Thailand Statistician
January 2019; 17(1): 1-15
http://statassoc.or.th
Contributed paper

Some Properties and Interval Estimation Based on Maximum Likelihood Estimators of the Crack Lifetime Distribution

Monthira Duangsaphon*
Department of Mathematics and Statistics, Faculty of Science and Technology,
Thammasat University, Pathum Thani 12121, Thailand.
*Corresponding author; e-mail: monthira.stat@gmail.com

Received: 13 August 2017
Revised: 9 February 2018
Accepted: 18 February 2018

Abstract

The crack lifetime distribution is used for modeling lifetime data. This paper proposes some properties of the crack lifetime distribution. The confidence intervals of parameters based on the maximum likelihood estimators (MLEs) for crack lifetime distribution are considered. The Fisher information matrix is provided for constructing the asymptotic confidence interval. Moreover, the confidence intervals are computed using the two parametric bootstrap methods: the bootstrap-p and the bootstrap-t methods. Monte Carlo simulations are performed to investigate the performance of the three different interval estimation methods in terms of coverage probabilities and average width. Finally, a real data set is analyzed for illustrative purposes. Results indicate that the asymptotic confidence intervals behave very well for moderate and large sample sizes, while the bootstrap-p intervals work generally well. Moreover, the bootstrap-t intervals also perform quite satisfactorily for small sample sizes.

Keywords: Asymptotic distribution, bootstrap confidence interval, coverage probability, average width, maximum likelihood estimation.

1. Introduction

Lifetime distribution gives a benefit and applied information that motivates practitioners to prevent damages of machines, systems, manufacturing, industry and finance that occur after the lifetime is terminated and such that they might occur also accidentally. The engineering interpretation of a crack random variable is the time after a crack has started to develop in a machine element because of a cyclic or non-cyclic loading until the crack achieves the critical value. At the beginning, there may be a small crack in the machine, but the element could still work. However, when it achieves the critical point, its tolerance is exceeded and the element does not work anymore.

The three-parameter crack lifetime distribution was first introduced by Bowonrattanaset and Budsaba (2011). This distribution is performed by adding weighted parameters and combining the inverse Gaussian distribution and length biased inverse Gaussian distribution. The probability density function of the crack lifetime distribution is given by:

$$f_X(x)=\begin{cases}\dfrac{1}{\theta\sqrt{2\pi}}\left[p\lambda\left(\dfrac{\theta}{x}\right)^{\frac{3}{2}}+(1-p)\left(\dfrac{\theta}{x}\right)^{\frac{1}{2}}\right]\exp\left[-\dfrac{1}{2}\left(\sqrt{\dfrac{x}{\theta}}-\lambda\sqrt{\dfrac{\theta}{x}}\right)^2\right] & ,x>0 \\ 0 & ,\text{otherwise}\end{cases} \quad (1)$$

which we will denote by $X\sim CR(\lambda,\theta,p)$, where $\lambda>0,\theta>0$ and $0\le p\le 1$ are parameters corresponding to the thickness of a machine element, the nominal treatment pressure on a machine element, and the weighted parameter, respectively. Thus, the crack lifetime distribution contains three known distributions as special cases: the Birnbaum-Saunders distribution, where $p=1/2$; the inverse Gaussian distribution, where $p=1$; and the length biased inverse Gaussian distribution, where $p=0$. Also, Bowonrattanaset and Budsaba (2011) established the properties of this distribution, which are the characteristic function, the moment generating function and the cumulative distribution function of the crack lifetime distribution.

For a random sample $X_1,X_2,...,X_n$ from $CR(\lambda,\theta,p)$, the MLEs of parameters λ,θ and p were proposed by Duangsaphon et al. (2016). These MLEs cannot be computed in closed form; however, one can use the Newton-Raphson method, which is the direct approach for obtaining the MLEs by maximizing the likelihood function. This method involves the calculation of the first and second derivatives of the log of the likelihood function with respect to parameters. This method works well in general; unfortunately, it fails to converge in some cases under this setup. The Newton-Raphson method was applied by using the default function of R software, called the "nlminb" function. Also, simulation results of the MLEs for large sample sizes performed very well. Unfortunately, the estimator of p was far from the true value for small and moderate sample sizes. On the other hand, when p was assumed to be known, simulation results of the two MLEs worked very well for almost all situations. Moreover, the Bayesian estimation was proposed, based on the Lindley's approximation and the Gibbs sampling procedure, and the performances of Bayes estimators and MLEs were also compared. The Bayes estimators behave much better than the MLEs.

One of the disadvantages of Bayesian inferences is that there is no correct way to select a prior distribution, which means we must generate misleading results. Often, the results produced by the maximum likelihood estimation are very similar to the results produced by the Bayesian estimation when the sample size is sufficiently large. Thus, the maximum likelihood estimation is a very popular estimation for parametric models. Interval estimates are often more useful than point estimates. The interval estimation for lifetime distributions based on the MLEs are also popular, since the MLEs are usually quite satisfactory. Some of the recent references for the confidence interval estimation of parameters based on asymptotic distributions are Miller (1981), Ng et al. (2006), Gamsmi and Berzig (2011), Panahi and Asadi (2011), and Kohansal and Rezakhah (2012). Moreover, one may refer to Efron (1982) and Hall (1988), who first suggested the bootstrap-p and bootstrap-t methods. While the bootstrap-p method is the most widely used method, the bootstrap-t method is the most efficient method theoretically. Furthermore, Pandey and Bandyopadhyay (2012) proposed the two bootstrap methods for interval estimation; the bootstrap-p and the bootstrap-t methods of inverse Gaussian distribution. The question of interval estimation and other interesting properties of the crack lifetime distribution remain to be considered.

The aim of this paper is to propose some properties of the crack lifetime distribution. We provide the confidence interval estimation of the parameters based on the maximum likelihood estimators by constructing from the asymptotic distribution and the two parametric bootstrap methods mentioned earlier. Moreover, we also compare the performances of the suggest methods.

