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Moments of Generalized Record Values from Kumaraswamy-log-logistic Distribution and Related Inferences

Bavita Singh, Rafiqullah Khan* and Mohd. A.R. Khan
Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh-202 002, India
*Corresponding author; e-mail: aruke@rediffmail.com

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Abstract

In this paper, explicit expression for single moments and some recurrence relations satisfied by single and product moments of generalized upper record statistics or k-th upper record values from the Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution are derived. These relations can be used to obtain the higher order moments from those of the lower order. The results obtained are deduced for moments of upper record statistics. Further, conditional expectation, recurrence relations for the single as well as product moments and truncated moment are used to characterize this distribution.

Keywords: Order statistics, generalized upper record values, record values, Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution, single moments, product moments, recurrence relations, truncated moment, characterization.

1. Introduction

A random variable X is said to have Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution (Huang and Oluyede 2014), if its probability density function (pdf) is of the form

$$f(x)=\alpha\beta\lambda \; x^{-\beta-1}(1+\lambda \; x^{-\beta})^{-2}[1-(1+\lambda \; x^{-\beta})^{-1}]^{\alpha-1}, \; x>0, \; \alpha>0, \; \beta>0 \text{ and } \lambda>0 \tag{1}$$

with corresponding distribution function (df)

$$F(x)=1-[1-(1+\lambda \; x^{-\beta})^{-1}]^{\alpha}, \; x>0, \; \alpha>0, \; \beta>0 \text{ and } \lambda>0. \tag{2}$$

In view of (1) and (2), it is easy to see that

$$\alpha\beta\bar{F}(x)=(x+\lambda x^{-\beta+1})f(x), \tag{3}$$

where $\bar{F}(x)=1-F(x)$.

It is observed in Huang and Oluyede (2014) that this distribution has desirable features of exhibiting a non-monotone failure rate, thereby accommodating different shapes for the hazard rate function and should be an attractive choice for survival and reliability data analysis.

The statistical study of record values in a sequence of independent and identically distributed (iid) continuous random variables was first carried out by Chandler (1952). For a survey on important results in this area one may refer to Ahsanullah (1995), Arnold et al. (1998) and Ahsanullah and Navzorov (2015). Dziubdziela and Kopociński (1976) have generalized the concept of record values of Chandler (1952) by random variables of a more generalized nature and called them the k-th record values. Later, Minimol and Thomas (2013) called the record values defined by Dziubdziela and Kopociński (1976) also as the generalized record values, since the r-th member of the sequence of the ordinary record values is also known as the r-th record value. Setting $k=1,$ we obtain ordinary record statistics.

Several applications of k-th record values can be found in the literature, for instance, see the examples cited in Kamps (1995) or Danielak and Raqab (2004) in reliability theory. Suppose that a technical system or piece of equipment is subject to shocks, e.g. peaks of voltages. If the shocks are viewed as realizations of an iid sequence, then the model of ordinary records is adequate. If it is not the records themselves, but second or third values are of special interest, then the model of k-th record values is adequate. When record values themselves are viewed as outliers, then the second or third largest values are of special interest. Record statistics are applied in estimating strength of materials, predicting natural disasters, sport achievements, etc. For statistical inference based on ordinary records, serious difficulties arise if expected values of inter arrival time of records is infinite and occurrences of records are very rare in practice. This problem is avoided once we consider the model of k-th record statistics.

For some recent developments on generalized upper record values or k-th upper record values with special reference to those arising from exponential, Gumble, Pareto, generalized Pareto, Burr, Weibull, Gompertz, Makeham, modified Weibull, exponential-Weibull and additive Weibull distributions, see Grudzień and Szynal (1983, 1997), Pawlas and Szynal (1998, 1999, 2000), Minimol and Thomas (2013, 2014), Khan and Khan (2016) and Khan, et al. (2015, 2017), respectively. In this paper we mainly focus on the study of generalized upper record values arising from the Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution and discussed exact explicit expressions as well as several recurrence relations satisfied by single and product moments. In addition, conditional expectation and recurrence relations for single moments of k-th upper record values and truncated moment are used to characterize this distribution.

