpha^2(1-\alpha+\mu_{\epsilon,1})\mu_{\mathrm{x},2} + \alpha\big((1-\alpha)(1-2\alpha) + 3(1-\alpha)\mu_{\epsilon,1} + 3\mu_{\epsilon,2}\big)\mu_{\mathrm{x},1} + \mu_{\epsilon,3}\Big].$$

$$\begin{aligned}(d)\mu_{\mathrm{x},4} := EX_t^4 = \frac{1}{1-\alpha^4}\Big[&\big(6\alpha^3(1-\alpha) + 4\alpha^3\mu_{\epsilon,1}\big)\mu_{\mathrm{x},3}\\ &+\big(\alpha^2(1-\alpha)(7-11\alpha) + 12\alpha^2(1-\alpha)\mu_{\epsilon,1} + 6\alpha^2\mu_{\epsilon,2}\big)\mu_{\mathrm{x},2}\\ &+\big(\alpha(1-\alpha)(1-6\alpha+6\alpha^2) + 4\alpha(1-\alpha)(1-2\alpha)\mu_{\epsilon,1} + 6\alpha(1-\alpha)\mu_{\epsilon,2} + 4\alpha\mu_{\epsilon,3}\big)\mu_{\mathrm{x},1}\\ &+\mu_{\epsilon,4}\Big].\end{aligned}$$

$$(e)\mu_{\mathrm{x}_{(s)}}(u) := E(X_tX_{t+us}) = \alpha\mu_{\mathrm{x}_{(s)}}(u-1) + \frac{\mu_{\epsilon,1}^2}{(1-\alpha)}.$$

$$(f)\mu_{\mathrm{x}_{(s)}}(0,u) := E\big(X_t^2X_{t+us}\big) = \alpha\mu_{\mathrm{x}_{(s)}}(0,u-1) + \mu_{\epsilon,1}\mu_{\mathrm{x},2}.$$

$$(g)\mu_{\mathrm{x}_{(s)}}(u,u) := E\big(X_tX_{t+us}^2\big) = \alpha^2\mu_{\mathrm{x}_{(s)}}(u-1,u-1) + \big(\alpha(1-\alpha) + 2\alpha\mu_{\epsilon,1}\big)\mu_{\mathrm{x}_{(s)}}(u-1) + \mu_{\mathrm{x},1}\mu_{\epsilon,2}.$$

$$(h)\mu_{\mathrm{x}_{(s)}}(u,v) := E\big(X_tX_{t+us}X_{t+vs}\big) = \alpha\mu_{\mathrm{x}_{(s)}}(u,v-1) + \mu_{\epsilon,1}\mu_{\mathrm{x}_{(s)}}(u).$$

$$(i)\mu_{\mathrm{x}_{(s)}}(0,0,u) := E\big(X_t^3X_{t+us}\big) = \alpha\mu_{\mathrm{x}_{(s)}}(0,0,u-1) + \mu_{\epsilon,1}\mu_{\mathrm{x},3}.$$

$$(j)\mu_{\mathrm{x}_{(s)}}(0,u,u) := E\big(X_t^2X_{t+us}^2\big) = \alpha^2\mu_{\mathrm{x}_{(s)}}(0,u-1,u-1) + \mu_{\epsilon,2}\mu_{\mathrm{x},2} + 2\alpha\mu_{\epsilon,1}\mu_{\mathrm{x},1} + \alpha(1-\alpha)\mu_{\mathrm{x}_{(s)}}(0,u-1).$$

$$\begin{aligned}(k)\mu_{\mathrm{x}_{(s)}}(0,u,v) := E\big(X_t^2X_{t+us}X_{t+vs}\big) &= \alpha\mu_{\mathrm{x}_{(s)}}(0,u,v-1) + \mu_{\epsilon,1}\mu_{\mathrm{x}_{(s)}}(0,u).\\ &+\big(\alpha(1-\alpha) + 2\alpha\mu_{\epsilon,1}\big)\mu_{\mathrm{x}_{(s)}}(u,v-1) + \mu_{\epsilon,2}\mu_{\mathrm{x}_{(s)}}(u).\end{aligned}$$

$$\begin{aligned}(l)\mu_{\mathrm{x}_{(s)}}(u,u,u) := E\big(X_tX_{t+us}^3\big) &= \alpha^3\mu_{\mathrm{x}_{(s)}}(u-1,u-1,u-1) + 3\alpha\big(\mu_{\epsilon,1} + \alpha(1-\alpha)\big)\mu_{\mathrm{x}_{(s)}}(u-1,u-1)\\ &+\big(\alpha(1-3\alpha+2\alpha^2) + 3\mu_{\epsilon,2}\big)\mu_{\mathrm{x}_{(s)}}(u-1) + \mu_{\epsilon,3}.\end{aligned}$$

$$(m)\mu_{\mathrm{x}_{(s)}}(u,u,v) := E\big(X_tX_{t+us}^2X_{t+vs}\big) = \alpha\mu_{\mathrm{x}_{(s)}}(u,u,v-1) + \mu_{\epsilon,1}\mu_{\mathrm{x}_{(s)}}(u,u).$$

$$\begin{aligned}(n)\mu_{\mathrm{x}_{(s)}}(u,v,v) := E\big(X_tX_{t+us}X_{t+vs}^2\big) &= \alpha^2\mu_{\mathrm{x}_{(s)}}(u,v-1,v-1)\\ &+\big(\alpha(1-\alpha) + 2\alpha\mu_{\epsilon,1}\big)\mu_{\mathrm{x}_{(s)}}(u,v-1) + \mu_{\epsilon,2}\mu_{\mathrm{x}_{(s)}}(u).\end{aligned}$$

$$(o)\mu_{\mathrm{x}_{(s)}}(u,v,w) := E\big(X_tX_{t+us}X_{t+vs}X_{t+ws}\big) = \alpha\mu_{\mathrm{x}_{(s)}}(u,v,w-1) + \mu_{\epsilon,1}\mu_{\mathrm{x}_{(s)}}(u,v).$$

Given observations $X_{-s+1}, X_{-s+2}, \dots, X_n$ from the stationary INAR(1)s process with $E\epsilon_t = \mu_\epsilon (> 0)$, $Var(\epsilon_t) = \sigma_\epsilon^2 (> 0)$ and autocorrelation $\alpha \in [0,1)$, the CLSE, $\hat{\theta}_n = (\hat{\alpha}_n, \widehat{\mu_{\epsilon}}_n)$, of parameter $\theta = (\alpha, \mu_\epsilon)$ can be obtained by minimizing the conditional sum of squares $S_n(\theta) := \sum_{t=1}^{n} \left(X_t - \alpha X_{t-s} - \mu_\epsilon \right)^2$ over $\theta \in \{0 \le \alpha < 1,\ \mu_\epsilon > 0\}$, that is,

$$\hat{\alpha}_n = \frac{n\sum_{t=1}^{n} X_t X_{t-s} - \sum_{t=1}^{n} X_{t-s} \sum_{t=1}^{n} X_t}{n\sum_{t=1}^{n} X_{t-s}^2 - (\sum_{t=1}^{n} X_{t-s})^2}, \tag{4}$$

$$\widehat{\mu_{\epsilon}}_n = \frac{1}{n}\left(\sum_{t=1}^{n} X_t - \hat{\alpha}_n \sum_{t=1}^{n} X_{t-s} \right), \tag{5}$$

where $\hat{\alpha}_n$ is set to be 0 if its denominator is 0. By checking the regularity conditions in Klimko and Nelson (1978), one can see that the CLSE $\hat{\theta}_n$ enjoys the property of consistency and asymptotic normality.