The rest of the paper is arranged as follows. In Section 2, we propose some properties of the crack distribution. The confidence interval estimation by constructing from the asymptotic distribution and the two parametric bootstrap methods are suggested in Section 3. Numerical results from simulation studies are presented in Section 4. In Section 5, we illustrate some numerical examples of all the methods for inference developed here. Finally, we conclude the paper in Section 6.

2. Some Properties of the Three-Parameter Crack Lifetime Distribution

In this section, we prove some new properties of the three-parameter crack lifetime distribution. Assume that X has $CR(\lambda,\theta,p)$ distribution. Clearly cX, with $c>0$, this is the same distribution as X, replacing θ with $c\theta$. Moreover, the distribution of X^{-1} is the three-parameter crack lifetime distribution, with λ, $(\lambda^2\theta)^{-1}$ and $1-p$ parameters. Additionally, we find the distribution of $X+Y$ where $X\sim CR(\lambda_1,\theta,p)$ and $Y\sim IG(\lambda_2,\theta)$ which are independent random variables. The distribution of $X+Y$ is the three- parameter crack lifetime distribution, with $\lambda_1+\lambda_2,\theta$ and p parameters. In the following theorems it is shown that $cX\sim CR(\lambda,c\theta,p)$, $X^{-1}\sim CR(\lambda,(\lambda^2\theta)^{-1},1-p)$ and $X+Y\sim CR(\lambda_1+\lambda_2,\theta,p)$.

Theorem 1 *Let* $X\sim CR(\lambda,\theta,p)$. *Then,*

i) For $c>0$, $cX\sim CR(\lambda,c\theta,p)$,

ii) $X^{-1}\sim CR\left(\lambda,\left(\lambda^2\theta\right)^{-1},1-p\right)$.

Proof:

i) Let $c>0$ and $Y=cX$. Then $X=Y/c$. Therefore, the Jacobian is

$$|J|=\left|\frac{d}{dY}X\right|=\left|\frac{d}{dY}\frac{Y}{c}\right|=\frac{1}{c}. \tag{2}$$

Thus, the probability density function of Y is

$$\begin{aligned} f_Y(y)&=f_X\left(\frac{y}{c}\right)|J| \\ &=\frac{1}{\theta\sqrt{2\pi}}\left[p\lambda\left(\frac{c\theta}{y}\right)^{\frac{3}{2}}+(1-p)\left(\frac{c\theta}{y}\right)^{\frac{1}{2}}\right]\exp\left[-\frac{1}{2}\left(\sqrt{\frac{y}{c\theta}}-\lambda\sqrt{\frac{c\theta}{y}}\right)^2\right]\left(\frac{1}{c}\right) \qquad (3) \\ &=\frac{1}{c\theta\sqrt{2\pi}}\left[p\lambda\left(\frac{c\theta}{y}\right)^{\frac{3}{2}}+(1-p)\left(\frac{c\theta}{y}\right)^{\frac{1}{2}}\right]\exp\left[-\frac{1}{2}\left(\sqrt{\frac{y}{c\theta}}-\lambda\sqrt{\frac{c\theta}{y}}\right)^2\right]. \end{aligned}$$

Therefore, $cX\sim CR(\lambda,c\theta,p)$.

ii) Let $Y=X^{-1}$, then $x=y^{-1}$. Therefore, the Jacobian is

$$|J|=\left|\frac{d}{dy}x\right|=\left|\frac{d}{dy}\left(\frac{1}{y}\right)\right|=\left|-\frac{1}{y^2}\right|=\frac{1}{y^2}. \tag{4}$$

Thus, the probability density function of Y is

$$f_Y(y)=f_X\left(\frac{1}{y}\right)|J|$$

$$=\frac{1}{\theta\sqrt{2\pi}}\left[p\lambda\left(y\theta\right)^{\frac{3}{2}}+(1-p)\left(y\theta\right)^{\frac{1}{2}}\right]\exp\left[-\frac{1}{2}\left(\sqrt{\frac{1}{y\theta}}-\lambda\sqrt{y\theta}\right)^2\right]\left(\frac{1}{y^2}\right)$$

$$=\frac{1}{\sqrt{2\pi}}\left[p\lambda\theta^{\frac{1}{2}}\left(\frac{1}{y}\right)^{\frac{1}{2}}+(1-p)\theta^{-\frac{1}{2}}\left(\frac{1}{y}\right)^{\frac{3}{2}}\right]\exp\left[-\frac{1}{2}\left(\sqrt{y\lambda^2\theta}-\sqrt{\frac{1}{y\theta}}\right)^2\right]$$

$$=\frac{\lambda^2\theta}{\sqrt{2\pi}}\left[p\left(\frac{1}{\lambda\theta^{\frac{1}{2}}}\right)\left(\frac{1}{y}\right)^{\frac{1}{2}}+(1-p)\left(\frac{1}{\lambda^2\theta^{\frac{3}{2}}}\right)\left(\frac{1}{y}\right)^{\frac{3}{2}}\right]\times\exp\left[-\frac{1}{2}\left(\sqrt{\frac{y}{\left(\lambda^2\theta\right)^{-1}}}-\lambda\sqrt{\frac{\left(\lambda^2\theta\right)^{-1}}{y}}\right)^2\right]$$

$$=\frac{1}{\left(\lambda^2\theta\right)^{-1}\sqrt{2\pi}}\left[(1-p)\lambda\left(\frac{1}{\lambda^3\theta^{\frac{3}{2}}}\right)\left(\frac{1}{y}\right)^{\frac{3}{2}}+p\left(\frac{1}{\left(\lambda^2\theta\right)^{\frac{1}{2}}}\right)\left(\frac{1}{y}\right)^{\frac{1}{2}}\right]$$

$$\times\exp\left[-\frac{1}{2}\left(\sqrt{\frac{y}{\left(\lambda^2\theta\right)^{-1}}}-\lambda\sqrt{\frac{\left(\lambda^2\theta\right)^{-1}}{y}}\right)^2\right]$$