Let $\{X_n, n\geq 1\}$ be a sequence of iid random variables with df $F(x)$ and pdf $f(x)$. The j-th order statistic of a sample $X_1, X_2, \ldots, X_n$ is denoted by $X_{j:n}$. For a fixed positive integer $k,$ we define the sequence $\{U_n^{(k)}, n\geq 1\}$ of k-th upper record times of $\{X_n, n\geq 1\}$ as follows:

$$U_1^{(k)}=1,$$

$$U_{n+1}^{(k)}=\min\{j>U_n^{(k)}: X_{j:j+k-1}>X_{U_n^{(k)}:U_n^{(k)}+k-1}\}.$$

The sequence $\{Y_n^{(k)}, n\geq 1\},$ where $Y_n^{(k)}=X_{U_n^{(k)}}$ is called the sequence of generalized upper record values or k-th upper record values of $\{X_n, n\geq 1\}$. Note that for $k=1,$ we have $Y_n^{(1)}=X_{U_n},$ $n\geq 1,$ which are the record values of $\{X_n, n\geq 1\}$ as defined in Ahsanullah (1995).

The pdf of $Y_n^{(k)}$ and the joint pdf of $Y_m^{(k)}$ and $Y_n^{(k)}$ are given by (see Dziubdziela and Kopociński 1976, Grudzień 1982)

$$f_{Y_n^{(k)}}(x)=\frac{k^n}{(n-1)!}[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x), \quad n\geq 1, \tag{4}$$

$$f_{Y_m^{(k)},Y_n^{(k)}}(x,y)=\frac{k^n}{(m-1)!(n-m-1)!}[-\ln \bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}$$
$$\times[\ln \bar{F}(x)-\ln \bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1}f(y),\ \ x<y,\ 1\le m<n,\ n\ge 2, \tag{5}$$

and the conditional pdf of $Y_n^{(k)}$ given $Y_m^{(k)}=x$, is

$$f_{Y_n^{(k)}\ |\ Y_m^{(k)}}(y\,|\,x)\ =\frac{k^{n-m}}{(n-m-1)!}[\ln \bar{F}(x)-\ln \bar{F}(y)]^{n-m-1}\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{k-1}\frac{f(y)}{\bar{F}(x)},\ \ x<y. \tag{6}$$

2. Relations for Single Moments

We shall first establish the exact expression for single moments of k -th upper record statistics in the following theorem.

Theorem 1 *For the Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution given by* (2)*, for any fixed positive integer* k $(1\le k\le n)$*, and* $j=0,1,2,...$

$$E(Y_n^{(k)})^j\ =\frac{(k\alpha)^n(-\lambda)^{j/\beta}}{(n-1)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(j/\beta)_p\,a_q\,(n-1)\Gamma(k\alpha)\Gamma(n+q-p)}{p!\ \ \Gamma(n+k\alpha+q-p)}. \tag{7}$$

Proof: From (4) and (2), we have

$$E(Y_n^{(k)})^j\ =\frac{k^n(-\lambda)^{j/\beta}}{(n-1)!}\int_0^{\infty}[1-(1-\bar{F}(x)^{1/\alpha})^{-1}]^{-j/\beta}[-\ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)\,dx\cdot \tag{8}$$

On using Maclaurin series expansion

$$(1-z)^{-t}\ =\sum_{p=0}^{\infty}\frac{(t)_p\,z^p}{p!}, \tag{9}$$

where

$$(t)_p\ =\begin{cases}t(t+1)...(t+p-1), & p=1,2,...\\ 1, & p=0,\end{cases}$$

and simplifying the resulting expression , we get

$$E(Y_n^{(k)})^j\ =\frac{k^n(-\lambda)^{j/\beta}}{(n-1)!}\sum_{p=0}^{\infty}\frac{(j/\beta)_p}{p!}\int_0^{\infty}(1-\bar{F}(x)^{1/\alpha})^{-p}[-\ln\bar{F}(x)]^{n-1}\ [\bar{F}(x)]^{k-1}f(x)dx\,. \tag{10}$$