Since Page (1955), the change point problem has been widely appreciated as an important issue in the time series analysis context, because ignorance of parameter changes leads to a false conclusion. The CUSUM test has been popular due to the ease at its usage among researchers: see Chen and Gupta (2011) for a general review and Lee et al. (2003) for time series models. In this study, we consider to utilize the CLSE-based CUSUM test in Lee and Na (2005) and Kang and Lee (2009) for detecting the change of

We set up the null and alternative hypothesis as follows:
$H_0 : \theta_0 = (\alpha_0, \mu_{\epsilon 0})$ does not changes over $X_{-s+1}, \dots, X_n$ vs. H_1: not H_0

Put

$$W = \begin{pmatrix} E\left(X_{t-s}^2 (X_t - \alpha_0 X_{t-s} - \mu_{\epsilon 0})^2 \right) & E\left(X_{t-s} (X_t - \alpha_0 X_{t-s} - \mu_{\epsilon 0})^2 \right) \\ E\left(X_{t-s} (X_t - \alpha_0 X_{t-s} - \mu_{\epsilon 0})^2 \right) & E(X_t - \alpha_0 X_{t-s} - \mu_{\epsilon 0})^2 \end{pmatrix}, \tag{6}$$

$$V = \begin{pmatrix} EX_t^2 & EX_t \\ EX_t & 1 \end{pmatrix}. \tag{7}$$

Define

$$T_n^{cls} := \max_{\nu \le k \le n} T_{n,k}^{cls} = \max_{\nu \le k \le n} \frac{k^2}{n} \left(\hat{\theta}_k - \hat{\theta}_n \right)^\top \hat{V}_n \hat{W}_n^{-1} \hat{V}_n \left(\hat{\theta}_k - \hat{\theta}_n \right),$$

where ν is a positive integer,

$$\hat{W}_n = \begin{pmatrix} \frac{1}{n}\sum_{t=1}^{n} X_{t-s}^2 (X_t - \hat{\alpha} X_{t-s} - \hat{\mu}_\epsilon)^2 & \frac{1}{n}\sum_{t=1}^{n} X_{t-s} (X_t - \hat{\alpha} X_{t-s} - \hat{\mu}_\epsilon)^2 \\ \frac{1}{n}\sum_{t=1}^{n} X_{t-s} (X_t - \hat{\alpha} X_{t-s} - \hat{\mu}_\epsilon)^2 & \frac{1}{n}\sum_{t=1}^{n} (X_t - \hat{\alpha} X_{t-s} - \hat{\mu}_\epsilon)^2 \end{pmatrix}. \tag{8}$$

$$\hat{V}_n = \begin{pmatrix} \frac{1}{n}\sum_{t=1}^{n} X_{t-s}^2 & \frac{1}{n}\sum_{t=1}^{n} X_{t-s} \\ \frac{1}{n}\sum_{t=1}^{n} X_{t-s} & 1 \end{pmatrix}. \tag{9}$$

Following the arguments in Lee and Na (2005) and Kang and Lee (2009), we can verify the following, the proof of which is omitted for brevity:

Theorem 2.1 *Under* H_0, *as* $n \to \infty$,

$$T_n^{cls} \xrightarrow{w} \sup_{0\le u\le 1} \left\| \mathbf{B}_2^{\circ}(u) \right\|^2,$$

where $\mathbf{B}_2^{\circ}(u) = (B_1^{\circ}(u), B_2^{\circ}(u))^{\top}$ *is a 2-dimensional Brownian bridge.*

Tables 1 and 2 show that the performance of the CLSE-CUSUM statistics with $\lambda = \mu_{\epsilon 0} + \delta\sqrt{\mu_{\epsilon 0}}$, $\alpha = \alpha_0 + \delta$ at the nominal level of 0.05 when the innovations follow a Poisson distribution with mean λ: the corresponding critical value is 2.408 (see Lee et al. 2003). Here, we use 1,000 repetitions. When n is small ($n = 250, 500$), the sizes are somewhat over-estimated, which becomes more prominent when α is higher ($\alpha = 0.75$). However, as n increases, the size approaches the predetermined level. Moreover, the power tends to increase gradually when δ and n increase. Note that when there is a shift in λ, the performance in the case of $\delta < 0$ is slightly better in terms of power than in the other case. It can be also seen that for shift in α, the performance in the case of $\delta > 0$ is much better than in the other case. Overall, the results confirm the validity of the CLSE-CUSUM statistic.

Table 1 Empirical sizes and power of the CLSE-CUSUM statistic for INAR(1) process with shift in $\lambda = \mu_{\epsilon 0} + \delta\sqrt{\mu_{\epsilon 0}}$ at the level of 0.05

s	$\mu_{\varepsilon 0}$	α_0	n	δ								
				-0.6	-0.45	-0.3	-0.15	0	0.15	0.3	0.45	0.6
4	5	0.25	250	0.97	0.78	0.43	0.16	0.07	0.15	0.38	0.68	0.89
			500	1.00	0.99	0.72	0.23	0.06	0.21	0.65	0.95	1.00
			1000	1.00	1.00	0.97	0.41	0.05	0.38	0.93	1.00	1.00
			1500	1.00	1.00	1.00	0.56	0.05	0.54	0.99	1.00	1.00
			2000	1.00	1.00	1.00	0.71	0.05	0.68	1.00	1.00	1.00
		0.5	250	0.96	0.77	0.44	0.18	0.10	0.17	0.36	0.59	0.83
			500	1.00	0.96	0.66	0.23	0.09	0.21	0.57	0.88	0.99
			1000	1.00	1.00	0.93	0.35	0.06	0.33	0.87	1.00	1.00
			1500	1.00	1.00	0.99	0.50	0.05	0.47	0.97	1.00	1.00
			2000	1.00	1.00	1.00	0.64	0.05	0.58	0.99	1.00	1.00
		0.75	250	0.97	0.81	0.52	0.26	0.11	0.24	0.38	0.57	0.77
			500	1.00	0.97	0.68	0.28	0.11	0.24	0.53	0.82	0.96
			1000	1.00	1.00	0.92	0.38	0.10	0.32	0.80	0.98	1.00
			1500	1.00	1.00	0.98	0.50	0.07	0.41	0.94	1.00	1.00
			2000	1.00	1.00	1.00	0.60	0.05	0.51	0.98	1.00	1.00