$$=\frac{1}{\left(\lambda^2\theta\right)^{-1}\sqrt{2\pi}}\left[(1-p)\lambda\left(\frac{1}{\left(\lambda^2\theta\right)^{\frac{3}{2}}}\right)\left(\frac{1}{y}\right)^{\frac{3}{2}}+p\left(\frac{1}{\left(\lambda^2\theta\right)^{\frac{1}{2}}}\right)\left(\frac{1}{y}\right)^{\frac{1}{2}}\right]$$

$$\times\exp\left[-\frac{1}{2}\left(\sqrt{\frac{y}{\left(\lambda^2\theta\right)^{-1}}}-\lambda\sqrt{\frac{\left(\lambda^2\theta\right)^{-1}}{y}}\right)^2\right]$$

$$=\frac{1}{\left(\lambda^2\theta\right)^{-1}\sqrt{2\pi}}\left[(1-p)\lambda\left(\frac{\left(\lambda^2\theta\right)^{-1}}{y}\right)^{\frac{3}{2}}+p\left(\frac{\left(\lambda^2\theta\right)^{-1}}{y}\right)^{\frac{1}{2}}\right]$$

$$\times\exp\left[-\frac{1}{2}\left(\sqrt{\frac{y}{\left(\lambda^2\theta\right)^{-1}}}-\lambda\sqrt{\frac{\left(\lambda^2\theta\right)^{-1}}{y}}\right)^2\right]. \tag{5}$$

Therefore, $X^{-1}\sim CR\left(\lambda,\left(\lambda^2\theta\right)^{-1},1-p\right)$.

Theorem 2 *Let* $X \sim CR(\lambda_1, \theta, p)$ *and* $Y \sim IG(\lambda_2, \theta)$ *be independent random variables. Then,* $X + Y \sim CR(\lambda_1 + \lambda_2, \theta, p).$

Proof:

The characteristic function of $X \sim CR(\lambda_1, \theta, p)$ is

$$\varphi_X(t) = \frac{e^{\lambda_1\left(1-\sqrt{1-2\theta ti}\right)}}{\sqrt{1-2\theta ti}}\left[1 - p\left(1-\sqrt{1-2\theta ti}\right)\right], \tag{6}$$

and it is defined for all $-\infty < t < \infty,$ (Bowonrattanaset and Budsaba 2011).

The inverse Gaussian distribution is a special case of the three-parameter crack lifetime distribution with $p = 1.$ Hence, the characteristic function of $Y \sim IG(\lambda_2, \theta)$ is

$$\varphi_Y(t) = \frac{e^{\lambda_2\left(1-\sqrt{1-2\theta ti}\right)}}{\sqrt{1-2\theta ti}}\left[\sqrt{1-2\theta ti}\right] = e^{\lambda_2\left(1-\sqrt{1-2\theta ti}\right)}. \tag{7}$$

Since X and Y are independent random variables, the characteristic function of $X + Y$ can be written as $\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t).$ Thus, the characteristic function of $X + Y$ is

$$\begin{aligned}\varphi_{X+Y}(t) &= \frac{e^{\lambda_1\left(1-\sqrt{1-2\theta ti}\right)}}{\sqrt{1-2\theta ti}}\left[1 - p\left(1-\sqrt{1-2\theta ti}\right)\right]\left(e^{\lambda_2\left(1-\sqrt{1-2\theta ti}\right)}\right) \\ &= \frac{e^{\lambda_1\left(1-\sqrt{1-2\theta ti}\right)+\lambda_2\left(1-\sqrt{1-2\theta ti}\right)}}{\sqrt{1-2\theta ti}}\left[1 - p\left(1-\sqrt{1-2\theta ti}\right)\right] \\ &= \frac{e^{(\lambda_1+\lambda_2)\left(1-\sqrt{1-2\theta ti}\right)}}{\sqrt{1-2\theta ti}}\left[1 - p\left(1-\sqrt{1-2\theta ti}\right)\right]. \end{aligned} \tag{8}$$

Therefore, $X + Y \sim CR(\lambda_1 + \lambda_2, \theta, p).$

3. Interval Estimation

In this section, we describe different methods of constructing confidence intervals for λ, θ and $p.$ The first method of construction is based on the asymptotic distribution of estimates $\hat{\lambda},\ \hat{\theta}$ and $\hat{p}$ obtained by the method of maximum likelihood (MLE). Other methods of construction are based on two parametric bootstrap methods: the percentile bootstrap-p and bootstrap-t methods.

Before describing these three methods, we provide the MLEs of λ, θ and $p.$ Let $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$ denote a random sample of size n from the crack lifetime distribution. The likelihood function is given by

$$L(\lambda, \theta, p \mid \underset{\sim}{x}) = (2\pi)^{-\frac{n}{2}}\theta^{-\frac{n}{2}}e^{n\lambda - \frac{1}{2\theta}\sum_{i=1}^{n}x_i - \frac{\lambda^2\theta}{2}\sum_{i=1}^{n}\frac{1}{x_i}}\prod_{i=1}^{n}\frac{1}{x_i^{\frac{1}{2}}}\left[1 - p + \frac{\lambda\theta p}{x_i}\right]. \tag{9}$$

Let $l(\lambda, \theta, p)$ be the log likelihood function

$$\begin{aligned} l(\lambda,\theta,p) &= \ln L(\lambda,\theta,p\,|\,\underset{\sim}{x}) \\ &= \ln\left[(2\pi)^{-\frac{n}{2}}\theta^{-\frac{n}{2}}e^{n\lambda-\frac{1}{2\theta}\sum_{i=1}^{n}x_i-\frac{\lambda^2\theta}{2}\sum_{i=1}^{n}\frac{1}{x_i}}\prod_{i=1}^{n}\frac{1}{x_i^{\frac{1}{2}}}\left[1-p+\frac{\lambda\theta p}{x_i}\right]\right] \\ &= -\frac{n}{2}\ln 2\pi-\frac{n}{2}\ln\theta+n\lambda-\frac{1}{2\theta}\sum_{i=1}^{n}x_i-\frac{\lambda^2\theta}{2}\sum_{i=1}^{n}\frac{1}{x_i} \\ &\quad+\sum_{i=1}^{n}\ln\frac{1}{x_i^{\frac{1}{2}}}\left[1-p+\frac{\lambda\theta p}{x_i}\right]. \end{aligned} \tag{10}$$

The MLEs $\hat{\lambda}$, $\hat{\theta}$ and $\hat{p}$ of λ, θ and p are obtained from a numerical maximization of the log likelihood function, as the solution of the algorithm presented in Duangsaphon et al. (2016).

