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Average Run Length with a Practical Investigation of Estimating Parameters of the EWMA Control Chart on the Long Memory AFRIMA Process

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Abstract

An appropriate control chart for practical observations should be designed from optimal parameters. In this research, the main objectives are to estimate the optimal smoothing parameter of the EWMA control chart and fractional differencing parameter to evaluate the Average Run Length (ARL) and compare among analytical EWMA ARL, numerical EWMA ARL, and analytical CUSUM ARL. Also, the analytical EWMA ARL is derived and numerical EWMA ARL is evaluated and illustrated. The time intervals in days between explosions in mines in Great Britain during 1875 to 1951 are an example of practical observations of a long memory ARFIMA process with exponential white noise. The findings showed that the method for evaluating analytical EWMA ARL is an alternative for measurement of the efficiency of the EWMA control chart due to the good performance.

Keywords: ARFIMA process, average run length, EWMA and CUSUM control chart, integral equation.

1. Introduction

In statistical estimation and forecasting, observations that often occur in a natural process such as a stochastic process are interesting to study. The stochastic process relates to natural phenomena, which generates random attributes causing an ineffective and inconsistent statistical estimator or unbiased estimator. In particular, these attributes are formed in serially correlated observations, or auto-correlated observations that are commonly practical observations. In order to model the auto-correlated attributes from collected observations, econometrics models, which are suitable for such the observations, such as Auto-Regressive (AR), Moving Average (MA), Auto-Regressive Integrated Moving Average (ARIMA), Auto-Regressive Fractionally Integrated Moving Average (ARFIMA), etc. have emerged as tools for decision making. Not only do serially correlated observations occur naturally, they are also important in fields such as finance, economics, production, etc. Statistical Process Control (SPC) as a quality control technique, especially the Exponential Weighted Moving Average (EWMA) control chart, is employed in decision making with respect to the behavior of auto-correlated observations.

Therefore, several researchers have studied the construction of forecasting models for serially correlated observations. This research has implications for statistical quality control, or the EWMA control chart, with serially correlated observations. In finance and agricultural economics, a forecasting model that combines ARIMA and the Moving Average (MA), the Weighted Moving Average (WMA), and the EWMA model to estimate the daily closing stock price of the PALTEL Company in Palestine was provided by Samir and Issam (2013); meanwhile, the ARIMA models for forecasting selected 1950-2010 annual agricultural productivity was used by Padhan (2012). The selected ARIMA model in minimum of Akaike Information Criterion (AIC) and lowest Mean Absolute Percentage Error (MAPE) was proposed. The results showed that the ARIMA model for tea productivity provided the lowest MAPE; meanwhile, for cardamom it provided the lowest AIC. Also, forecasting models for natural rubber ribbed smoked sheets No.3 (RSS3) in the agricultural futures market of Thailand and the effect on the rubber futures price were presented by Suppanunta (2009). The best model was selected by the minimum of least mean squared error.

The characteristic in the time series of practical observations often involves a long memory process (see Liubov and Wolfgang (2016)). The ARFIMA process presented by Granger and Joyeux (1980) and Hosking (1981) was an initial example of a long memory stationary process. This process plays an important role in prediction and estimation, especially in quality control. In construction of the ARFIMA model, the estimation of the fractional differencing parameter (d) is very important. The maximum likelihood estimator method (see Palma and Chan (1997)), parametric approaches (see Barbara P. Olbermann et al. (2006)), the GPH estimator method (see Geweke and Porter-Hudak (1983)), and spectral regression (see Barkoulas and Baum (1997)) were used to estimate d value. Additionally, there has been some research on the application of the ARFIMA model to practical observations. ARFIMA models in forecasting the Air Pollution Index (API) in Shah Alam, Selangor, Malaysia were used by Lim Ying Siew et al. (2008). Moreover, a comparison between ARFIMA and AR models for unemployment rates in Japan was undertaken by Kurita (2010).