Table 2 Empirical sizes and power of the CLSE-CUSUM statistic for INAR(1) process with shift in $\alpha = \alpha_0 + \delta$ at the level of 0.05

s	$\mu_{\varepsilon 0}$	α_0	n	δ								
				-0.2	-0.15	-0.1	-0.05	0	0.05	0.1	0.15	0.2
12	7	0.25	250	0.86	0.60	0.29	0.10	0.06	0.19	0.56	0.89	0.99
			500	1.00	0.93	0.57	0.18	0.06	0.30	0.83	1.00	1.00
			1000	1.00	1.00	0.93	0.34	0.05	0.51	0.99	1.00	1.00
			1500	1.00	1.00	0.99	0.53	0.05	0.66	1.00	1.00	1.00
			2000	1.00	1.00	1.00	0.68	0.05	0.81	1.00	1.00	1.00
		0.5	250	0.86	0.64	0.35	0.13	0.07	0.25	0.69	0.97	1.00
			500	1.00	0.95	0.64	0.19	0.07	0.39	0.94	1.00	1.00
			1000	1.00	1.00	0.96	0.39	0.05	0.62	1.00	1.00	1.00
			1500	1.00	1.00	1.00	0.59	0.05	0.80	1.00	1.00	1.00
			2000	1.00	1.00	1.00	0.73	0.05	0.89	1.00	1.00	1.00
		0.75	250	0.93	0.78	0.50	0.21	0.10	0.43	0.94	1.00	1.00
			500	1.00	0.97	0.79	0.32	0.10	0.65	1.00	1.00	1.00
			1000	1.00	1.00	0.99	0.57	0.08	0.90	1.00	1.00	1.00
			1500	1.00	1.00	1.00	0.77	0.06	0.97	1.00	1.00	1.00
			2000	1.00	1.00	1.00	0.91	0.05	1.00	1.00	1.00	1.00

3. Monitoring of Mean Shift based on CLSE-CUSUM Statistic

In this section, we modify the CUSUM test in Theorem 2.1 that can detect a parameter shift more efficiently in stationary INAR(1)s processes. Given predetermined positive integer $l > (s+1)$, playing a role such as the length of virtual in-control data, we put $\nu = l - s$ and define

$$C_n(\nu) = \max_{1\le k\le n} l'_{n,k}(\nu) = \max_{1\le k\le n} \frac{(k+\nu)^2}{n+\nu}\left(\hat{\theta}'_k - \hat{\theta}'_n\right)^\top VW^{-1}V\left(\hat{\theta}'_k - \hat{\theta}'_n\right), \tag{10}$$

where $\hat{\theta}'_k = (\hat{\alpha}'_k, \widehat{\mu_{\epsilon}}'_k)$ with

$$\hat{\alpha}'_k = \frac{(\nu+k)\sum_{t=-\nu+1}^{k} X_t X_{t-s} - \sum_{t=-\nu+1}^{k} X_{t-s} \sum_{t=-\nu+1}^{k} X_t}{(\nu+k)\sum_{t=-\nu+1}^{k} X_{t-s}^2 - (\sum_{t=-\nu+1}^{k} X_{t-s})^2}, \quad \widehat{\mu_{\epsilon}}'_k = \frac{1}{(\nu+k)}\left(\sum_{t=-\nu+1}^{k} X_t - \hat{\alpha}'_k \sum_{t=-\nu+1}^{k} X_{t-s}\right);$$

Initial sums such as $\sum_{t=-\nu+1}^{0} X_t X_{t-s}$, $\sum_{t=-\nu+1}^{0} X_{t-s}$, $\sum_{t=-\nu+1}^{0} X_t$, and $\sum_{t=-\nu+1}^{0} X_{t-s}^2$ are replaced with $\nu E X_t X_{t-s}$, $\nu E X_t$, $\nu E X_t$, and $\nu E X_t^2$, respectively; W and V are calculated using the formula in Proposition 2.2, (6) and (7), or replaced by those in (8) and (9). The same convergence result as in Theorem 2.1 also holds for $C_n(\nu)$, regardless of ν.

To compare the performance with the CUSUM chart, we use the Poisson INAR(1) process in (1) because it is the most widely used INAR(1) process in literatures. We evaluate the CUSUM statistic with the Poisson INAR(1) with mean λ_0. In this case, the marginal mean is $\mu_0 = \lambda_0/(1-\alpha_0)$, W and V are obtained as:

$$V = \begin{pmatrix} \mu_0 + \mu_0^2 & \mu_0 \\ \mu_0 & 1 \end{pmatrix}, \qquad W = \begin{pmatrix} \mu_0(W_{1,1}+W_{1,2}+W_{1,3}) & \mu_0(W_{2,1}+W_{2,2}+W_{2,3}) \\ \mu_0(W_{2,1}+W_{2,2}+W_{2,3}) & W_{3,1}+W_{3,2} \end{pmatrix},$$

where

$W_{1,1} = (1+\mu_0)(\lambda_0^2 + \mu_0 - 2\lambda_0\mu_0 + \mu_0^2)$, $W_{1,2} = \alpha_0(1 + 2(1+\lambda_0)\mu_0 - 2(1-\lambda_0)\mu_0^2 - 2\mu_0^3)$,

$W_{1,3} = -\alpha_0^2(1 + 3\mu_0 - \mu_0^3)$, $W_{2,1} = \lambda_0^2 + \mu_0 - 2\lambda_0\mu_0 + \mu_0^2$, $W_{2,2} = \alpha_0(1 + 2\lambda_0\mu_0 - 2\mu_0^2)$,

$W_{2,3} = -\alpha_0^2(1+\mu_0-\mu_0^2)$, $W_{3,1} = \lambda_0^2 - 2\lambda_0\mu_0 + \mu_0 + 2\alpha_0\lambda_0\mu_0 + \mu_0^2$,

$W_{3,2} = \alpha_0^2\mu_0(1+\mu_0) - 2\alpha_0\mu_0(\alpha_0+\mu_0)$.

