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Construction of Second Order Slope Rotatable Designs under Tri-Diagonal Correlated Structure of Errors Using Balanced Incomplete Block Designs

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Abstract

In this paper, second order slope rotatable design (SOSRD) under tri-diagonal correlated structure of errors using balanced incomplete block designs (BIBD) is suggested by following the works of Das (2003a, 2003b). Further, the variance of the estimated slopes for different values of the tri-diagonal correlated coefficient ρ (–0.9 to 0.9) for "v factors 3 to 8" using BIBD is studied and observed that for some factors SOSRD under correlated structure of errors using BIBD has less number of design points than the corresponding SOSRD under tri-diagonal correlated structure of errors using central composite designs (CCD).

Keywords: Second order slope rotatable designs (SOSRD), tri-diagonal correlated errors.

1. Introduction

Box and Hunter (1957) introduced rotatable designs for the exploration of response surface designs. Das and Narasimham (1962) constructed second order rotatable designs (SORD) through balanced incomplete block designs (BIBD). Panda and Das (1994) introduced first order rotatable designs with correlated errors. Das (1997) introduced robust second order rotatable designs (RSORD). Das (1999, 2003a) studied RSORD.

In response surface methodology, good estimation of the derivatives of the response function may be as important or perhaps more important than estimation of mean response. Estimation of differences in responses at two different points in the factor space will often be of great importance. If difference in responses at two points close together is of interest then estimation of local slope (rate of change) of the response is required. Estimation of slopes occurs frequently in practical situations. For instance, there are cases in which we want to estimate rate of reaction in chemical experiment, rate of change in the yield of a crop to various fertilizer doses, rate of disintegration of radioactive material in animal etc., (Park 1987).

Hader and Park (1978) introduced slope rotatable central composite designs (SRCCD). Park (1987) studied a class of multifactor designs for estimating the slope of response surfaces. Victorbabu and Narasimham (1991) constructed second order slope rotatable design (SOSRD) using BIBD. Das (2003b) introduced slope rotatability with correlated errors. Das and Park (2009) studied measure of robust slope rotatability for second order response surface experimental designs. Das et al. (2010) suggested on D-optimal robust second order slope rotatable designs (RSOSRD). Rajyalakshmi and Victorbabu (2014) studied SOSRD under tri-diagonal correlated structure of errors using central composite designs (CCD). Rajyalakshmi and Victorbabu (2015) studied SOSRD under tri-diagonal correlated structure of errors using pairwise balanced designs. Rajyalakshmi and Victorbabu (2016) studied SORD under tri-diagonal correlated structure of errors using incomplete block designs. Rajyalakshmi and Victorbabu (2018) studied SOSRD under tri-diagonal correlated structure of errors using symmetrical unequal block arrangements with two unequal block sizes. In this paper following the works of Das (2003a, 2003b), SOSRD under tri-diagonal correlated structure of errors using BIBD is suggested. Further, the variance of the estimated slopes for different values of the tri-diagonal correlated coefficient ρ (–0.9 to 0.9) for "v factors 3 to 8" using BIBD is studied.

Tri-diagonal correlation structure: It is a covariance structure of errors which is a relaxation of intra-class structure or log model covariance structure of errors and is given by

$$w_0 = \left\{ D(e) = \sigma^2 \left[\begin{bmatrix} I_n & I_n \\ I_n & I_n \end{bmatrix} \times \frac{1+\rho}{2} + \begin{bmatrix} I_n & -I_n \\ -I_n & I_n \end{bmatrix} \times \frac{1-\rho}{2} \right] = W_{2n\times 2n}(\rho) \right\}, \quad \text{(i)}$$

$$W^{-1}_{2n\times 2n}(\rho) = (\sigma^2)^{-1} \left[\begin{bmatrix} I_n & I_n \\ I_n & I_n \end{bmatrix} \times \frac{1}{2(1+\rho)} + \begin{bmatrix} I_n & -I_n \\ -I_n & I_n \end{bmatrix} \times \frac{1}{2(1-\rho)} \right]. \quad \text{(ii)}$$