Similarly, the EWMA control chart has been applied to practical observations. The EWMA control chart for monitoring air quality of the Ambient Ozone Levels of urban and industrial areas in Muscat was applied by Muhammad Idrees Ahmad (2015). The observations for monitoring were weekly 8-hour maximum concentrations of Carbon Monoxide (CO) in both areas. The results showed that the observations followed the AR model and the air quality around both areas was at the international standard limit. The SPC based on the capacity index for fulfillment of the gap between ISO 1400 and Total Quality Environmental Management (TQEM) was used to measure and evaluate the environmental performance by Charles and Jeh-Nan (2002). The EWMA control chart, as one of the effective quality control tools for reducing the variability of production process, was applied to bottle manufacturing observations by Saravanan and Nagarajan (2013). The control chart for auto-correlated observations, which was long-memory air quality data from Taiwan, was used Jeh-Nan Pan and Su-Tsu Chen (2008). The practical observations were modeled as the ARFIMA and ARIMA process. The comparison between both models showed that the ARFIMA model was more appropriate than the ARIMA model. A quality control method for analysis and evaluation of observations on dissolved heavy metals in water in Peninsular Malaysia was studied by Fawaz et al. (2016). Furthermore, the important point for the EWMA control chart is that the optimal smoothing parameter (λ) is considered and estimated. The optimal λ value leads to an effective EWMA control chart, so some researchers have studied the optimal λ value and effective control chart in several applications. Polunchenko et al. (2014) used the optimal λ value and initial value for the EWMA control chart and compared the design with two optimal methods based on the following criteria: Pollak's minimax and Shiryaev's multi-cyclic setup. The results showed that the conventional optimal λ value of the EWMA control chart could be a competitive method. The optimal λ value of the EWMA statistic and the width factor of the control limits was evaluated by Petar and Sanja (2011). Then, the optimal EWMA control chart was applied to monitor the rate of occurrences of intrusion events on computer network traffic. A comparison between the optimal EWMA control chart and the CUSUM control chart as measured by the performances of both control charts based on the standard deviation of the run length (SDRL) was proposed by Lee et al. (2013). The results showed that the simulation observations were applied to the control chart. When out-of-control, the optimal EWMA control chart had a better performance than the optimal CUSUM control chart. However, when in-control, the optimal CUSUM control chart performed better. Therefore, an optimal quality control method results in an efficient control chart.

The efficiency of a control chart for sensitive detection of small changes or shifts is measured by Average Run Length (ARL) which is the expected value of the number of in-control observations before the control chart finds the signal out of control. There are two types of ARL: ARL_0 for an in-control state and ARL_1 for an out-of control state. The EWMA control chart for detecting changes in the mean of a long memory process, which is assumed the ARFIMA (1, *d*, 1) model and extended also to the modified EWMA control chart, was employed by Liubov and Wolfgang (2016). The results make a measurement of the effectiveness as ARL. The ARL expression on the EWMA statistic for AR(1) observations with error following exponential white noise process was derived by Suriyakat et al. (2012). The computational method integral equation for ARL compared with the derived ARL was demonstrated.

However, error or noise occurs during the working process, so the collected practical observations from the process kept the noise in the observations. When the model from such observations is constructed, the noise is an important factor. Commonly, noise follows normal white noise, but noise sometimes follows exponential white noise. Some researchers have studied exponential white noise, and proposed methods for testing white noise and exponential distribution as follows: first-order Autoregressive model white error following exponential white noise was tested by Hocine Fellag (2001). The size of the test is concentrated when the exponential white error occurs by varying the size of the test. The test is sensitive to the varied size of the test. Statistics for testing normal and exponential distributions estimated by Monte Carlo simulations of reliability data were presented by Duan and Chen (1983). Test statistics: Kolmogorov-Smirnov, Cramer-von Mises, and Anderson-Darling, for testing exponential data were presented by Diane et al. (2008). The parameter of exponential distribution was estimated by the maximum likelihood method. Furthermore, a simpler testing statistic for the white noise process was constructed by Lobato and Velasco (2004). Practical observations with white noise, which forms the ARMA process, was applied to the test for residual white noise.