Setting $z=[\bar{F}(x)]^{1/\alpha}$ in (10), we obtain

$$E(Y_n^{(k)})^j\ =\frac{k^n\alpha(-\lambda)^{j/\beta}}{(n-1)!}\sum_{p=0}^{\infty}\frac{(j/\beta)_p}{p!}\int_0^1(1-z)^{-p}[-\ln z^{\alpha}]^{n-1}z^{k\alpha-1}dz$$
$$=\frac{(k\alpha)^n(-\lambda)^{j/\beta}}{(n-1)!}\sum_{p=0}^{\infty}\frac{(j/\beta)_p}{p!}\int_0^1(1-z)^{-p}[-\ln\{1-(1-z)\}]^{n-1}z^{k\alpha-1}dz. \tag{11}$$

In view of Balakrishnan and Cohen (1991), note that

$$[-\ln(1-t)]^j=\left(\sum_{p=1}^{\infty}\frac{t^p}{p}\right)^j\ =\sum_{p=0}^{\infty}a_p(j)\,t^{j+p}\,,\ |t|<1\,, \tag{12}$$

where $a_p(j)$ is the coefficient of t^{j+p} in the expansion of $\left(\sum_{p=1}^{\infty}\frac{t^p}{p}\right)^j$.

On substituting (12) in (11), we have

$$E(Y_n^{(k)})^j = \frac{(k\alpha)^n(-\lambda)^{j/\beta}}{(n-1)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(j/\beta)_p a_q(n-1)}{p!}\int_0^1 (1-z)^{n+q-p-1}z^{k\alpha-1}dz$$

and hence the result given in (7).

Corollary 1 *The explicit expressions for the single moments of upper record values from the Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution has the form*

$$E(Y_n^{(1)})^j = E(X_{U_n}^j) = \frac{\alpha^n(-\lambda)^{j/\beta}}{(n-1)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(j/\beta)_p a_q(n-1)\Gamma(\alpha)\Gamma(n+q-p)}{p!\Gamma(n+\alpha+q-p)}.$$

Now, we obtain the recurrence relations for single moments of the k-th upper record values from the Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution in the following theorem.

Theorem 2 *For the Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution given by* (2)*, for any fixed positive integer* k $(1 \le k \le n)$*, and* $j = 0,1,2,...$

$$\left(1-\frac{j}{\alpha\beta k}\right)E(Y_n^{(k)})^j = E(Y_{n-1}^{(k)})^j + \frac{j\lambda}{\alpha\beta k}E(Y_n^{(k)})^{j-\beta}. \tag{13}$$

Proof: In view of Khan et al. (2017), note that

$$E(Y_n^{(k)})^j - E(Y_{n-1}^{(k)})^j = \frac{jk^{n-1}}{(n-1)!}\int_\alpha^\beta x^{j-1}[-\ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^k\,dx. \tag{14}$$

Using (3) in (14), we have

$$E(Y_n^{(k)})^j - E(Y_{n-1}^{(k)})^j = \frac{j}{\alpha\beta k}\left(\frac{k^n}{(n-1)!}\int_0^\infty x^j[-\ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx\right)$$
$$+\frac{j\lambda}{\alpha\beta k}\left(\frac{k^n}{(n-1)!}\int_0^\infty x^{j-\beta}[-\ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx\right)$$

and hence we obtained the result given in (13).

Corollary 2 *The recurrence relation for single moments of upper record values from the Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution has the form*

$$\left(1-\frac{j}{\alpha\beta}\right)EX_{U_n}^j = EX_{U_{n-1}}^j + \frac{j\lambda}{\alpha\beta}EX_{U_n}^{j-\beta}. \tag{15}$$

3. Relations for Product Moments

This section gives the recurrence relations for product moments of the k-th upper record values from Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution.