Note that in the case of INAR(1) process with Poisson innovations, the elements of W can be also obtained from Proposition 1 in Weiß (2012). Tables 3 and 4 show examples of ARL profiles, compared with the two-sided CUSUM chart (CUSUM(k^+, h^+, k^-, h^-) with $c_0^+ = c_0^- = 0$) when the parameters are changed to $\mu = \lambda/(1-\alpha)$, $\lambda = \lambda_0 + \delta\sqrt{\lambda_0}$ and $\alpha = \alpha_0 + \delta$. Here, we use 30,000 repetitions. The tables show that the CLSE-CUSUM control procedure outperforms the two-sided CUSUM chart regardless of ν when there is an up-shift in λ. Overall, the ARL performance appears to be the best at $\nu = 250$ when there is an up-shift in λ. However, the performance of the proposed procedure performs poorly when there is a down-shift in λ or α, which becomes clearer as ν gets smaller. When there is an up-shift in α, the performance of the two charts does not differ significantly expect $\nu = 100$. For $\nu = 100$, there are cases such that the out-of-control ARL is larger than the in-control ARL when there is a down-shift in λ or α, although this phenomenon is mitigated as ν increases. Such a case often occurs when the difference between the in-control ARLs of the two one-sided charts is large, especially when the in-control ARL of the upper one-sided chart is smaller than the lower one-sided chart: see, for example, Yontay et al. (2013). We can easily guess that for the given control limit c, the out-of-control state signals including false alarms are mainly caused by the up-shifts of parameter. Hence, if one uses the one control limit for the two-sided monitoring, it may lead to a biased ARL performance. This problem could be avoided by using proper lower and upper one-sided control limits.

Based on our findings mentioned above, we design the one-sided CLSE-CUSUM control procedure as follows: given observations $X_{-s+1}, X_{-s+2}, \ldots, X_n$ from the stationary INAR(1)s process with $\alpha_0 \in [0,1)$, $EX_t = \mu_0 > 0$ and $EX_t^4 < \infty$, let $c_p(\nu, n) = \underset{1\le k\le n}{\arg\max}\, l'_{n,k}(\nu)$, where $l'_{n,k}(\cdot)$ are in (10) and $\bar{X}_{c_p(\nu,n)} = \sum_{k=c_p(\nu,n)}^{n} X_k$ for $\nu \ge 2$. Note that $c_p(\nu,n)$ is an estimated location of change point when it exists. The upper one-sided CLSE-CUSUM control statistic is then:

$$C_n^u(\nu_u) = C_n(\nu_u) I(\bar{X}_{c_p(\nu_u,n)} > \mu_0),\ n \in \mathbf{N}, \tag{11}$$

while the lower one-sided CLSE-CUSUM control statistic is:

$$C_n^l(\nu_l) = C_n(\nu_l) I(\bar{X}_{c_p(\nu_l,n)} < \mu_0),\ n \in \mathbf{N},$$

where $C_n(\cdot)$ is the one in (10), $I(\cdot)$ is an indicator function, $\nu_u \ge 2$ and $\nu_l \ge 2$ are predetermined positive integers. Given control limits $c_u > 0$ and $c_l > 0$, we determine that the process is out-of-control when $C_n^u(\nu_u) \ge c_u$ or $C_n^l(\nu_l) \ge c_l$ is signaled, that is, the signal of out-of-control state is triggered by a mean increase (the former case) or a mean decrease (the latter case). The fundamental difference between the proposed and conventional methods lies in that ours uses additional information in estimation when determining the status of the process of interest.

Table 3 ARLs of the CLSE-CUSUM test statistic and conventional two-sided CUSUM chart for INAR(1) process with shift in $\lambda = \lambda_0 + \delta\sqrt{\lambda_0}$

μ_0	α_0	ν	c	δ								
				-0.2	-0.15	-0.1	-0.05	0	0.05	0.1	0.15	0.2
2.5	0.25		CUSUM									
			(3,19,2,15):	155.7	229.4	338.8	459.5	510.7	440.8	322.1	224.8	158.3
		100	1.374	298.6	448.6	510.5	559.1	510.8	398.6	309.5	213.6	147.1
		250	1.151	149.3	227.9	355.0	505.0	510.7	387.5	263.8	177.7	127.6
		500	0.955	147.6	203.6	301.1	466.4	510.6	396.2	261.7	183.2	136.5
		750	0.828	153.9	207.0	297.3	453.5	510.8	398.3	270.3	192.1	145.0
2.5	0.5		CUSUM									
			(3,29,2,22):	144.4	210.8	307.8	429.2	492.4	438.6	325.3	231.5	165.0
		100	1.38	390.4	509.8	573.4	530.2	492.1	403.0	305.1	217.6	159.5
		250	1.14	166.9	254.0	381.4	486.8	493.4	385.7	266.1	184.6	134.1
		500	0.942	158.5	221.9	324.2	452.1	492.6	388.8	268.0	189.6	144.6
		750	0.809	162.2	222.1	313.5	436.4	492.9	386.1	273.4	197.1	153.5
2.5	0.75		CUSUM									
			(4,22,2,40):	143.8	202.2	296.1	413.5	497.6	477.4	386.1	292.1	216.5
		100	1.401	580.4	656.9	624.0	590.2	497.5	381.4	275.5	203.5	149.8
		250	1.146	197.5	303.8	440.0	528.4	497.6	384.7	264.6	189.5	139.5
		500	0.937	171.5	238.5	352.0	480.9	497.6	382.3	268.2	195.5	150.1
		750	0.808	173.4	235.3	334.9	467.8	497.6	389.9	277.0	204.5	159.2

Table 4 ARLs of the CLSE-CUSUM test statistic and conventional two-sided CUSUM chart for INAR(1) process with shift in $\alpha = \alpha_0 + \delta$

μ_0	α_0	ν	c	δ								
				-0.2	-0.15	-0.1	-0.05	0	0.05	0.1	0.15	0.2
2.5	0.25		CUSUM									
			(3,19,2,15):	108.3	171.6	292.9	489.0	512.3	263.5	119.9	63.5	38.9
		100	1.374	87.9	145.5	333.9	525.4	510.8	299.3	129.3	61.9	36.8
		250	1.151	84.1	111.3	179.0	375.3	510.7	253.9	111.1	62.3	40.5
		500	0.955	94.7	120.6	174.7	320.6	510.6	253.8	119.3	69.7	45.9
		750	0.828	102.8	129.2	181.7	313.8	510.8	260.4	126.5	75.1	49.7
2.5	0.5		CUSUM									
			(3,29,2,22):	78.7	117.2	200.2	386.2	491.9	230.0	94.9	48.9	30.0
		100	1.38	90.1	138.4	280.5	519.3	492.1	269.4	110.0	50.6	27.8
		250	1.14	84.8	109.9	167.6	355.1	493.4	222.5	93.4	50.0	30.0
		500	0.942	95.2	119.4	167.6	304.5	492.6	224.2	99.9	55.3	33.4
		750	0.809	102.6	127.2	174.1	297.2	492.9	229.7	105.7	58.8	35.5
2.5	0.75		CUSUM									
			(4,22,2,40):	65.5	86.5	134.3	279.7	500.6	185.5	57.0	25.3	15.3
		100	1.401	91.5	125.2	216.4	464.5	497.5	223.6	64.6	25.6	14.3
		250	1.146	83.4	103.4	149.0	291.2	497.6	163.8	56.6	25.9	15.0
		500	0.937	91.8	111.0	151.4	258.6	497.6	162.6	59.9	27.6	16.0
		750	0.808	98.6	118.5	159.0	261.5	497.6	167.9	62.7	28.8	16.5