3.1. Asymptotic confidence intervals

We cannot obtain $\hat{\lambda}$, $\hat{\theta}$ and $\hat{p}$ in closed form and hence it is impossible to derive the exact distributions of $\hat{\lambda}$, $\hat{\theta}$ and $\hat{p}$. In order to construct the asymptotic confidence intervals for parameters λ,θ and p using the MLEs, we need the asymptotic variance-covariance matrix of MLEs which can be obtained by inverting the observed Fisher information matrix.

Let $I(\lambda,\theta,p)$ be the observed Fisher information matrix of λ,θ and p. The elements of 3×3 matrix $I(\lambda,\theta,p)$ are the negative of second order derivatives of $l(\lambda,\theta,p)$ with respect to parameters. Hence, the observed information matrix is as follows:

$$I(\lambda,\theta,p)=\begin{pmatrix} I_{\lambda\lambda} & I_{\lambda\theta} & I_{\lambda p} \\ I_{\theta\lambda} & I_{\theta\theta} & I_{\theta p} \\ I_{p\lambda} & I_{p\theta} & I_{pp} \end{pmatrix}, \tag{11}$$

where

$$I_{\lambda\lambda}=-\frac{\partial^2 l(\lambda,\theta,p)}{\partial\lambda^2}=\theta\sum_{i=1}^{n}\frac{1}{x_i}+\theta^2p^2\sum_{i=1}^{n}\frac{1}{x_i^2\left(1-p+\frac{\lambda\theta p}{x_i}\right)^2}, \tag{12}$$

$$I_{\theta\theta}=-\frac{\partial^2 l(\lambda,\theta,p)}{\partial\theta^2}=-\frac{n}{2\theta^2}+\frac{1}{\theta^3}\sum_{i=1}^{n}x_i+\lambda^2p^2\sum_{i=1}^{n}\frac{1}{x_i^2\left(1-p+\frac{\lambda\theta p}{x_i}\right)^2}, \tag{13}$$

$$I_{pp}=-\frac{\partial^2 L(\lambda,\theta,p)}{\partial p^2}=\sum_{i=1}^{n}\frac{\left(-1+\frac{\lambda\theta}{x_i}\right)^2}{\left(1-p+\frac{\lambda\theta p}{x_i}\right)^2}, \tag{14}$$

$$
\begin{aligned}
I_{\lambda\theta} &= I_{\theta\lambda} \\
&= -\frac{\partial^2 L(\lambda,\theta,p)}{\partial\lambda\partial\theta} \\
&= \lambda\sum_{i=1}^{n}\frac{1}{x_i} + \lambda\theta p^2\sum_{i=1}^{n}\frac{1}{x_i^2\left(1-p+\dfrac{\lambda\theta p}{x_i}\right)^2} \\
&\quad - p\sum_{i=1}^{n}\frac{1}{x_i\left(1-p+\dfrac{\lambda\theta p}{x_i}\right)},
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
I_{p\theta} &= I_{\theta p} \\
&= -\frac{\partial^2 L(\lambda,\theta,p)}{\partial p\partial\theta} \\
&= \lambda p\sum_{i=1}^{n}\frac{\left(-1+\dfrac{\lambda\theta}{x_i}\right)}{x_i\left(1-p+\dfrac{\lambda\theta p}{x_i}\right)^2} - \lambda\sum_{i=1}^{n}\frac{1}{x_i\left(1-p+\dfrac{\lambda\theta p}{x_i}\right)},
\end{aligned} \tag{16}
$$

and

$$
\begin{aligned}
I_{p\lambda} &= I_{\lambda p} \\
&= -\frac{\partial^2 L(\lambda,\theta,p)}{\partial p\partial\lambda} \\
&= \theta p\sum_{i=1}^{n}\frac{\left(-1+\dfrac{\lambda\theta}{x_i}\right)}{x_i\left(1-p+\dfrac{\lambda\theta p}{x_i}\right)^2} - \theta\sum_{i=1}^{n}\frac{1}{x_i\left(1-p+\dfrac{\lambda\theta p}{x_i}\right)}.
\end{aligned} \tag{17}
$$

The variance–covariance matrix can be represented as

$$
\Sigma = \begin{pmatrix} \sigma_{\lambda\lambda} & \sigma_{\lambda\theta} & \sigma_{\lambda p} \\ \sigma_{\theta\lambda} & \sigma_{\theta\theta} & \sigma_{\theta p} \\ \sigma_{p\lambda} & \sigma_{p\theta} & \sigma_{pp} \end{pmatrix} = I^{-1}(\lambda,\theta,p). \tag{18}
$$

The asymptotic joint distribution of $\hat{\lambda}$, $\hat{\theta}$ and $\hat{p}$ is normal and is given by

$$
\begin{pmatrix} \hat{\lambda} \\ \hat{\theta} \\ \hat{p} \end{pmatrix} \sim N\left(\begin{pmatrix} \lambda \\ \theta \\ p \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda\lambda} & \sigma_{\lambda\theta} & \sigma_{\lambda p} \\ \sigma_{\theta\lambda} & \sigma_{\theta\theta} & \sigma_{\theta p} \\ \sigma_{p\lambda} & \sigma_{p\theta} & \sigma_{pp} \end{pmatrix}\right). \tag{19}
$$