Theorem 3 *For the distribution given in* (2) *and* $m \ge 1$, $m \ge k$, $i, j = 0,1,2,...$

$$\left(1-\frac{j}{\alpha\beta k}\right)E[(Y_m^{(k)})^i(Y_{m+1}^{(k)})^j] = E[(Y_m^{(k)})^{i+j}] + \frac{j\lambda}{\alpha\beta k}E[(Y_m^{(k)})^i(Y_{m+1}^{(k)})^{j-\beta}]. \tag{16}$$

and for $1 \le m \le n-2$, $i, j = 0,1,2,...$

$$\left(1-\frac{j}{\alpha\beta k}\right)E[(Y_m^{(k)})^i(Y_n^{(k)})^j] = E[(Y_m^{(k)})^i(Y_{n-1}^{(k)})^j] + \frac{j\lambda}{\alpha\beta k}E[(Y_m^{(k)})^i(Y_n^{(k)})^{j-\beta}]. \tag{17}$$

Proof: From (5) and (3), we have

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \frac{j \quad k^{n-1}}{\alpha\beta(m-1)!(n-m-1)!}\int_0^\infty \int_x^\infty x^i y^j [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)}$$
$$\times[-\ln \bar{F}(y) + \ln(\bar{F}(x))]^{n-m-1} [\bar{F}(y)]^{k-1} f(y) dy dx$$
$$+\frac{j\lambda \quad k^{n-1}}{\alpha\beta(m-1)!(n-m-1)!}\int_0^\infty \int_x^\infty x^i y^{j-\beta} [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)}$$
$$\times[-\ln \bar{F}(y) + \ln(\bar{F}(x))]^{n-m-1} [\bar{F}(y)]^{k-1} f(y) dy dx.$$

(17) can be proved in view of Khan et al. (2017), by noting that

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] - E[(Y_m^{(k)})^i (Y_{n-1}^{(k)})^j] = \frac{jk^{n-1}}{(m-1)!(n-m-1)!}\int_\alpha^\beta \int_x^\beta x^i y^{j-1}$$
$$\times[-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [-\ln \bar{F}(y) + \ln(\bar{F}(x))]^{n-m-1} [\bar{F}(y)]^{k-1} f(y) dy dx.$$

Now putting $n = m+1$ in (17) and noting that $E[(Y_m^{(k)})^i (Y_m^{(k)})^j = E[(Y_m^{(k)})^{i+j}$, the recurrence relation given in (16) can be easily established. One can also note that Theorem 2 can be deduced from Theorem 3 by putting $i = 0$.

Corollary 3 *The recurrence relation for product moments of upper record values from the Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution has the form*

$$\left(1 - \frac{j}{\alpha\beta}\right) E(X_{U_m}^i X_{U_n}^j) = E(X_{U_m}^i X_{U_{n-1}}^j) + \frac{j\lambda}{\alpha\beta} E(X_{U_m}^i X_{U_n}^{j-\beta}).$$

4. Characterizations

Theorem 4 *Let X be a non-negative random variable having an absolutely continuous df $F(x)$ with $F(0) = 0$ and $0 \le F(x) \le 1$ for all $x > 0$, then for $1 \le l < s \le n$,*

$$E[\xi(Y_n^{(k)}) \mid (Y_l^{(k)}) = x] = \xi(x) \left(\frac{k}{k+1}\right)^{n-l}, \quad l = m, \; m+1 \tag{18}$$

if and only if

$$F(x) = 1 - [1 - (1 + \lambda \quad x^{-\beta})^{-1}]^\alpha, \; x > 0, \; \alpha > 0, \; \beta > 0 \text{ and } \lambda > 0,$$

where $\xi(y) = [1 - (1 + \lambda y^{-\beta})^{-1}]^\alpha$.

Proof: From (6), we have

$$E[\xi(Y_n^{(k)}) \mid (Y_m^{(k)}) = x] = \frac{k^{n-m}}{(n-m-1)!}\int_x^\infty [1 - (1 + \lambda y^{-\beta})^{-1}]^\alpha$$
$$\times[-\ln \bar{F}(y) + \ln \bar{F}(x)]^{n-m-1} \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{k-1} \frac{f(y)}{\bar{F}(x)} dy. \tag{19}$$