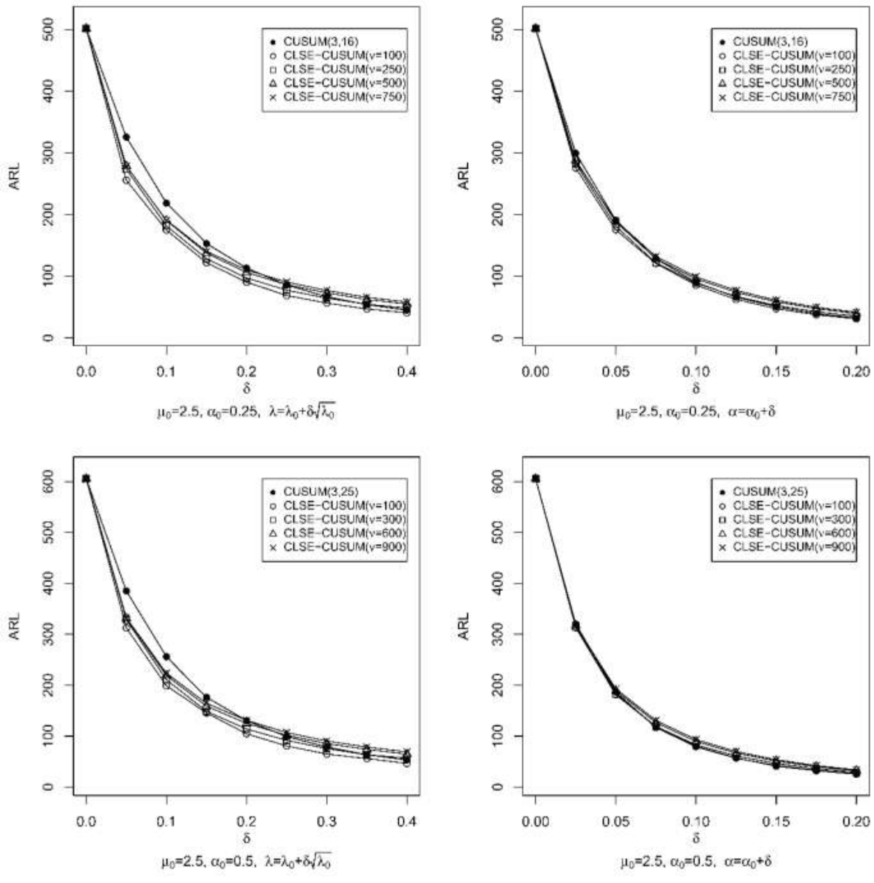

Figure 1 ARLs of upper one-sided CLSE-CUSUM and conventional CUSUM chart

4. Performance Comparison

We focus on the upper one-sided control chart for detecting a mean increase, since this case receives more attention in practice. To compare with the upper one-sided CUSUM chart (CUSUM (k,h) with $c_0 = 0$), we adopt the reference value and corresponding control limit in Weiss and Testik (2009). Let $\{X_t\}_{t\in\mathbb{Z}}$ be a stationary INAR(1) process with $\alpha_0 \in [0,1)$ and Poisson innovations with mean λ_0. In this case, the marginal mean is obtained as $\mu_0 = \lambda_0/(1-\alpha_0)$. Numerical experiments show that the performance of our procedure and the CUSUM chart is not much affected by the values of μ and λ for fixed α, so we only take account of the case that $\mu_0 = \lambda_0/(1-\alpha_0) = 2.5$ and $\alpha_0 \in \{0.25, 0.5, 0.75\}$. These values are assumed to change to $\mu = \lambda/(1-\alpha)$ with $\lambda = \lambda_0 + \delta\sqrt{\lambda_0}$. and $\alpha = \alpha_0 + \delta$. Figures 1 shows the performance, in terms of ARL, of the CUSUM chart and our procedure with several ν. The specific values in the figures can be found in Tables 5 and 6. The ARLs, SDs and medians are obtained using $30,000$ repetitions and the same random seed is used for fairness.

It is clear that the upper one-sided CLSE-CUSUM procedure with $\nu = 100$ outperforms the CUSUM chart in term of ARL, when there are small to moderate shifts in λ. For $\nu \approx 0.5\text{ARL}$, the proposed procedure has a better ARL performance when there are small shifts (about $\delta \le 0.3$) in λ. But for some moderate shifts ($\delta = 0.35, 0.4$) in λ, the CLSE-CUSUM chart with $\nu \approx 0.5\text{ARL}$ shows a slightly lower performance. This phenomenon is more apparent when $\nu \approx \text{ARL}, 1.5\text{ARL}$. Moreover, for given δ, the ARL also decreases as the ν decreases. The value of α does not have a significant impact on the ARL performance in monitoring up-shift in λ.

When there is an up-shift in $\alpha = 0.25$, the CLSE-CUSUM procedure with $\nu = 100$ shows a better ARL performance. However, for other α and ν, the CLSE-CUSUM procedure shows a similar or slightly lower performance than the CUSUM chart, which becomes more significant as α gets higher. This may be due to the poor performance of the CLSE when high autocorrelations exist.

Tables 7-8 show the results comparing the performance in terms of SD. For given δ, the SD tends to decreases as the ν increases. For $\nu = 100$, the in-control SD and out-of-control SD performance appear to be worse than that of the CUSUM chart although the difference gets smaller as δ increases. For $\nu \approx 0.5\text{ARL}$, the in-control SD of the CLSE-CUSUM shows a worse performance. However, the out-of-control SD shows a better performance, except for $\delta = 0.05$, when there is a shift in λ. For $\nu \approx \text{ARL}, 1.5\text{ARL}$, the out-of-control SD in our procedure shows a better performance while the in-control SD looks reasonable for all the cases with a shift in λ. For the shift in α, a similar conclusion can be reached except for $\alpha = 0.75$. For $\alpha = 0.75$, even the out-of-control SD performance is no better than that of the CUSUM chart.

Tables 9 and 10 show that when the median is used, a similar conclusion to the ARL case can be made. The median performance in our procedure tends to be smaller as the ν decreases.