2. Conditions for SORD under Tri-Diagonal Correlated Structure of Errors (Das 2003a, 2003b and Das et al. 2010)

A second order response surface design $D = ((x_{iu}))$ for fitting

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i=1}^{v} \sum_{i \pounds j=1}^{v} b_{ij} x_{iu} x_{ju} + e_u, \quad (1)$$

where x_{iu} denotes the level of the i^{th} factor $(i = 1,2,...,v)$ in the u^{th} run $(u = 1,2,...,2n)$ of the experiment, e_u's are correlated random errors, is said to be a SORD under tri-diagonal correlated structure of errors, if the variance of the estimated response of $\hat{Y}_u$ from the fitted surface is only a function of the distance, $\left(d^2 = \sum x_{iu}^2\right)$ of the point $x_{1u}, x_{2u},..., x_{vu}$ from the origin (centre) of the design, i.e. $V[\hat{Y}_u] = g(d^2)$. Such a spherical variance function $g(d^2)$ for estimation of responses in the second order response surface is achieved if the design points satisfy the following conditions.

Here $b_{0,} b_i, b_{ii}, b_{ij}$ are the parameters of the model and Y_u is the response observed at u^{th} design point. The parameters in the response relation are estimated using the least squares technique. Further we impose the simple symmetry conditions on the design points to simplify the solutions of the normal equations.

$$\sum_{u=1}^{2n}\prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0 \text{ if any } \alpha_i \text{ is odd, for } \sum \alpha_i \le 4, \tag{2}$$

$$\sum_{u=1}^{2n} x_{iu}^2 = \text{constant} = 2n\lambda_2, \tag{3}$$

$$\sum_{u=1}^{2n} x_{iu}^4 = \text{constant} = 3(2n)\lambda_4, \text{ for all } i, \tag{4}$$

$$\sum_{u=1}^{2n} x_{iu}^2 x_{ju}^2 = \text{constant} = 2n\lambda_4, \text{ for all } i \ne j, \tag{5}$$

$$\sum_{u=1}^{2n} x_{iu}^4 = 3\sum_{u=1}^{2n} x_{iu}^2 x_{ju}^2, \tag{6}$$

$$\frac{\lambda_4}{\lambda_2^2} > \frac{v(1-\rho)}{(v+2)} \quad \text{(non-singularity condition)}, \tag{7}$$

where λ_2, λ_4 are constants. The summation is over the design points, and the correlated coefficient $\rho \in (-0.9, 0.9)$. If the non-singularity condition (7) exists then only the design exists.

Using the above solutions, the variances and covariances of the estimated parameters are,

$$V(\hat{b}_0) = \frac{\lambda_4(v+2)(1+\rho)\sigma^2}{2n\Delta},$$

$$V(\hat{b}_i) = \frac{\sigma^2(1-\rho^2)}{2n\lambda_2},$$

$$V(\hat{b}_{ij}) = \frac{\sigma^2(1-\rho^2)}{2n\lambda_4},$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2(1-\rho^2)[\lambda_4(v+1)-(v-1)\lambda_2^2(1-\rho)]}{2(2n)\lambda_4\Delta},$$

$$Cov(\hat{b}_0, \hat{b}_{ii}) = \frac{-\lambda_2\sigma^2(1-\rho^2)}{2n\Delta},$$

$$Cov(\hat{b}_{ii}, \hat{b}_{ij}) = \frac{\sigma^2(1-\rho^2)[\lambda_2^2(1-\rho)-\lambda_4]}{2(2n)\lambda_4\Delta}, \tag{8}$$

where $\Delta = [\lambda_4(V+2) - V\lambda_2^2(1-\rho)]$ and other covariances are zero.

3. Conditions for SOSRD under Tri-Diagonal Correlated Structure of Errors

A second order response surface design $D = ((x_{iu}))$ for fitting

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i=1}^{v}\sum_{i£j=1}^{v} b_{ij} x_{iu} x_{ju} + e_u, \tag{9}$$

where x_{iu} denotes the level of the i^{th} factor $(i = 1, 2, ..., v)$ in the u^{th} run $(u = 1, 2, ..., 2n)$ of the experiment, e_u's are correlated random errors, is said to be a SOSRD under tri-diagonal correlated structure of errors, if the variance of the estimate of first order partial derivative of $Y_u(x_1, x_2, ..., x_v)$ with respect to each of independent variable (x_i) is only a function of the distance $d^2 = \sum_{i=1}^{v} x_i^2$ of the point $(x_1, x_2, ..., x_v)$ from the origin (centre of the design), i.e. $V\left(\frac{\partial \widehat{Y}_u}{\partial x_i}\right) = h(d^2)$. Such a spherical variance function $h(d^2)$ for estimation of slopes in the second order response surface is achieved if the design points satisfy the following conditions.