By setting $u = \frac{\bar{F}(y)}{\bar{F}(x)} = \frac{[1 - (1 + \lambda y^{-\beta})^{-1}]^\alpha}{[1 - (1 + \lambda x^{-\beta})^{-1}]^\alpha}$ from (2) in (19), we get

$$E[\xi(Y_n^{(k)}) \mid (Y_m^{(k)}) = x] = \frac{k^{n-m}}{(n-m-1)!}[1 - (1 + \lambda x^{-\beta})^{-1}]^\alpha \int_0^1 u^k [-\ln u]^{n-m-1} du. \tag{20}$$

We have Gradshteyn and Ryzhik (2007, p.551)

$$\int_0^1 (-\ln x)^{\mu-1} x^{\upsilon-1} dx = \frac{\Gamma\mu}{\upsilon^{\mu}}, \quad \mu > 0, \ \upsilon > 0. \tag{21}$$

On using (21) in (20), we have the result given in (18). To prove sufficient part, we have

$$\frac{k^{n-m}}{(n-m-1)!}\int_x^{\infty}[1-(1+\lambda y^{-\beta})^{-1}]^{\alpha}[\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1}f(y)dy = [\bar{F}(x)]^k g_{n|m}(x), \tag{22}$$

where

$$g_{n\ |\ m}(x) = [1-(1+\lambda x^{-\beta})^{-1}]^{\alpha}\left(\frac{k}{k+1}\right)^{n-m}.$$

Differentiating (22) both sides with respect to x, we get

$$-\frac{k^{n-m}f(x)}{\bar{F}(x)(n-m-2)!}\int_x^{\infty}[1-(1+\lambda y^{-\beta})^{-1}]^{\alpha}[\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-2}\times[\bar{F}(y)]^{k-1}f(y)dy$$
$$= g'_{n\ |\ m}(x)[\bar{F}(x)]^k - kg_{n\ |\ m}(x)[\bar{F}(x)]^{k-1}f(x)$$

or

$$-kg_{n\ |\ m+1}(x)[\bar{F}(x)]^{k-1}f(x) = g'_{n\ |\ m}(x)[\bar{F}(x)]^k - kg_{n\ |\ m}(x)[\bar{F}(x)]^{k-1}f(x).$$

Therefore,

$$\frac{f(x)}{\bar{F}(x)} = -\frac{g'_{n\ |\ m}(x)}{k[g_{n\ |\ m+1}(x)-g_{n\ |\ m}(x)]}$$
$$= \frac{\alpha\beta\lambda x^{-\beta-1}(1+\lambda x^{-\beta})^{-2}}{[1-(1+\lambda x^{-\beta})^{-1}]}, \tag{23}$$

where

$$g'_{n\ |\ m}(x) = -\alpha\beta\lambda x^{-\beta-1}(1+\lambda x^{-\beta})^{-2}[1-(1+\lambda x^{-\beta})^{-1}]^{\alpha-1}\left(\frac{k}{k+1}\right)^{n-m},$$

$$g_{n\ |\ m+1}(x) - g_{n\ |m}(x) = \frac{1}{k}[1-(1+\lambda x^{-\beta})^{-1}]^{\alpha}\left(\frac{k}{k+1}\right)^{n-m}.$$

Now integrating both the sides of (23) with respect to x between $(0, y)$, the sufficiency part is proved.

Remark 1 If $k = 1$, in (18), we obtain the following characterization of the Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution based on upper record values

$$E[\xi(X_{U_n}) \,|\, (X_{U_l}) = x] = \xi(x) \ \ (1/2)^{n-l}, \ \ l = m, \ m+1,$$

where

$$\xi(x) = [1-(1+\lambda \ \ x^{-\beta})^{-1}]^{\alpha}.$$

Following theorems deal with the characterization of the Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution by a recurrence relations for the single and product moments of k-th upper record statistics.