From these findings, we conclude that our method can be comparable with the conventional CUSUM chart if one is interested in an effective detection of the small mean increase with maintained autocorrelation, which actually attracts more attention from the researchers: see, for instance, Yontay et al. (2013) and Kim and Lee (2017). In practice, the choice of ν could be an important issue. An optimal ν in term of ARL, SD and median might be obtained based on Monte Carlo simulations. However, this approach is not always feasible in practice. We recommend to choose $\nu \approx 0.5\text{ARL}, \text{ARL}$ or the values between these two. Notice that if ν is smaller than 0.5ARL, the performance is poor in terms of the in-control SD. On the other hand, if the ν is much larger than ARL, the overall performance would not be satisfactory in terms of ARL. The performance does not vary much according to the type of innovations, e.g. the Katz innovation (Kim and Lee, 2017) and other parameter settings. The result is not reported here for brevity.

Table 5 ARLs of the upper one-sided CLSE-CUSUM chart and conventional CUSUM chart for INAR(1) process with shift in $\lambda = \lambda_0 + \delta\sqrt{\lambda_0}$

μ_0	α_0	ν	c_u	δ								
				0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4
2.5	0.25		CUSUM(3,16):	501.4	325.6	218.6	153.1	113.2	85.7	66.8	54.1	45.0
		100	1.110	501.4	255.6	174.8	122.0	90.4	68.8	56.6	46.9	40.8
		250	0.917	501.4	273.5	181.6	128.5	97.1	77.4	64.6	54.6	48.3
		500	0.719	501.4	278.7	190.2	137.5	106.1	86.3	73.1	62.3	55.1
		750	0.583	501.4	278.3	190.8	140.4	110.1	90.5	77.0	65.8	58.4
2.5	0.5		CUSUM(3,25):	605.5	385.1	256.0	176.3	130.4	99.2	78.5	63.5	53.2
		100	1.169	605.5	312.6	198.8	145.6	104.6	80.7	64.8	55.7	46.5
		300	0.931	605.5	328.6	210.6	148.7	114.1	91.6	75.6	63.2	56.7
		600	0.724	605.5	331.3	219.2	158.8	125.1	101.7	85.2	73.9	64.6
		900	0.590	604.5	332.5	223.5	164.5	131.0	107.1	90.3	78.6	68.9
2.5	0.75		CUSUM(3,39):	505.6	321.0	221.0	158.6	120.1	94.2	75.8	64.2	54.9
		100	1.149	505.6	269.4	184.6	131.5	101.0	79.9	65.9	55.7	47.7
		250	0.924	505.6	277.8	190.2	137.4	107.3	87.3	73.0	62.9	54.5
		500	0.720	505.6	289.0	199.8	147.2	116.5	96.4	81.8	70.5	61.6
		750	0.583	505.6	290.7	201.8	150.9	120.0	100.2	85.7	74.0	64.8

Table 6 ARLs of the upper one-sided CLSE-CUSUM chart and conventional CUSUM chart for INAR(1) process with shift in $\alpha = \alpha_0 + \delta$

μ_0	α_0	ν	c_u	δ								
				0	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2
2.5	0.25		CUSUM(3,16):	501.4	299.5	190.6	128.4	90.0	66.2	50.6	39.6	32.3
		100	1.110	501.4	275.3	175.2	120.8	85.6	62.6	47.5	37.6	31.1
		250	0.917	501.4	282.6	180.9	121.7	88.8	67.2	52.7	42.4	35.4
		500	0.719	501.4	287.9	188.0	128.4	95.5	74.0	58.5	47.5	39.5
		750	0.583	501.4	285.8	189.9	132.1	99.0	77.1	61.2	49.8	41.6
2.5	0.5		CUSUM(3,25):	605.5	320.0	186.4	117.0	79.1	56.5	42.3	33.0	26.5
		100	1.169	605.5	313.1	181.1	118.9	80.1	57.0	41.1	32.1	25.4
		300	0.931	605.5	314.6	183.0	119.0	82.7	60.9	46.3	36.3	29.1
		600	0.724	605.5	316.5	189.4	126.6	89.5	66.7	51.0	40.1	32.1
		900	0.590	604.5	316.9	192.9	130.7	93.3	69.8	53.6	42.1	33.7
2.5	0.75		CUSUM(3,39):	505.6	221.5	115.0	68.6	45.0	32.1	24.4	19.6	16.4
		100	1.149	505.6	233.8	134.4	78.6	48.9	32.8	23.0	17.1	13.5
		250	0.924	505.6	227.8	125.5	75.2	48.1	33.3	24.0	18.0	14.3
		500	0.720	505.6	233.4	129.4	78.7	51.1	35.3	25.4	19.0	15.1
		750	0.583	505.6	233.1	131.2	80.9	52.5	36.3	26.2	19.5	15.4

Table 7 SDs of the upper one-sided CLSE-CUSUM chart and conventional CUSUM chart for INAR(1) process with shift in $\lambda = \lambda_0 + \delta\sqrt{\lambda_0}$

μ_0	α_0	ν	c_u	δ								
				0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4
2.5	0.25		CUSUM(3,16):	488.9	311.2	206.7	142.6	100.8	73.6	55.6	43.0	34.1
		100	1.110	1508.2	443.2	262.1	160.5	99.6	63.1	46.7	35.7	28.6
		250	0.917	981.8	328.9	182.5	111.2	71.9	51.3	40.0	31.9	26.5
		500	0.719	617.3	252.8	154.6	95.5	64.9	48.8	38.5	31.8	26.7
		750	0.583	560.4	228.4	138.3	88.7	62.1	47.6	38.2	31.6	27.0
2.5	0.5		CUSUM(3,25):	595.0	364.8	238.7	159.4	112.0	82.0	61.8	47.6	37.9
		100	1.169	1987.5	770.7	304.8	210.5	116.7	81.5	57.2	45.5	33.7
		300	0.931	1036.3	414.5	207.3	124.5	83.6	60.8	47.0	37.8	32.4
		600	0.724	770.9	306.6	167.1	107.9	76.5	57.6	45.3	38.2	32.0
		900	0.590	691.2	277.3	156.2	102.3	74.1	56.6	45.3	38.1	31.6
2.5	0.75		CUSUM(3,39):	482.5	302.0	198.2	136.8	98.1	74.0	56.9	45.5	37.1
		100	1.149	1638.8	585.8	279.1	168.1	111.2	73.9	56.5	43.9	35.3
		250	0.924	1240.1	323.9	187.9	117.8	85.8	60.9	47.9	39.6	33.2
		500	0.720	602.0	264.8	162.0	106.0	77.9	58.0	47.4	40.2	34.1
		750	0.583	565.4	242.3	150.2	100.3	74.2	57.6	47.5	40.5	34.8