Following Box and Hunter (1957), Hader and Park (1978) and Victorbabu and Narasimham (1991) the general conditions for second order slope rotatability can be obtained as follows. To simplify the fit of the second order polynomial from design points D through the method of least squares, we impose the simple symmetry conditions on D to facilitate easy solutions of the normal equations:

$$\sum_{u=1}^{2n} \prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0, \text{ if any } \alpha_i \text{ is odd, for } \sum \alpha_i \le 4, \quad (10)$$

$$\sum_{u=1}^{2n} x_{iu}^2 = \text{constant} = 2n\lambda_2, \quad (11)$$

$$\sum_{u=1}^{2n} x_{iu}^4 = \text{constant} = c(2n)\lambda_4, \text{ for all } i, \quad (12)$$

$$\sum_{u=1}^{2n} x_{iu}^2 x_{ju}^2 = \text{constant} = 2n\lambda_4, \text{ for all } i \ne j, \quad (13)$$

$$\sum_{u=1}^{2n} x_{iu}^4 = c \sum_{u=1}^{2n} x_{iu}^2 x_{ju}^2, \quad (14)$$

where c, λ_2 and λ_4 are constants and the summation is over the design points.

The variances and covariances of the estimated parameters are,

$$V(\widehat{b}_0) = \frac{\lambda_4 (c+v-1)(1+\rho)\sigma^2}{2n\Delta},$$

$$V(\widehat{b}_i) = \frac{\sigma^2(1-\rho^2)}{2n\lambda_2},$$

$$V(\widehat{b}_{ij}) = \frac{\sigma^2(1-\rho^2)}{2n\lambda_4},$$

$$V(\widehat{b}_{ii}) = \frac{\sigma^2(1-\rho^2)[\lambda_4(c+v-2)-(v-1)\lambda_2^2(1-\rho)]}{(c-1)(2n)\lambda_4\Delta},$$

$$Cov(\widehat{b}_0, \widehat{b}_{ii}) = \frac{-\lambda_2 \sigma^2 (1-\rho^2)}{2n\Delta},$$

$$Cov(\widehat{b}_{ii}, \widehat{b}_{ij}) = \frac{\sigma^2(1-\rho^2)[\lambda_2^2(1-\rho)-\lambda_4]}{(c-1)(2n)\lambda_4\Delta}, \quad (15)$$

where $\Delta = [\lambda_4(c+v-1) - v\lambda_2^2(1-\rho)]$ and other covariances are zero.

An inspection of the variance of $\hat{b}_0$ shows that a necessary condition for the existence of a non-singular second order slope rotatable design under tri-diagonal structure is

$$[\lambda_4(v+2)-v\lambda_2^2(1-\rho)]>0. \quad (16)$$

If the non-singularity condition (16) exists then only the design exists.

$$\frac{\lambda_4}{\lambda_2^2}>\frac{v(1-\rho)}{(c+v-1)} \text{ (non-singularity condition).} \quad (17)$$

For the second order model

$$\frac{\partial\widehat{y}_u}{\partial x_i}=\widehat{b}_i+2\widehat{b}_{ii}x_i+\sum_{i=1;j\neq i}^{v}\widehat{b}_{ij}x_j, \quad (18)$$

$$v\left(\frac{\partial\widehat{y}_u}{\partial x_i}\right)=v(\widehat{b}_i)+4x_i^2v(\widehat{b}_{ii})+\sum_{i=1;j\neq i}^{v}x_j^2v(\widehat{b}_{ij}). \quad (19)$$

The condition for right hand side of Equation (19) to be a function of $\left(d^2=\sum x_{iu}^2\right)$ alone (for slope rotatability) is clearly,

$$v(\widehat{b}_{ii})=\frac{1}{4}v(\widehat{b}_{ij}), \quad (20)$$

Equations (10) to (15) and (20) lead to condition,

$$\left(\frac{cN\lambda_4}{(1-\rho^2)(1+\rho)}\right)\left[4N-(c+v-2)N+v\left(\frac{N\lambda_2^2(1-\rho)}{\lambda_4}\right)\right],$$

$$\frac{N^2\lambda_4}{(1-\rho^2)(1+\rho)}[5v-9]-N^2\lambda_2^2\left[\frac{5v-4}{(1+\rho)^2}\right]=0, \quad (21)$$

where $N=2n$.