Theorem 5 *Fix a positive integer* $k \geq 1$ *and let* j *be a non-negative integer. A necessary and sufficient condition for a random variable* X *to be distributed with pdf given by* (1) *is that*

$$\left(1-\frac{j}{\alpha\beta k}\right)E(Y_n^{(k)})^j = E(Y_{n-1}^{(k)})^j + \frac{j\lambda}{\alpha\beta k}E(Y_n^{(k)})^{j-\beta}, \tag{24}$$

for $n = 1,2,...,n-k.$

Proof: The necessary part follows from (13). On the other hand if the recurrence relation in (24) is satisfied, then on using (4), we have

$$\frac{k^n}{(n-1)!}\int_0^\infty x^j[-\ln \bar{F}(x)]^{n-2}[\bar{F}(x)]^{k-1}f(x)\ \left\{-\ln \bar{F}(x)-\frac{n-1}{k}\right\}\ dx$$

$$= \frac{j}{\alpha\beta k}\left\{\frac{k^n}{(n-1)!}\int_0^\infty x^j[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx\right.$$

$$\left.+\frac{\lambda k^n}{(n-1)!}\int_0^\infty x^{j-\beta}[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx\right\}. \tag{25}$$

Let

$$h(x) = -\frac{1}{k}[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^k. \tag{26}$$

Differentiating both the sides of (26), we get

$$h'(x) = [-\ln \bar{F}(x)]^{n-2}[\bar{F}(x)]^{k-1}f(x)\ \left\{-\ln \bar{F}(x)-\frac{n-1}{k}\right\}.$$

Thus,

$$\frac{k^n}{(n-1)!}\int_0^\infty x^j h'(x)dx = \frac{j}{\alpha\beta}\left\{\frac{k^{n-1}}{(n-1)!}\int_0^\infty x^j[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx\right.$$

$$\left.+\frac{\lambda k^{n-1}}{(n-1)!}\int_0^\infty x^{j-\beta}[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx\right\}. \tag{27}$$

Integrating left hand side in (27) by parts and using the value of $h(x)$ from (26), we find that

$$\frac{jk^{n-1}}{(n-1)!}\int_0^\infty x^j[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}\left\{\frac{\bar{F}(x)}{x}-\frac{f(x)}{\alpha\beta}-\frac{\lambda f(x)}{\alpha\beta x^\beta}\right\}dx = 0. \tag{28}$$

Now applying the generalization of the Müntz-Szász Theorem (see for example Hwang and Lin 1984) to (28), we get

$$\alpha\beta\bar{F}(x) = (x+\lambda x^{-\beta+1})f(x),$$

which proves that

$$F(x) = 1-[1-(1+\lambda x^{-\beta})^{-1}]^\alpha,\ \ x>0,\ \ \alpha>0,\ \ \beta>0\ \text{ and }\ \lambda>0.$$

Remark 2 *For* $k=1$ *in* (24)*, we obtain the following characterizing result based on upper record values for Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution as*

$$\left(1-\frac{j}{\alpha\beta}\right)EX_{U_n}^j = EX_{U_{n-1}}^j + \frac{j\lambda}{\alpha\beta}EX_{U_n}^{j-\beta}.$$

Theorem 6 *For a positive integer* k, i *and* j *be a non-negative integer, a necessary and sufficient condition for a random variable* X *to be distributed with pdf given by* (1) *is that*

$$\left(1-\frac{j}{\alpha\beta k}\right)\ E[(Y_m^{(k)})^i(Y_n^{(k)})^j]=E[(Y_m^{(k)})^i(Y_{n-1}^{(k)})^j]+\frac{j\lambda}{\alpha\beta k}E[(Y_m^{(k)})^i(Y_n^{(k)})^{j-\beta}]. \tag{29}$$

Proof: The necessary part follows from (17). On the other hand if the relation in (29) is satisfied, then on using (5), we have

$$\frac{k^n}{(m-1)!(n-m-1)!}\int_0^\infty\int_x^\infty x^i y^j[-\ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{n-m-2}[\bar{F}(y)]^{k-1}$$
$$\times\left\{-\ln\bar{F}(y)+\ln\bar{F}(x)-\frac{(n-m-1)}{k}\right\}f(y)dydx=\frac{jk^{n-1}}{(m-1)!(n-m-1)!\alpha\beta}\int_0^\infty\int_x^\infty x^i y^j$$
$$\times[-\ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{n-m-1}[\bar{F}(y)]^{k-1}f(y)dy\,dx+\frac{j\lambda\,k^{n-1}}{(m-1)!(n-m-1)!\alpha\beta}$$
$$\times\int_0^\infty\int_x^\infty x^i y^{j-\beta}[-\ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[-\ln\bar{F}(y)+\ln[\bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1}f(y)dydx. \tag{30}$$