Table 8 SDs of the upper one-sided CLSE-CUSUM chart and conventional CUSUM chart for INAR(1) process with shift in $\alpha = \alpha_0 + \delta$

μ_0	α_0	ν	c_u	δ								
				0	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2
2.5	0.25		CUSUM(3,16):	488.9	287.7	177.9	117.6	79.6	56.1	41.1	30.8	23.9
		100	1.110	1508.2	580.9	277.6	166.0	102.3	59.6	41.7	30.3	23.9
		250	0.917	981.8	348.5	181.6	104.9	70.2	47.4	36.4	27.6	23.0
		500	0.719	617.3	267.1	149.9	92.4	64.3	46.0	35.5	28.2	23.3
		750	0.583	560.4	239.0	138.6	88.1	62.8	45.2	35.4	28.3	23.7
2.5	0.5		CUSUM(3,25):	595.0	302.9	169.6	101.7	64.4	43.4	30.6	22.2	16.7
		100	1.169	1987.5	602.6	310.7	187.5	88.5	53.1	34.5	24.3	18.6
		300	0.931	1036.3	374.3	173.9	97.1	60.9	41.4	30.4	23.1	18.3
		600	0.724	770.9	294.0	146.7	88.5	58.1	40.9	31.0	23.8	19.1
		900	0.590	691.2	263.5	137.6	85.0	57.3	41.1	31.4	24.3	19.5
2.5	0.75		CUSUM(3,39):	482.5	200.7	96.1	51.8	30.3	19.0	12.5	8.6	6.2
		100	1.149	1638.8	411.4	241.3	82.9	44.5	25.0	15.3	9.8	6.8
		250	0.924	1240.1	254.3	115.9	58.4	34.1	22.0	14.7	9.7	6.9
		500	0.720	602.0	206.8	99.6	55.0	33.9	22.2	15.1	9.9	7.0
		750	0.583	565.4	189.6	95.1	54.2	33.9	22.4	15.2	10.1	7.1

Table 9 Medians of the upper one-sided CLSE-CUSUM chart and conventional CUSUM chart for INAR(1) process with shift in $\lambda = \lambda_0 + \delta\sqrt{\lambda_0}$

μ_0	α_0	ν	c_u	δ								
				0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4
2.5	0.25		CUSUM(3,16):	349	230	156	111	82	64	51	42	35
		100	1.110	189	136	101	76	62	51	44	38	34
		250	0.917	268	178	128	97	79	65	56	48	44
		500	0.719	315	204	147	113	92	76	67	58	51
		750	0.583	329	213	155	120	97	82	71	62	55
2.5	0.5		CUSUM(3,25):	425	272	180	130	96	75	61	50	43
		100	1.169	209	149	113	87	71	58	50	44	38
		300	0.931	321	209	152	114	93	78	66	58	52
		600	0.724	375	240	174	132	108	91	78	69	61
		900	0.590	392	252	184	141	116	98	84	74	66
2.5	0.75		CUSUM(3,39):	360	229	160	117	91	73	60	52	45
		100	1.149	207	143	108	86	71	59	51	46	40
		250	0.924	276	180	134	105	86	73	63	56	49
		500	0.720	320	208	154	123	100	86	75	65	58
		750	0.583	334	219	163	129	106	91	80	69	62

Table 10 Medians of the upper one-sided CLSE-CUSUM chart and conventional CUSUM chart for INAR(1) process with shift in $\alpha = \alpha_0 + \delta$

μ_0	α_0	ν	c_u	δ								
				0	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2
2.5	0.25		CUSUM(3,16):	349	211	136	92	66	50	39	31	26
		100	1.110	189	139	100	75	58	46	36	30	25
		250	0.917	268	179	128	92	70	56	45	37	31
		500	0.719	315	206	146	105	80	64	51	42	35
		750	0.583	329	214	153	112	85	68	54	45	37
2.5	0.5		CUSUM(3,25):	425	228	135	86	60	44	34	27	22
		100	1.169	209	149	105	75	55	43	31	26	21
		300	0.931	321	203	132	93	68	52	40	32	25
		600	0.724	375	231	150	106	77	59	45	36	28
		900	0.590	392	241	158	112	82	63	48	38	30
2.5	0.75		CUSUM(3,39):	360	161	86	54	37	27	22	18	15
		100	1.149	207	128	82	55	37	26	19	15	12
		250	0.924	276	153	93	61	40	28	21	16	13
		500	0.720	320	175	105	67	44	30	22	17	14
		750	0.583	334	181	109	70	45	31	23	17	14

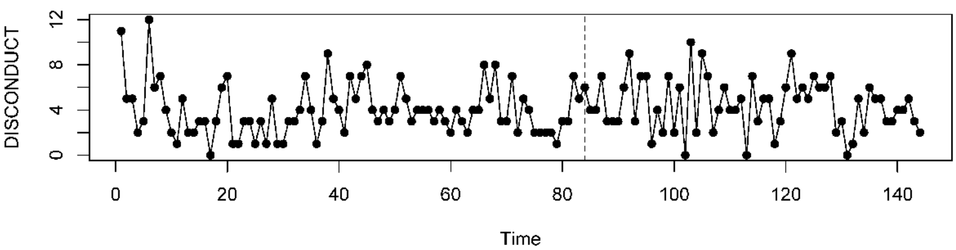

Figure 2 The sample path of disorderly conduct data

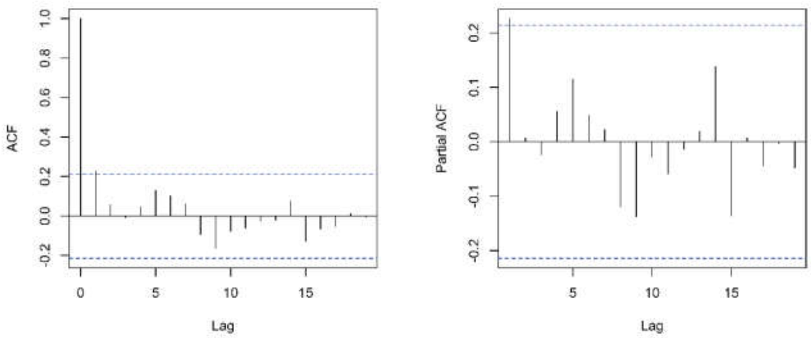

Figure 3 The ACF and PACF plot of disorderly conduct data (from 1990 to 1996)