As mentioned in the above literature review, the estimation of *d* and λ value are important in modelling. Therefore, in this research, the ARL with optimal *d* and λ value with applications to practical observations following the long memory ARFIMA process is proposed. The rest of this paper is organized as follows. In the next section, the definition and fundamental knowledge about control charts and the ARFIMA process are given. The estimation in λ value of the EWMA statistic, the *d* value of the ARFIMA model, and exponential white noise testing of the error are illustrated in Section 3. The application and comparison of the ARL are contained in Section 4. Section 5 is the conclusion and discussion of this research.

2. Fundamental Definitions and Preliminaries about Control Charts and the ARFIMA Long Memory Process

In this section, some fundamental concepts and knowledge that are essential for dealing with control charts and parameter estimation are shown as follows:

2.1. Mathematical definitions about white noise process, likelihood function, and EWMA and CUSUM control chart

Definition 1 (White Noise Process) *The process* $X_t\,; t \in (0,T]$ *is called white noise if* $\mathrm{E}(X_t) = \mu$ *and correlation function,* $\mathrm{E}(X_t X_{t+\tau}) = \delta(\tau)$ *where* $\delta(\tau)$ *is Dirac delta function defined as*

$$\delta(\tau) = \begin{cases} \infty; & \tau = 0 \\ 0; & \tau \neq 0 \end{cases}.$$

That is to say, the white noise process is the process with a constant mean and variance of independent identically random variables.

Definition 2 (Likelihood Function) *Let independent identically random samples* $x_i\,; i = 1,2,3,...,n$ *from the probability density function (pdf),* $f(x_i)$. *Joint density function of* x_i, $L(\theta) = f(x_1, x_2, ..., x_n; \theta)$ *which is called likelihood function.*

Definition 3 (EWMA Statistic) *The EWMA statistic at time* t *is defined as*

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t \quad ; t = 1,2,3,...,n \tag{1}$$

where Z_t *is an EWMA statistic starting at the process mean,* $Z_0 = \mu$ *and* X_t *is generated from ARFIMA process.*

Definition 4 (EWMA Control Chart) *The control limits of EWMA chart (see Areepong and Sukparungsee (2010)) with a smoothing parameter* λ, *a width* K *of the control limit, a process mean* μ, *a standard deviation process* σ, *and process variance* σ^2 *consist of*:

Upper control limit: $UCL = \mu + K\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2t}\right]}$,

Center Line: $CL = \mu$,

Lower control limit: $LCL = \mu - K\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2t}\right]}$.

An optimal smoothing parameter is a smoothing parameter λ *with the minimum sum of squared error (SSE) between the EWMA statistic and observations. Furthermore, the smoothing parameter* λ *is sometimes called* λ *value which desires to be optimal.*

Definition 5 (CUSUM Control Chart) *The Cumulative Sum (CUSUM) control limits are recursively defined as*

$$C_t = \max(C_{t-1} + X_t - r,\, 0), \;\; t = 1,2,\ldots,\; C_0 = u \tag{2}$$

where C_t *is the CUSUM statistic,* r *is a positive constant reference value,* u *is an initial value and* X_t *is generated from ARFIMA process. Additionally, the reference value (*r*) is the mean-shift detection constant which equals to the average between the acceptable quality level and unacceptable quality level for the CUSUM control chart.*

Definition 6 (Absolute Percentage Relative Error) *Absolute percentage relative error (APRE) of ARL is defined as*

$$APRE = \frac{\left|L(u) - \tilde{L}(u)\right|}{L(u)} \times 100\% \tag{3}$$

where $L(u)$ *is analytical ARL and* $\tilde{L}(u)$ *is numerical ARL.*

2.2. ARFIMA long memory process with exponential white noise

In time series, the characteristic of some observations in practical applications obviously involves the long memory process with autocorrelation of square error such that, $\rho(k) \sim ck^{2d-1}$ as $k \to \infty,\ c \neq 0,$ decays slowly to zero where $0 < d < \frac{1}{2}$ (see Liubov and Wolfgang (2016)). This characteristic plays an important role in prediction and estimation, especially in quality control.