Let

$$g(y)=-\frac{1}{k}[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{n-m-1}[\bar{F}(y)]^k\ . \tag{31}$$

Differentiating both the sides of (31), we get

$$g'(x)=[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{n-m-2}[\bar{F}(y)]^{k-1}f(y)\left\{-\ln\bar{F}(y)+\ln\bar{F}(x)-\frac{(n-m-1)}{k}\right\}. \tag{32}$$

Thus,

$$\frac{k^n}{(m-1)!(n-m-1)!}\int_0^\infty\int_x^\infty x^i y^j[-\ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}g'(x)\,dy\,dx=\frac{jk^{n-1}}{(m-1)!(n-m-1)!\alpha\beta}$$
$$\times\int_0^\infty\int_x^\infty x^i y^j[-\ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{n-m-1}[\bar{F}(y)]^{k-1}f(y)\,dy\,dx$$
$$+\frac{j\lambda k^{n-1}}{(m-1)!(n-m-1)!\alpha\beta}\int_0^\infty\int_x^\infty x^i y^{j-\beta}[-\ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}$$
$$\times[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{n-m-1}[\bar{F}(y)]^{k-1}f(y)dydx\ . \tag{33}$$

Now consider

$$I(x)=\int_x^\infty y^j g'(x)dy. \tag{34}$$

Integrating (34) by parts and using the value of $g(y)$ from (31), we have

$$I(x)=\frac{j}{k}\int_x^\infty y^{j-1}[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{n-m-1}[\bar{F}(y)]^k\,dy. \tag{35}$$

On using (35) in (33), we find that

$$\frac{jk^{n-1}}{(m-1)!(n-m-1)!}\int_0^\infty\int_x^\infty x^i y^{j-1}[-\ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{n-m-1}[\bar{F}(y)]^k\,dy$$
$$=\frac{jk^{n-1}}{(m-1)!(n-m-1)!\alpha\beta}\int_0^\infty\int_x^\infty x^i y^j[-\ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[-\ln\bar{F}(y)+\ln\bar{F}(x)]^{n-m-1}$$
$$\times[\bar{F}(y)]^{k-1}f(y)dydx+\frac{j\lambda k^{n-1}}{(m-1)!(n-m-1)!\ \ \alpha\beta}\int_0^\infty\int_x^\infty x^i y^{j-\beta}[-\ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}$$

$$\times[-\ln \bar{F}(y)+\ln \bar{F}(x)]^{n-m-1}[\bar{F}(y)]^{k-1} f(y)dydx.$$

$$=\frac{jk^{n-1}}{(m-1)!(n-m-1)1 \quad \alpha\beta}\int_0^{\infty}\int_x^{\infty} x^i y^{j-1}[-\ln \bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[-\ln \bar{F}(y)+\ln \bar{F}(x)]^{n-m-1}$$

$$\times[\bar{F}(y)]^{k-1}\{\alpha\beta\bar{F}(y)-(y+\lambda y^{1-\beta})f(y)\}dydx=0. \tag{36}$$

Applying the extension of Müntz-Szász Theorem, (see for example Hwang and Lin 1984) to (36), we get

$$\alpha\beta\bar{F}(y)=(y+\lambda y^{1-\beta})f(y),$$

which proves that $f(y)$ has the form as in (3).

Following theorem deals with the characterization of the Kumaraswamy-Fisk or Kumaraswamy-log-logistic distribution through truncated moment.