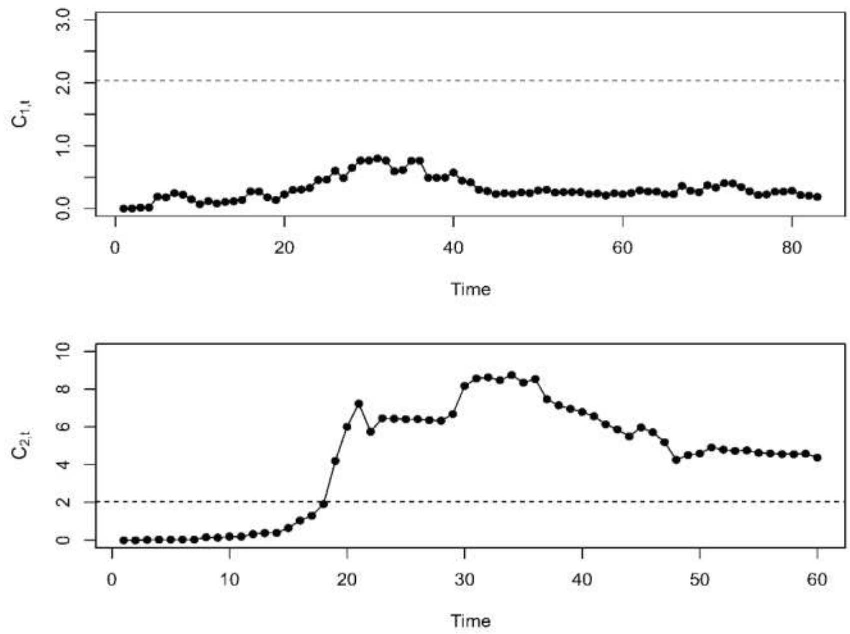

Figure 4 The upper one-sided CLSE-CUSUM chart of disorderly conduct data

5. A real data example

In order to showcase an application of CLSE-CUSUM charts to monitoring INAR(1)s processes, we consider the monthly number of disorderly conduct reported in the 44th police car beat in Pittsburgh from 1990 to 2001 in Kim and Lee (2017). We use the data from 1990 to 1996 as an in-control sample. The CLSE-CUSUM control chart is then applied to the data from 1997 to detect whether the mean increase occurs or not. The sample path plot is given in Figure 2, wherein the dashed line denotes December 1996, and the ACF and PACF plots are presented in Figure 3. For in-control data, the sample mean, variance, and autocorrelation are given as 3.9643, 5.3602 and 0.2261, respectively. From the sample path plot, one cannot easily check whether the mean increases or not after the dashed line.

Kim and Lee (2017) demonstrate that for the data from 1990 to 1996, the first order INAR(1) process with the Katz family innovation (INARKF(1)). Based on the results from Kim and Lee (2017) assuming that in-control data follows an INARKF(1) process with $\theta_1 = 2.2080$, $\theta_2 = 0.2537$ and $\alpha = 0.2511$. Kim and Lee (2017) apply the upper one-sided CUSUM chart defined as in (2) and (3) with $k = 4$, $h = 34$ wherein the in-control ARL is computed as 205.4 and the out-of-control signal occurs in August 2000. We apply the proposed upper one-sided CLSE-CUSUM procedure in (11) to this data. Note that our procedure has the advantage of not requiring a specific distributional assumptions on the innovation process.

To obtain the control limit that renders the in-control ARL near 205.4, we use $\nu = 100$ following the recommendation in Section 4 and replace V and W by their estimates from the in-control sample given as in (8) and (9). From numerical experiments with $30{,}000$ repetitions, the control limit c_u is calculated as 2.035. The plot of CLSE-CUSUM statistic is presented in Figure 4, wherein the dashed line denotes $c_u = 2.035$, and the $C_{1,t}$ and $C_{2,t}$ stand for the CLSE-CUSUM statistics with $\nu = 100$ of the disorderly conduct data from 1990 to 1996 and from 1997 to 2001, respectively. It can be observed that the data obtained from 1990 to 1996 is in-control because the maximum value of the statistic $C_{1,t}$ is less than 2.035. In the meantime, the plot of $C_{2,t}$, shows that the control statistic has an increasing trend and an out-of-control signal occurs at $t=19$ (July 1998). The estimated change point appears to be 8 (August 1997), indicating an earlier detection in comparison of the CUSUM chart. It can be also seen that the sample mean of the data from August 1997 to July 1998 appears to be 4.833, which is greater than that of the in-control data.

6. Proof of Proposition 2.1

Assume that $X_0 = 0$. For $t \in \mathbf{N}$, we can obtain the following by using the mathematical induction. Notice that $EX_{ts} \le C_k \sum_{i=0}^{t-1} \alpha^{ik}$, where

$$C_1 = \mu_{\epsilon,1},$$

$$C_2 = \left(\alpha + \frac{2\alpha\mu_{\epsilon,1}}{1-\alpha}\right) C_1 + \mu_{\epsilon,2},$$

$$C_3 = 3\alpha^2\left(1-\alpha+\mu_\epsilon\right)\frac{C_2}{1-\alpha^2} + \alpha\left(1-2\alpha+3\mu_\epsilon+\frac{3\mu_{\epsilon,2}}{1-\alpha}\right)C_1 + \mu_{\epsilon,3},$$

$$C_4 = \alpha^3\left(6(1-\alpha)+4\mu_{\epsilon,3}\right)\frac{C_3}{1-\alpha^3} + \alpha^2\left((1-\alpha)(7-11\alpha)+12(1-\alpha)\mu_{\epsilon,3}+6\mu_{\epsilon,2}\right)\frac{C_2}{1-\alpha^2}$$
$$+\alpha\left(1-6\alpha+\alpha^2+4\left(1-2\alpha+\frac{1}{1-\alpha}\right)\mu_{\epsilon,3}+6\mu_{\epsilon,2}\right)C_1+\mu_{\epsilon,4}.$$

Since $\alpha \in [0,1)$, we have $EX_{ts}^k \le C := \max\{C_k/(1-\alpha^k) : k = 1,2,3,4\}$. Proposition 1 from Bourguignon et al. (2016) indicates that X_t converges in distribution to a unique stationary marginal distribution X. Thus, owing to the Portmanteau lemma (cf. Theorem 29.1 of Billingsley (1979)), we get $EX^k \le \lim_{t\to\infty} EX_{ts}^k \le C,\ k = 1,2,3,4.$

7. Concluding Remarks

In this paper, we proposed a new control procedure based on the CLSE-CUSUM statistic. Our method outperforms the CUSUM chart when there are small to moderate mean increases of innovation processes in terms of ARL, and also, SD and median. Moreover, it merits to give additional information on the location of a shift and to set in-control ARLs at one's disposal. The proposed procedure can be applied to other distributions and probabilistic structures without serious difficulties. The task of statistical design in more complicated models such as INGARCH process based on various performance measures such as ARL, SD is left as our future project.

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