The process X_t ; $t = 1,2,3,...,n$ with the process mean μ of n observations, and the initial value $X_0 = \mu,$ given the notation as ARFIMA(*p*, *d*, *q*) (see Liubov and Wolfgang (2016)) such that $0 < d < \frac{1}{2}$ called long-memory, or long-term dependence ARFIMA process with exponential white noise is defined as,

$$\phi(B)\nabla^d X_t = \mu + \theta(B)\varepsilon_t \tag{4}$$

where $\varepsilon_t \sim \exp(\beta)$ is an exponential white noise or error with $\beta > 0$ which is called *shift parameter* in the quality control, or *rate parameter* of the exponential distribution in the general. i.e., $f(\varepsilon) = \frac{1}{\beta} e^{-\frac{\varepsilon}{\beta}}.$

In Equation (4), B is a backward shift operator such that $B^k \varepsilon_t = \varepsilon_{t-k}$ with k^{th} order, $\phi(B) = (1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p),\ \theta(B) = (1 - \theta_1 B - \theta_2 B^2 - ... - \theta_q B^q)$, and

$$\nabla^d = (1-B)^d = \sum_{i=0}^{\infty} \binom{d}{i} (-B)^i = 1 - dB - \frac{1}{2!} d(1-d)B^2 - \frac{1}{3!} d(1-d)(2-\mathrm{d})B^3 - ...$$

where $\binom{d}{i} = \frac{\Gamma(d+1)}{\Gamma(d-i+1)\Gamma(i+1)};$ $\Gamma(\cdot)$ is gamma function.

3. Method for Estimation of Parameters in the EWMA Control Chart and Long Memory ARFIMA Process

3.1. Estimation smoothing parameter and differencing parameter

In this section, the estimation of important parameters, λ and *d*, is proposed as follows:

The method for estimation of λ value in the EWMA statistic is the minimum sum of squared error (SSE) (see Petar and Sanja (2011)). Let S_t be smoothing value by virtue of the EWMA statistic as in Equation (1) with initial value $S_2 = x_1$. Thus,

$$S_t = \lambda x_{t-1} + (1-\lambda) S_{t-1};\quad \lambda \in (0,1],\ t \geq 3.$$

By the iterative approach for the sum of squared errors (SSE): $\sum_{t=3}^{n}(S_{t-1}-x_{t-1})^2$ as shown for the smoothing scheme in Table 1, the desired SSE is a minimization of the SSE of the estimated λ value of the EWMA statistic.

Table 1 Exponential smoothing scheme by sum of squared error

Observation (x_t)	S_t	Error ($(S_t - x_t)$)	Squared Error
x_1	-	-	-
x_2	x_1	E_2	$(E_2)^2$
x_3	S_3	E_3	$(E_3)^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	S_n	E_n	$(E_n)^2$
$SSE = \sum_{t=3}^{n}(S_{t-1}-x_{t-1})^2$			

The optimal d value and the other parameters of ARFIMA(*p, d, q*) model as in Equation (4) was identified and calculated by the method of Haslett and Raftery (1989) which is based on maximum likelihood estimation for the parameter of ARIMA model. The fracdiff package in R program is applied to calculate the optimal *d* value (see Chris and Washington (2003)).

3.2. Rate parameter estimation and exponential white noise testing

In this section, the maximum likelihood estimation method for the rate parameter is carried out and exponential white noise process testing are follows.

Let $X_i;\, i = 1,2,3,...,n$ be random variables with an identically independent distribution (i.i.d.) from the exponential distribution of probability density function (pdf) $f_{X_i}(x_i;\beta) = \frac{1}{\beta}e^{-\frac{x}{\beta}};\, x > 0.$ The likelihood function for random samples $x_i;\, i = 1,2,3,...,n$ from exponential function, $L(\beta) = \beta^n e^{-\beta\sum_{i=1}^{n}x_i}$. Thus, the maximum likelihood estimator of the exponential random variables is $\hat{\beta} = \bar{x}$.

The exponential white noise process is the process that follows exponential distribution and white noise process.

In this research, the white noise process is tested by the concept of the Lobato-Velasco white noise test (see Lobato and Velasco (2004)). The null hypothesis is that the observations are white noise. The testing method based on a two-tail test against the normal distribution with mean 0 and variance 4 is computed with the Cramer von Mises (CVM) test statistic. The *normwhn.test* package including *whitenoise.test()* function in R program is applied to test the white noise process (see Peter Wickham (2012)).