Theorem 7 *Suppose an absolutely continuous* (*with respect to Lebesque measure*) *random variable* X *has the df* $F(x)$ *and pdf* $f(x)$ *for* $0<x<\infty$, *such that* $f'(x)$ *and* $E(X \mid X \leq x)$ *exist for all* x, $0<x<\infty$, *then*

$$E(X \mid X \leq x)=g(x)\eta(x), \tag{37}$$

where

$$\eta(x)=\frac{f(x)}{F(x)} \text{ and}$$

$$g(x)=-\frac{x^{\beta+2}[1-(1+\lambda x^{-\beta})^{-1}]}{\alpha\beta\lambda(1+\lambda x^{-\beta})^{-2}}+\frac{x^{\beta+1}\int_0^x[1-(1+\lambda u^{-\beta})^{-1}]^{\alpha}du}{\alpha\beta\lambda(1+\lambda x^{-\beta})^{-2}[1-(1+\lambda x^{-\beta})^{-1}]^{\alpha-1}},$$

if and only if

$$f(x)=\alpha\beta\lambda x^{-\beta-1}(1+\lambda x^{-\beta})^{-2}[1-(1+\lambda x^{-\beta})^{-1}]^{\alpha-1}, \quad x>0, \quad \alpha>0, \quad \beta>0 \text{ and } \lambda>0.$$

Proof: From (1), we have

$$E(X \mid X \leq x)=\frac{1}{F(x)}\int_0^x u\{\alpha\beta\lambda u^{-\beta-1}(1+\lambda u^{-\beta})^{-2}[1-(1+\lambda u^{-\beta})^{-1}]^{\alpha-1}\}du. \tag{38}$$

Integrating by parts, taking "$\alpha\beta\lambda u^{-\beta-1}(1+\lambda u^{-\beta})^{-2}[1-(1+\lambda u^{-\beta})^{-1}]^{\alpha-1}$" as the part to be integrated and rest of the integrand for differentiation, we get

$$E(X \mid X \leq x)=\frac{1}{F(x)}[-x[1-(1+\lambda x^{-\beta})^{-1}]^{\alpha}+\int_0^x[1-(1+\lambda u^{-\beta})^{-1}]^{\alpha}du], \tag{39}$$

multiplying and dividing by $f(x)$ in (39), we have result given in (37).

To prove sufficient part, we have from (37)

$$\frac{1}{F(x)}\int_0^x uf(u)du=\frac{g(x)f(x)}{F(x)} \quad \text{or} \quad \int_0^x u\,f(u)du=g(x)f(x)\cdot \tag{40}$$

Differentiating (40) on both sides with respect to x, we find that

$$xf(x)=g'(x)f(x)+g(x)f'(x).$$

Therefore,

$$\frac{f'(x)}{f(x)}=\frac{x-g'(x)}{g(x)} \quad \text{(Ahsanullah et al. 2016)}$$

$$=-\frac{(\beta+1)}{x}+\frac{2\lambda\beta\ \ x^{-\beta-1}}{(1+\lambda x^{-\beta})}+\frac{\lambda\beta(1-\alpha)x^{-\beta-1}(1+\lambda x^{-\beta})^{-2}}{(1-(1+\lambda x^{-\beta})^{-1})}, \tag{41}$$

where

$$g'(x)=x-g(x)\left\{\frac{(\beta+1)}{x}+\frac{2\lambda\beta x^{-\beta-1}}{(1+\lambda x^{-\beta})}+\frac{\lambda\beta\ \ (1-\alpha)x^{-\beta-1}(1+\lambda x^{-\beta})^{-2}}{(1-(1+\lambda x^{-\beta})^{-1})}\right\}.$$

Integrating both the sides (41) with respect to x

$$f(x)=cx^{-\beta-1}(1+\lambda x^{-\beta})^{-2}[1-(1+\lambda x^{-\beta})^{-1}]^{\alpha-1}.$$

It is known that

$$\int_0^\infty f(x)dx=1.$$

Thus,

$$\frac{1}{c}=\int_0^\infty x^{-\beta-1}(1+\lambda x^{-\beta})^{-2}[1-(1+\lambda x^{-\beta})^{-1}]^{\alpha-1}dx=\frac{1}{\alpha\beta\lambda}.$$

which proves that

$$f(x)=\alpha\beta\lambda x^{-\beta-1}(1+\lambda x^{-\beta})^{-2}[1-(1+\lambda x^{-\beta})^{-1}]^{\alpha-1},\ \ x\ge 0,\ \ \beta>0\ \text{ and }\ \lambda>0.$$

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