For $\rho=0$, Equation (21) reduces to

$$\lambda_4[v(5-c)-(c-3)^2]+\lambda_2^2[v(c-5)+4]=0. \quad (22)$$

This is similar to the SOSRD condition of Victorbabu and Narasimham (1991).

Therefore, Equations (10) to (15), (17) and (21) give a set of conditions for SOSRD under tri-diagonal correlated structure of errors for any general second order response surface design.

Further,

$$v\left(\frac{\partial\widehat{y}_u}{\partial x_i}\right)=\frac{(1-\rho^2)}{N}\left(\frac{1}{\lambda_2}+\frac{d^2}{\lambda_4}\right). \quad (23)$$

4. Construction of SOSRD under Tri-Diagonal Correlated Structure of Errors Using BIBD

Following Hader and Park (1978), Victorbabu and Narasimham (1991), Das (1997, 2003b), Das et al. (2010) methods of constructions, here a study on SOSRD under tri-diagonal correlated structure of errors using BIBD is studied.

Let (v,b,r,k,λ) denote a BIBD, where v is the number of factors, b is the number of blocks, r is the number of times replicated each treatment, k is the block size, λ is the pairwise replication of each treatment and $2^{t(k)}$ denote a fractional replicate of 2^k in +1 and −1 levels, in which no interaction with less than five factors is confounded. $[1-(v,b,r,k,\lambda)]$ denote the design points generated from the transpose of incidence matrix of BIBD. $[1-(v,b,r,k,\lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from the BIBD by "multiplication" (Raghavarao 1971), $(a,0,0,...,0)2^1$ denote the design points generated from $(a,0,0,...,0)$ point set, and $\cup$ denotes combination of the design points generated from different sets of points. n_0 denote the number of central points (Victorbabu and Narasimham 1991).

Consider SOSRD using BIBD (Victorbabu and Narasimham 1991) having ‘ n ’ $(n = b2^{t(k)} + 2v)$ non-central design points. The set of ‘ n ’-non central design points are extended to $2n$ design points by adding ‘ n ’ $(n_0 = n)$ central points just below or above the ‘ n ’ non-central design points. Hence $2n$ be the total number of design points of the SOSRD under tri-diagonal correlated structure of errors using BIBD.

The method of construction of SOSRD under tri-diagonal correlated structure of errors using BIBD is given in Theorem 1.

Theorem 1 *The design points,* $\left[1-(v,b,r,k,\lambda)\right]2^{t(k)} \cup (a,0,0,...,0)2^1 \cup n_0$ *will give a* v *-dimensional SOSRD under tri-diagonal correlated structure of errors using BIBD in* $N = 2n$ *design points, where* a^2 *is positive real root of the fourth degree polynomial equation,*

$$\begin{gathered}[8v(1-\rho)-4N]a^8 + [8vr2^{t(k)}(1-\rho)]a^6 + \\ [2vr^2 2^{2t(k)}(1-\rho) + \{((12-2v)\lambda - 4r)N + (16\lambda - 20v\lambda + 4vr)(1-\rho)\}2^{t(k)}]a^4 + \\ [4vr^2 + (16-20v)r\lambda]2^{2t(k)}(1-\rho)]a^2 + \\ [((5v-9)\lambda^2 + (6-v)r\lambda - r^2)2^{2t(k)}N + (vr + 4\lambda - 5v\lambda)r^2 2^{3t(k)}(1-\rho)] = 0. \end{gathered} \tag{24}$$

If at least one positive real root for a^2 *exists in Equation (24) then the design exists.*

Proof: For the design points generated from the BIBD, simple symmetry conditions are true. Further we have

$$\sum_{u=1}^{2n} x_{iu}^2 = r2^{t(k)} + 2a^2 = \text{constant} = 2n\lambda_2, \text{ for all } i, \tag{25}$$

$$\sum_{u=1}^{2n} x_{iu}^4 = r2^{t(k)} + 2a^4 = \text{constant} = c2n\lambda_4, \text{ for all } i, \tag{26}$$

$$\sum_{u=1}^{2n} x_{iu}^2 x_{ju}^2 = \lambda 2^{t(k)} = \text{constant} = 2n\lambda_4, \text{ for all } i \neq j. \tag{27}$$

Substituting λ_2, λ_4 and c in Equation (21) and on simplification, we get Equation (24). The design exists only if at least one positive real root exits for Equation (24). Solving Equation (24) we get the SOSRD under tri-diagonal correlated structure of errors using BIBD.