The statistic for testing the hypothesis is D (see Diane et al. (2008)). The maximum value of absolute deviation (D) between the cumulative distribution $(F_n(x))$ of n observations normalized to uniform(0,1) and the theoretical cumulative distribution $(\hat{F}(x))$ is tested. i.e.,

$$D_n = \sup_x \{| F_n(x) - \hat{F}(x) |\}.$$

The *stats* package in R program including *ks.test ()* function was applied to calculate the statistic for testing the distribution of the observations (see R Core Team (2016)).

4. Application

This section focuses on the application in practical observations. The practical observations were obtained from a published research paper entitled "The time intervals between industrial accidents" (see Maguire et al. (1952)). The data are the time intervals in days between explosions in mines in Great Britain, in which more than 10 men were killed. The time period was from 6 December 1875 to 29 May 1951. Moreover, the process mean was 129 days per explosion and 339 days per explosion when the process mean changed. The ARFIMA (p, d, q) model and parameter estimation based on the practical observations fitted from the process and application to evaluate the ARL are given as follows.

4.1. The results of parameters estimation and ARFIMA modelling of a practical investigation

The parameters d and λ value are estimated for practical observations. The results and the interpretation of the model and testing are as follows.

Table 2 shows the parameters estimation for construction of the ARFIMA(*p, d, q*) model. The optimal d value estimated by the method of Haslett and Raftery (1989) is 0.1118246, which indicates that the time intervals in days between explosions in mines have a long memory time series property. The estimation of the coefficients of the ARFIMA(*p, d, q*) model are as follows: $\phi_1 = -0.8383, \phi_2 = -0.2517,\ \ \theta_1 = 0.1748, \theta_2 = -0.7089$. Thus, ARFIMA(*p, d, q*) model about the zero process mean for this practical observations as,

$$\begin{aligned}\text{ARFIMA}(2,\, 0.1118246,\, 2) &= 0.007867022529994364 x_{t-5} + 0.03870093378102443 x_{t-4} \\ &+ 0.10103172301452609 x_{t-3} - 0.10829750840258 x_{t-2} - 0.7264754 x_{t-1} + 0.7089 \hat{\varepsilon}_{t-2} \\ &+ 0.1748 \hat{\varepsilon}_{t-1} + \hat{\varepsilon}_t.\end{aligned}$$

Table 2 ARFIMA(*p, d, q*) model for the time intervals in days between explosions in mines in Great Britain during 1875 to 1951

Fractional differencing parameter = 0.1118246				
Coefficients:	ar1	ar2	ma1	ma2
	-0.8383	-0.2517	0.1748	-0.7089
s.e.	0.1619	0.1074	0.1340	0.1299
σ^2 estimated as 91518: log likelihood= -769.42				
AIC=1548.84 AICc=1549.43 BIC=1562.26				

Table 3 shows the results of testing the exponential white noise of the error between actual observations and the ARFIMA(2, 0.1118246, 2) model along with the rate parameter estimation based on the maximum likelihood estimation for exponential distribution. The hypotheses for the test consist of two parts as follows:

1) Testing based on the Kolmogorov-Smirnov test of error for exponential distribution,

Null hypothesis: The errors follow an exponential distribution.

versus Alternative hypothesis: The errors do not follow an exponential distribution.

p-value = 0.3734 $\geq$ 0.05, this leads to accepting the null hypothesis.

2) Testing based on the concept of the Lobato-Velasco method of error for a white noise process,
Null hypothesis: The errors follow the white noise process.
versus Alternative hypothesis: The errors do not follow the white noise process.
p-value = 0.5792938 $\geq$ 0.05, this leads to accepting the null hypothesis.

Therefore, both hypothesis testing shows that the errors follow exponential white noise with the rate parameter, 0.004051299.