Since Σ involves the parameters $\lambda,$ θ and $p,$ we substitute these parameters by the corresponding MLEs in order to obtain an estimate of $\Sigma,$ which denoted by

$$
\hat{\Sigma} = \begin{pmatrix} \hat{\sigma}_{\lambda\lambda} & \hat{\sigma}_{\lambda\theta} & \hat{\sigma}_{\lambda p} \\ \hat{\sigma}_{\theta\lambda} & \hat{\sigma}_{\theta\theta} & \hat{\sigma}_{\theta p} \\ \hat{\sigma}_{p\lambda} & \hat{\sigma}_{p\theta} & \hat{\sigma}_{pp} \end{pmatrix}. \tag{20}
$$

By using (19), approximate $100(1-\alpha)\%$ confidence intervals for λ, θ and p are given, respectively as

$$\left(\hat{\lambda}-z_{\alpha/2}\sqrt{\hat{\sigma}_{\lambda\lambda}}\ ,\hat{\lambda}+z_{\alpha/2}\sqrt{\hat{\sigma}_{\lambda\lambda}}\right),$$

$$\left(\hat{\theta}-z_{\alpha/2}\sqrt{\hat{\sigma}_{\theta\theta}}\ ,\hat{\theta}+z_{\alpha/2}\sqrt{\hat{\sigma}_{\theta\theta}}\right),$$

and
$$\left(\hat{p}-z_{\alpha/2}\sqrt{\hat{\sigma}_{pp}}\ ,\hat{p}+z_{\alpha/2}\sqrt{\hat{\sigma}_{pp}}\right),$$

where $z_{\alpha/2}$ is the $\alpha/2$ right-tail percentile of the standard normal. We denote $CI_{Asymp.}$ for asymptotic confidence intervals.

3.2. Bootstrap confidence intervals

In this subsection, we discuss two the parametric bootstrap methods for constructing confidence intervals which are the bootstrap-p and the bootstrap-t methods. We denote CI_p and CI_t for the bootstrap-p interval and the bootstrap-t interval, respectively.

The bootstrap-p confidence interval for λ, θ and p can be obtained as follows:

Step 1. Calculate $\hat{\lambda}$, $\hat{\theta}$ and $\hat{p}$ from the original sample $\underset{\sim}{x}$.

Step 2. Generate B_1 bootstrap samples $\underset{\sim}{x}^*$ of size n from $CR(\hat{\lambda},\hat{\theta},\hat{p})$, calculate $\hat{\lambda}_i^*$, $\hat{\theta}_i^*$ and $\hat{p}_i^*$ from $\underset{\sim}{x}^*$, $i=1,2,3,...,B_1$.

Step 3. The $100(1-\alpha)\%$ bootstrap-p confidence intervals for λ, θ and p are given by

$$\left(\hat{\lambda}_{B_1}^{(\alpha/2)}\ ,\ \hat{\lambda}_{B_1}^{(1-\alpha/2)}\right),$$

$$\left(\hat{\theta}_{B_1}^{(\alpha/2)}\ ,\ \hat{\theta}_{B_1}^{(1-\alpha/2)}\right),$$

and
$$\left(\hat{p}_{B_1}^{(\alpha/2)}\ ,\ \hat{p}_{B_1}^{(1-\alpha/2)}\right),$$

where $\hat{\lambda}_{B_1}^{(\alpha/2)},\hat{\theta}_{B_1}^{(\alpha/2)}$ and $\hat{p}_{B_1}^{(\alpha/2)}$ are $B_1\cdot(\alpha/2)$ th value in the ordered list of the B_1 replications of $\hat{\lambda}^*,\hat{\theta}^*$ and $\hat{p}^*$ respectively. Also, $\hat{\lambda}_{B_1}^{(1-\alpha/2)},\hat{\theta}_{B_1}^{(1-\alpha/2)}$ and $\hat{p}_{B_1}^{(1-\alpha/2)}$ are $B_1\cdot(1-\alpha/2)$ th value in the ordered list of the B_1 replications of $\hat{\lambda}^*,\hat{\theta}^*$ and $\hat{p}^*$, respectively.

The bootstrap-t confidence interval for λ, θ and p can be obtained as follows:

Step 1. Calculate $\hat{\lambda}$, $\hat{\theta}$ and $\hat{p}$ from the original sample $\underset{\sim}{x}$.

Step 2. Generate a bootstrap sample $\underset{\sim}{x}^*$ of size n from $CR(\hat{\lambda},\hat{\theta},\hat{p})$, calculate $\hat{\lambda}^*$, $\hat{\theta}^*$ and $\hat{p}^*$.

Step 3. Generate a bootstrap sample $\underset{\sim}{x}^{**}$ of size n from $CR(\hat{\lambda}^*,\hat{\theta}^*,\hat{p}^*)$, calculate $\hat{\lambda}^{**}$, $\hat{\theta}^{**}$ and $\hat{p}^{**}$.

Step 4. Repeat Step 3 B_2 times.

Step 5. Calculate

$$se(\hat{\lambda}^*)=\sqrt{\frac{1}{B_2-1}\sum_{j=1}^{B_2}(\hat{\lambda}_j^{**}-\overline{\lambda}^{**})^2},\quad \text{where } \overline{\lambda}^{**}=\sum_{j=1}^{B_2}\hat{\lambda}_j^{**}/B_2,$$

$$se(\hat{\theta}^*)=\sqrt{\frac{1}{B_2-1}\sum_{j=1}^{B_2}(\hat{\theta}_j^{**}-\overline{\theta}^{**})^2},\quad \text{where } \overline{\theta}^{**}=\sum_{j=1}^{B_2}\hat{\theta}_j^{**}/B_2$$

and

$$se(\hat{p}^*) = \sqrt{\frac{1}{B_2 - 1}\sum_{j=1}^{B_2}(\hat{p}_j^{**} - \overline{p}^{**})^2}, \quad \text{where } \overline{p}^{**} = \sum_{j=1}^{B_2}\hat{p}_j^{**} / B_2.$$

Step 6. Calculate $T_\lambda^* = \dfrac{\hat{\lambda}^* - \hat{\lambda}}{se(\hat{\lambda}^*)}$, $T_\theta^* = \dfrac{\hat{\theta}^* - \hat{\theta}}{se(\hat{\theta}^*)}$ and $T_p^* = \dfrac{\hat{p}^* - \hat{p}}{se(\hat{p}^*)}$.