Table 3 Testing of error for exponential white noise

Testing for exponential distribution of error: alternative hypothesis: two-sided	
One-sample Kolmogorov-Smirnov test	data: error
D = 0.089649	p-value = 0.3734
Testing based on Lobato-Velasco method for white noise of error: p- value = 0.5792938	
Rate parameter for exponential distribution of error : 0.004051299	

Next, the method for the minimum of the sum of squared error for optimal λ value is demonstrated as follows:

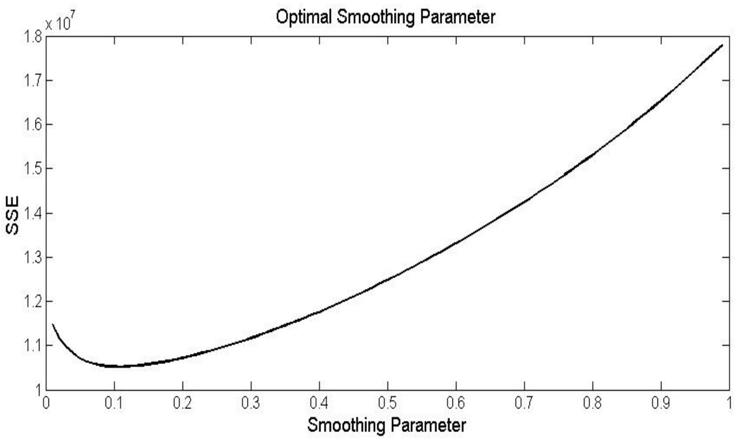

Figure 1 Optimal λ value of EWMA statistic

For the EWMA control chart based on practical observations, Figure 1 shows the optimal λ value at 0.11 and returns the minimum of SSE at 10519360.07.

4.2. ARL based on optimal parameters in a practical investigation

In this research, only the upper side of the EWMA control chart is considered, i.e. the central line is 0 and the UCL is $b > 0$.

The ARL consisting of analytical EWMA ARL notated by $L_E(u)$ and numerical EWMA ARL notated by $\tilde{L}_E(u)$ for the EWMA control chart of practical observations can be derived by the concept of Champ and Rigdon (1991) as follows:

With initial $Z_0 = u$ of the EWMA statistic around the zero process mean, the EWMA ARL denoted by $L(u)$ can be derived from a linear Fredholm integral equation of the second kind (see Kharab and Guenther (2012)) as

$$L(u)=1+\int_{\frac{-\lambda X_t-(1-\lambda)u}{\lambda}}^{\frac{b-\lambda X_t-(1-\lambda)u}{\lambda}} L(\lambda w+(1-\lambda)u+\lambda X_t)f\left(w\right)dw, \tag{5}$$

where X_t is long memory ARFIMA(*p, d, q*) process.

The analytical ARL of EWMA control chart for ARFIMA process can be carried out as follows. In the case of practical observations of the time intervals in days between explosions in mines in Great Britain, the ARFIMA(*p, d, q*) process is ARFIMA(2, 0.1118246, 2). Namely,

$$\begin{aligned}X_t = {} & 0.007867022529994364x_{t-5}+0.03870093378102443x_{t-4}+0.10103172301452609x_{t-3}\\ & -0.10829750840258x_{t-2}-0.7264754x_{t-1}+0.7089\hat{\varepsilon}_{t-2}+0.1748\hat{\varepsilon}_{t-1}+\hat{\varepsilon}_t.\end{aligned}$$

Thus, analytical EWMA ARL can be carried out as,

$$\begin{aligned}L_E(u)=1+\frac{1}{\lambda}\int_0^b L_E(y)f[\frac{y-(1-\lambda)u}{\lambda} & -\{0.007867022529994364x_{t-5}\\ & +0.03870093378102443x_{t-4}+0.10103172301452609x_{t-3}\\ & -0.10829750840258x_{t-2}-0.7264754x_{t-1}+0.7089\hat{\varepsilon}_{t-2}+0.1748\hat{\varepsilon}_{t-1}+\hat{\varepsilon}_t\}]dy\end{aligned}$$