Step 7. Repeat Steps 2-6 B_1 times.

Step 8. Calculate

$$se(\hat{\lambda}) = \sqrt{\frac{1}{B_1 - 1}\sum_{j=1}^{B_1}(\hat{\lambda}_j^* - \overline{\lambda}^*)^2}, \quad \text{where } \overline{\lambda}^* = \sum_{j=1}^{B_1}\hat{\lambda}_j^* / B_1,$$

$$se(\hat{\theta}) = \sqrt{\frac{1}{B_1 - 1}\sum_{j=1}^{B_1}(\hat{\theta}_j^* - \overline{\theta}^*)^2}, \quad \text{where } \overline{\theta}^* = \sum_{j=1}^{B_1}\hat{\theta}_j^* / B_1,$$

and

$$se(\hat{p}) = \sqrt{\frac{1}{B_1 - 1}\sum_{j=1}^{B_1}(\hat{p}_j^* - \overline{p}^*)^2}, \quad \text{where } \overline{p}^* = \sum_{j=1}^{B_1}\overline{p}_j^* / B_1.$$

Step 9. The $100(1-\alpha)\%$ bootstrap-t confidence intervals for λ, θ and p are given by

$$\left(\hat{\lambda} - T_{\lambda B_1}^{*(1-\alpha/2)} \cdot se(\hat{\lambda}),\, \hat{\lambda} - T_{\lambda B_1}^{*(\alpha/2)} \cdot se(\hat{\lambda})\right),$$

$$\left(\hat{\theta} - T_{\theta B_1}^{*(1-\alpha/2)} \cdot se(\hat{\theta}),\, \hat{\theta} - T_{\theta B_1}^{*(\alpha/2)} \cdot se(\hat{\theta})\right),$$

and

$$\left(\hat{p} - T_{p B_1}^{*(1-\alpha/2)} \cdot se(\hat{p}),\, \hat{p} - T_{p B_1}^{*(\alpha/2)} \cdot se(\hat{p})\right),$$

where $T_{\lambda B_1}^{*(1-\alpha/2)}$, $T_{\theta B_1}^{*(1-\alpha/2)}$ and $T_{p B_1}^{*(1-\alpha/2)}$ are $B_1(1-\alpha/2)$ th value in the ordered list of the B_1 replications of T_λ^*, T_θ^* and T_p^*, respectively. Also, $T_{\lambda B_1}^{*(\alpha/2)}$, $T_{\theta B_1}^{*(\alpha/2)}$ and $T_{p B_1}^{*(\alpha/2)}$ are $B_1(\alpha/2)$ th value in the ordered list of the B_1 replications of T_λ^*, T_θ^* and T_p^*, respectively.

4. Simulation Study

To compare the performance of the different confidence interval estimation methods which are described above, we shall assume the parameter p is known. We perform a Monte Carlo simulation study using different the samples of size $n = 25, 50, 100, 150$ from $CR(\lambda, \theta, p)$, where $\lambda = 4$, $\theta = 2$ and $p = 0.1, 0.3, 0.5, 0.7, 0.9$. A simulation was conducted using the R statistical software (version 3.3.2) based on 1,000 replications. The bootstrap procedure was applied with $B_1 = 1{,}000$ and $B_2 = 50$. The estimated mean square error (MSE) of MLEs coverage probabilities and expected lengths of the 95% confidence intervals are demonstrated in Tables 1-5.

As can be seen from Tables 1-5, the MSE decreases as the sample size increases (as it is expected). The average width of all intervals decreases when sample size increases. The confidence interval $CI_{Asymp.}$ of λ has the actual coverage probabilities close to the nominal level for all of situations. For the estimation of θ, we also observe the actual coverage probabilities close to the nominal level for moderate and large sample sizes $(n = 100, 150)$ for almost all situations, except for sample size 100 when the weighted parameter is 0.7 and 0.9. In addition, the actual coverage probabilities in small sample sizes $(n = 25, 50)$ become considerably much lower than the nominal level.

For moderate and large sample sizes $(n = 100, 150),$ the confidence interval CI_p for both λ and θ have actual coverage probability close to the nominal confidence level for almost all situations, except for sample size 100 when the weighted parameter is 0.7 and 0.9. But for small sample sizes $(n = 25, 50),$ the actual coverage probabilities are much lower than the nominal level for almost all situations, except for sample size 50 when the weighted parameter is 0.3.

Finally, the confidence interval CI_t of λ have actual coverage probability close to the nominal level for all of situations. Also, we observe that the actual coverage probabilities for confidence interval CI_t of θ are close to the nominal level for small sample sizes $(n = 25, 50)$ for almost all situations, except in when the weighted parameter is 0.1 and 0.9. Moreover, the actual coverage probabilities for moderate and large sample sizes $(n = 100, 150)$ is close to the nominal level when the weighted parameter is 0.1 and 0.9. But for in the weighted parameter from 0.3 to 0.7, we see that for almost all situations the actual coverage probability is much lower than the nominal level.

Regarding to the average width comparisons, the average width of $CI_{Asymp.}$ is the shortest for almost all situations. On the other hand, for the comparison the average width for small sample sizes $(n = 25,50)$ for θ we found that the average width of CI_p is the shortest for almost all situations when the weighted parameter is 0.1, 0.5 and 0.7. The average width of CI_p is the shortest for sample size 100 when the weighted parameter is 0.1 and 0.3. Moreover, the average width of CI_p is the shortest for large sample size 150 when the weighted parameter is form 0.1 to 0.7.