Let $s(u)=\exp\left\{\begin{array}{l}\frac{(1-\lambda)u}{\lambda\beta}+\frac{1}{\beta}[0.007867022529994364x_{t-5}+0.03870093378102443x_{t-4}\\ +0.10103172301452609x_{t-3}-0.10829750840258x_{t-2}-0.7264754x_{t-1}\\ +0.7089\hat{\varepsilon}_{t-2}+0.1748\hat{\varepsilon}_{t-1}+\hat{\varepsilon}_t]\end{array}\right\}$

and $w=\int_0^b L_E(y)\exp(-\frac{y}{\lambda\beta})dy$, so $L_E(u) = 1+\frac{s(u)}{\lambda\beta}w$,

$$w=\frac{\lambda\beta[1-\exp(-\frac{b}{\lambda\beta})]}{\left\{\begin{array}{l}1-\frac{1}{\lambda}[1-\exp(-\frac{b}{\beta})\exp(\frac{1}{\beta}[0.007867022529994364x_{t-5}\\ +0.03870093378102443x_{t-4}+0.10103172301452609x_{t-3}\\ -0.10829750840258x_{t-2}-0.7264754x_{t-1}+0.7089\hat{\varepsilon}_{t-2}\\ +0.1748\hat{\varepsilon}_{t-1}+\hat{\varepsilon}_t])\end{array}\right\}}.$$

Let $Nom(\beta)=\lambda\{1-\exp(-\frac{b}{\lambda\beta})\}\exp[\frac{(1-\lambda)u}{\lambda\beta}]$ and

$$Denom(\beta)=(1-\exp[-\frac{b}{\beta}])-\lambda\exp[-\frac{1}{\beta}\left\{\begin{array}{l}0.007867022529994364x_{t-5}\\ +0.03870093378102443x_{t-4}\\ +0.10103172301452609x_{t-3}\\ -0.10829750840258x_{t-2}\\ -0.7264754x_{t-1}+0.7089\hat{\varepsilon}_{t-2}\\ +0.1748\hat{\varepsilon}_{t-1}+\hat{\varepsilon}_t\end{array}\right\}].$$

Therefore,

$$L_E(u)=1-\frac{Nom(\beta)}{Denom(\beta)} \tag{6}$$

In Equation (6), the shift parameter is determined to $\beta = \beta_0$ in case the observations are in an in-control state and determined as $\beta = \beta_1 = \beta_0(1+\delta)$ in case the observations are in an out-of-control state with shift size (δ).

In order to compare the results of the EWMA ARL to CUSUM ARL, the analytical ARL of CUSUM control chart denoted by $L_C(u)$ with the first time outside the upper control limit (h) or stopping time ($\mathrm{T}_b = \inf\{t > 0;\, C_t > h\},\ u < h$) can defined as $L_C(u) = \mathbb{E}_u(\tau_b) < \infty$ (see Mititelu et al. (2010)),

$$L_C(u) = 1 + \frac{1}{\beta} e^{\frac{1}{\beta}(u-r+X_t)} \int_0^h L_C(y) e^{-\frac{1}{\beta}y} dy + \left[1 - e^{-\frac{1}{\beta}(r-u-X_t)} \right] L_C(0),\ u \in [0, r]. \tag{7}$$

Numerical ARL based on by the Composite Midpoint Rule (see e.g. Matheus and Dmitry (2008)) for Equation (5) with each m nodes weighted by $w_j = \frac{b}{m}$ and the interval $[0, b]$ is divided into a partition $0 = a_1 \le a_2 \le .... \le a_m = b,$ can be carried out as,

$$\widetilde{L}_E(u) \approx 1 + \frac{1}{\lambda} \sum_{j=1}^{m} w_j \widetilde{L_E}(a_j) f\left(\frac{a_j - (1-\lambda)u}{\lambda} - X_t \right). \tag{8}$$

Obviously, Equation (8) is attained by solving the system of linear eqations, which is the method for solving a Fredholm integral Equation (5) of the second kind.

The comparison of the efficiency of the EWMA and CUSUM control chart is measured by ARL, which compares the differences between the analytical and numerical ARL by APRE as follows:

Table 4 represents the numerical values of the analytical and numerical ARL starting at $\mathrm{ARL}_0 = 370$ with $m = 1{,}000$ nodes and $u = 1$ of practical observations, which follow the long memory ARFIMA(2, 0.1118246, 2) process and the APRE of the difference between the analytical and numerical EWMA ARL. Also, the numerical values of analytical CUSUM ARL for comparison with analytical and numerical EWMA ARL are shown in the last column.

From Table 4, when the shift size increases, there is a decrease in all analytical and numerical ARL. The numerical EWMA ARL is close to the analytical EWMA ARL in all different shift size levels. The APRE decreases in the same direction as the analytical and numerical EWMA ARL. Regarding the comparison between the analytical EWMA ARL and analytical CUSUM ARL, the results showed that the numerical values of EWMA ARL are less than the numerical values of CUSUM ARL at the same level of shift. That is to say, the EWMA control chart can detect a shift in the process mean faster than the CUSUM control chart. Turning to CPU time representing the numerical values in parentheses, the CPU time of EWMA numerical ARL increases when its ARL decreases; meanwhile, the CPU time of all analytical ARLs is steady between 0.014 to 0.015 minutes. On the whole, this results showed that the analytical method for EWMA ARL can detect a shift of the process mean better than the analytical method for the CUSUM control chart in the case of a long memory ARFIMA(2, 0.1118246, 2) process in a practical observation. Therefore, the analytical EWMA ARL is an alternative to measure the efficiency of the EWMA control chart because it is easier to calculate than the numerical method, which takes a lot of time.

Table 4 Comparison of the numerical values among the EWMA ARL with $b = 0.00488646$ and analytical CUSUM ARL with $r = 3$, $h = 4.24$ on ARFIMA (2, 0.1118246, 2) process

Shift size (δ) with $\beta_0 = 1$	EWMA Analytical ARL $L_E(u)$	EWMA Numerical ARL $\widetilde{L}_E(u)$	APRE (%) of EWMA	CUSUM Analytical ARL $L_C(u)$
0.00	369.99997774471956 (0.014)	369.9999777143371 (43.493)	8.211E-09	370.091 (0.015)
0.01	334.7798452043425 (0.015)	334.7798451774018 (87.141)	8.047E-09	345.497 (0.015)
0.03	275.6496446712831 (0.014)	275.64964464996905 (173.831)	8.171E+01	302.324 (0.015)
0.05	228.61973992205526 (0.014)	228.61973990505882 (217.683)	7.435E-09	265.915 (0.015)
0.10	147.48176267465314 (0.015)	147.48176266468786 (261.878)	6.757E-09	197.011 (0.015)
0.30	35.79639538632715 (0.014)	35.79639538463283 (304.731)	4.733E-09	75.533 (0.015)
0.40	20.640872122458013 (0.015)	20.640872121633492 (347.756)	3.995E-09	52.236 (0.015)
0.50	12.913962671660649 (0.014)	12.913962671225006 (390.984)	3.373E-09	38.156 (0.015)

5. Conclusion and Discussion

In this research, a method for estimating the optimal λ value of the EWMA control chart and d value for the long memory AFRIMA (*p, d, q*) process with applications to practical observations was proposed. Also, the long memory ARFIMA process with exponential white noise was estimated and tested by the maximum likelihood estimation method and the Lobato-Velasco white noise test. It was found that the optimal long memory AFRIMA (2, 0.1118246, 2) process was a suitable process for practical observations of time intervals in days between explosions in mines in Great Britain during 1875 to 1951. The main aim of this research is to apply the long memory ARFIMA process in a practical observation to measure the ARL for identifying the efficiency of a control chart in real situation quality control. The ARLs, analytical EWMA ARL, numerical EWMA ARL, and analytical CUSUM ARL as the shift sizes vary in different levels are shown and compared among all ARLs based on APRE. The results showed that the analytical method is the most effective method for comparing the numerical method in case of evaluating EWMA ARL. Furthermore, the analytical EWMA ARL was compared with the analytical CUSUM ARL. The results showed that the EWMA control chart is an effective control chart for comparing with the CUSUM chart based on analytical ARL criteria because the numerical values of the analytical EWMA ARL are less than the analytical CUSUM ARL. The results will help quality controllers chose appropriate control charts for practical observations. Also, choosing appropriate control charts will lessen computational and task time as well as result in more precise computation.

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