Based on this simulation study, one can apply the asymptotic confidence intervals for any sample size, as it performs better than the other two bootstrap intervals in a terms of coverage probabilities as well as the average width for estimation of $\lambda.$ Also, we can use the asymptotic confidence intervals and the bootstrap-p intervals for moderate and large sample sizes for estimation of $\theta.$ However, we can use the bootstrap-t intervals for small sample sizes because it performs well in a terms of coverage probabilities when the weighted parameter is from 0.3 to 0.7.

Table 1 MSE of $\hat{\lambda}$ and $\hat{\theta}$, coverage probabilities and average width of 95% confidence intervals for λ and θ with $p = 0.1$. The first entry in each cell corresponds to λ and the second to θ

n	MSE	Coverage Probabilities			Average Width		
		$CI_{Asymp.}$	CI_p	CI_t	$CI_{Asymp.}$	CI_p	CI_t
25	2.6606	0.968	0.877	0.970	5.3141	6.5975	5.4707
	0.3022	0.869	0.789	0.873	2.3147	2.0353	2.6854
50	0.9965	0.955	0.916	0.954	3.4688	3.8445	3.5490
	0.1652	0.907	0.908	0.846	1.9083	1.5142	1.7578
100	0.3699	0.956	0.944	0.961	2.3618	2.4802	2.4166
	0.0776	0.947	0.946	0.956	1.1166	1.0960	1.1947
150	0.2395	0.956	0.946	0.955	1.9174	1.9753	1.9581
	0.0501	0.949	0.944	0.953	0.9655	0.8937	0.9518

Table 2 MSE of $\hat{\lambda}$ and $\hat{\theta}$, coverage probabilities and average width of 95% confidence intervals for λ and θ with $p = 0.3$. The first entry in each cell corresponds to λ and the second to θ

n	MSE	Coverage Probabilities			Average Width		
		$CI_{Asymp.}$	CI_p	CI_t	$CI_{Asymp.}$	CI_p	CI_t
25	2.4131	0.964	0.897	0.970	5.0688	6.2835	5.2043
	0.3111	0.894	0.900	0.958	1.9239	2.1126	2.7941
50	0.7957	0.971	0.968	0.929	3.3768	3.4521	3.7469
	0.1489	0.926	0.972	0.927	1.0934	1.7837	1.5333
100	0.3461	0.959	0.946	0.960	2.2992	2.4142	2.3481
	0.0803	0.947	0.944	0.912	1.2567	1.1116	1.2123
150	0.2628	0.939	0.928	0.943	1.8773	1.9309	1.9116
	0.0603	0.932	0.926	0.896	0.9900	0.9049	0.9662

Table 3 MSE of $\hat{\lambda}$ and $\hat{\theta}$, coverage probabilities and average width of 95% confidence intervals for λ and θ with $p = 0.5$. The first entry in each cell corresponds to λ and the second to θ

n	MSE	Coverage Probabilities			Average Width		
		$CI_{Asymp.}$	CI_p	CI_t	$CI_{Asymp.}$	CI_p	CI_t
25	2.2328	0.963	0.899	0.968	5.0204	6.2364	5.1592
	0.3220	0.889	0.900	0.962	2.2616	2.1739	2.9051
50	1.0173	0.944	0.908	0.948	3.3775	3.7449	3.4556
	0.1845	0.897	0.901	0.937	1.6873	1.5895	1.8563
100	0.3747	0.950	0.929	0.958	2.2855	2.3936	2.3267
	0.0914	0.930	0.932	0.891	1.0391	1.1497	1.2568
150	0.2375	0.956	0.943	0.962	1.8436	1.9000	1.8810
	0.0589	0.939	0.938	0.948	0.9910	0.9453	1.0115

Table 4 MSE of $\hat{\lambda}$ and $\hat{\theta}$, coverage probabilities and average width of 95% confidence intervals for λ and θ with $p = 0.7$. The first entry in each cell corresponds to λ and the second to θ

n	MSE	Coverage Probabilities			Average Width		
		$CI_{Asymp.}$	CI_p	CI_t	$CI_{Asymp.}$	CI_p	CI_t
25	2.0282	0.971	0.895	0.968	5.0126	6.2160	5.1528
	0.3454	0.894	0.904	0.964	3.1362	2.3265	3.1631
50	0.9160	0.953	0.913	0.958	3.3669	3.3734	3.4437
	0.2041	0.905	0.912	0.954	1.3376	1.6943	1.9960
100	0.4385	0.937	0.911	0.938	2.3147	2.4311	2.3606
	0.1087	0.900	0.905	0.932	1.0660	1.2063	1.3250
150	0.2345	0.954	0.934	0.956	1.8646	1.9222	1.9041
	0.0650	0.943	0.944	0.911	0.9990	0.9944	1.0673

Table 5 MSE of $\hat{\lambda}$ and $\hat{\theta}$, coverage probabilities and average width of 95% confidence intervals for λ and θ with $p = 0.9$. The first entry in each cell corresponds to λ and the second to θ

n	MSE	Coverage Probabilities			Average Width		
		$CI_{Asymp.}$	CI_p	CI_t	$CI_{Asymp.}$	CI_p	CI_t
25	2.9955	0.957	0.874	0.962	5.3160	6.6158	5.4799
	0.4894	0.846	0.867	0.750	1.6473	2.4954	3.4581
50	0.8738	0.957	0.923	0.964	3.4592	3.8330	3.5293
	0.2319	0.910	0.919	0.825	1.7058	1.7973	2.1337
100	0.4012	0.953	0.925	0.956	2.3762	2.4966	2.4236
	0.1147	0.919	0.922	0.952	1.1144	1.2894	1.4236
150	0.2527	0.949	0.938	0.951	1.9191	1.9804	1.9574
	0.0743	0.935	0.942	0.955	0.9540	1.0552	1.1359

5. Illustrative Example

In this section, we consider a data set from Birnbaum and Saunders (1969) on the fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second (cps). The maintenance data set consists of 101 observations which are given below:

70	90	96	97	99	100	103	104	104	105
107	108	108	108	109	109	112	112	113	114
114	114	116	119	120	120	120	121	121	123
124	124	124	124	124	128	128	129	129	130
130	130	131	131	131	131	131	132	132	132
133	134	134	134	134	134	136	136	137	138
138	138	139	139	141	141	142	142	142	142
142	142	144	144	145	146	148	148	149	151
151	152	155	156	157	157	157	157	158	159
162	163	163	164	166	166	168	170	174	196
212									

For our analysis, we estimate two parameters, λ and θ with fixed $p = 0.3, 0.5, 0.7$ of the crack lifetime distribution by the maximum likelihood estimation in order to construct confidence intervals for λ and θ. The estimators of λ and θ are presented in Table 6. Moreover, the 95% confidence intervals for λ and θ are shown in Tables 7-9.

Table 6 The estimators for λ and θ with fixed $p = 0.3, 0.5, 0.7$

p	Estimators	
	$\hat{\lambda}$	$\hat{\theta}$
0.3	34.4065	3.8093
0.5	34.4460	3.8228
0.7	34.4066	3.8532

Table 7 The 95% confidence intervals with width for parameters with fixed $p = 0.3$. The first entry in each cell corresponds to λ and the second to θ

Methods	Confidence Intervals		Width
	Lower Limit	Upper Limit	
$CI_{Asymp.}$	24.90590	43.90710	19.00120
	2.75595	4.86271	2.10676
CI_p	26.82000	47.18310	20.36310
	2.79540	4.90200	2.10660
CI_t	24.82910	44.82750	19.99840
	2.90840	5.30200	2.39360

Table 8 The 95% confidence intervals with width for parameters with fixed $p = 0.5$. The first entry in each cell corresponds to λ and the second to θ

Methods	Confidence Intervals		Width
	Lower Limit	Upper Limit	
$CI_{Asymp.}$	24.9456	43.9466	19.0010
	2.7638	4.8899	2.12610
CI_p	26.6229	46.6885	20.0656
	2.8079	4.9604	2.1525
CI_t	25.0023	45.0768	20.0745
	2.9262	5.2629	2.3367

Table 9 The 95% confidence intervals with width for parameters with fixed $p = 0.7.$ The first entry in each cell corresponds to λ and the second to θ

Methods	Confidence Intervals		Width
	Lower Limit	Upper Limit	
$CI_{Asymp.}$	24.9059	43.9072	19.0013
	2.7755	4.9309	2.1554
CI_p	26.3734	46.0008	19.6274
	2.8663	5.0454	2.1791
CI_t	25.6579	44.3142	18.6563
	2.9599	5.2280	2.2681

In summary, it can be noticed that the width of $CI_{Asymp.}$ is the shortest in almost all situations. Moreover, the width of CI_p and $CI_{Asymp.}$ are the same for $\theta.$ This result confirms that the asymptotic confidence interval is the most efficient and also we can apply the bootstrap-p method for constructing the confidence intervals based on MLEs.

6. Conclusions

In this paper, we have proposed some properties of the crack lifetime distribution. We have also discussed different confidence interval estimation methods of the parameters based on MLEs. The first method of confidence estimation is based on the asymptotic distribution of MLEs. Other two methods are parametric bootstrap methods. The performance for the different methods is investigated in terms of coverage probabilities and average widths by the Monte Carlo simulations. A numerical example has also been presented to illustrate all the inferential results established. The simulation results indicated that the asymptotic confidence intervals behave very well for moderate and large sample sizes. Moreover, the bootstrap-p interval method works well for moderate and large sample sizes. Furthermore, the performance is quite satisfactory for bootstrap-t intervals for small sample sizes. But perhaps the results of two parametric bootstrap methods may be dependent on the bootstrap replication size. In the future research, the question of intervals based on the likelihood ratio, the test of hypothesis and the estimation based on censoring schemes still remains to be considered.

References

Birnbaum ZW, Saunders SC. Estimation for a family of life distributions with applications to fatigue. J Appl Prob. 1969; 6: 328-347.

Bowonrattanaset P, Budsaba K. Some properties of the three-parameter crack distribution. Thail Stat. 2011; 9: 195-203.

Duangsaphon M, Budsaba K, Volodin A. Improve statistical inference for three-parameter crack lifetime distribution. J Prob Stat Sci. 2016; 14: 239-251.

Efron B. The Jackknife, the bootstrap and other resampling plans. CBMS-NSF Regional Conference Series in Applied Mathematics. Philadelphia. SIAM; 1982.

Gamsmi S, Berzig M. Parameters estimation of the modified Weibull ditribution based on type I censored sample. Appl Math Sci. 2011; 5: 2899-2917.

Hall P. Theoretical comparison of bootstrap confidence intervals. Ann Stat. 1988; 16: 927-953.

Kohansal A, Rezakhah S. Parameter estimation of type-II hybrid censored weighted exponential distribution. Cornel University. 2012 [cited 2012 Mar 1]. Available from: https://arxiv.org/abs/1203.0094.

Miller RGJ. Survival analysis. New York: Wiley; 1981.

Ng HKT, Kundu D, Balakrishnan N. Point and interval estimations for the two-parameter Birnbaum-Saunders distribution based on type-II censored samples. Comput Stat Data Anal. 2006; 50: 3222-3242.

Panahi H, Asadi S. Estimation of the Weibull distribtion based on type-II censored samples. Appl Math Sci. 2011; 5: 2549-2558.

Pandey BD, Bandyopadhyay P. Bayesian estimation of inverse Gaussian distribution. Cornel University. 2012 [cited 2012 Oct 16]. Available from: https://arxiv. org/abs/1210.4524.