Example 1

We illustrate the above method with construction of SOSRD under tri-diagonal correlated structure of errors for 3-factors with the help of a BIBD with parameters $(v=3, b=3, r=2, k=2, \lambda=1)$.

The design points $\left[1-(3,3,2,2,1)\right]2^3 \cup (a,0,0,...,0)2^1 \cup (n_0 = 18)$ will give a SOSRD under tri-diagonal correlated structure of errors using BIBD in $N = 2n = 36$ design points for three factors.

$$\sum_{u=1}^{2n} x_{iu}^2 = 8 + 2a^2 = \text{constant} = 2n\lambda_2, \tag{28}$$

$$\sum_{u=1}^{2n} x_{iu}^4 = 8 + 2a^4 = \text{constant} = c2n\lambda_4, \tag{29}$$

$$\sum_{u=1}^{2n} x_{iu}^2 x_{ju}^2 = 4 = \text{constant} = 2n\lambda_4. \tag{30}$$

From Equations (29) and (30), we get $c = \frac{8+2a^4}{8}$. Substituting for λ_2, λ_4 and c in Equation (24) and on simplification, we get the following different biquadratic equations for each value of ρ in a^2 (viz.)

$$[24(1-\rho)-144]a^8 + [192(1-\rho)]a^6 + [384(1-\rho) + \{-72-20(1-\rho)\}X^4]a^4$$
$$[48-88]2^4(1-\rho)]a^2 + [4608-1280(1-\rho)] = 0 \tag{31}$$

Equation (31) has at least one positive real root for each value of ρ, $a^2 = 2.4687$ (by taking $\rho = 0.1$). It can be verified that Equation (17) is also satisfied. This can be alternatively written directly from Equation (24). Solving Equation (24), we get $a = 1.5712$ (by taking $\rho = 0.1$) Substituting 'a' value in Equations (28), (29) and (30) we obtain $\lambda_2 = 0.3594, \lambda_4 = 0.1111$ and $c = 5.0472.$ From Equation (15), we can obtain the variances and covariances. Further from Equation (23), we have, $v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right) = (0.0765 + 0.2475d^2)\sigma^2$ (at $\rho = 0.1$).

Example 2

We illustrate the above method with construction of SOSRD under tri-diagonal correlated structure of errors for 7-factors with the help of a BIBD with parameters $(v = 7, b = 7, r = 3, k = 3, \lambda = 1)$.

The design points, $[1-(7,7,3,3,1)]2^3 \cup (a,0,0,...,0)2^1 \cup (n_0 = 70)$ will give a SOSRD under tri-diagonal correlated structure of errors using BIBD in $N = 2n = 141$ design points for seven factors.

$$\sum_{u=1}^{2n} x_{iu}^2 = 24 + 2a^2 = N^*\lambda_2, \tag{32}$$

$$\sum_{u=1}^{2n} x_{iu}^2 = 24 + 2a^4 = \text{constant} = cN^*\lambda_4, \tag{33}$$

$$\sum_{u=1}^{2n} x_{iu}^2 x_{ju}^2 = 8 = \text{constant} = N^*\lambda_4. \tag{34}$$

From Equations (33) and (34), we get $c = \frac{24+2a^4}{8}$. Substituting for λ_2, λ_4 and c in Equation (24) and on simplification, we get the following different biquadratic equations for each value of ρ in a^2 (viz.)

$$[56(1-\rho)-560]a^8 + [1344(1-\rho)]a^6 + [8064(1-\rho) + \{-1960-20(1-\rho)\}X^8]a^4$$
$$[252-372]2^6(1-\rho)]a^2 + [125440-46080(1-\rho)] = 0 \tag{35}$$

Equation (35) has at least one positive real root for each value of ρ, $a^2 = 2.6781$. (by taking $\rho = 0.1$). It can be verified that Equation (17) is also satisfied. This can be alternatively written directly from equation (24). Solving Equation (24), we get $a = 1.6365$ (by taking $\rho = 0.1$). Substituting ' a ' value in Equations (32), (33) and (34) we obtain $\lambda_2 = 0.2097$, $\lambda_4 = 0.0571$ and $c = 4.7931$. From Equation (15), we can obtain the variances and covariances. Further from Equation (23), we have, $v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right) = (0.0337 + 0.1238d^2)\sigma^2$ (at $\rho = 0.1$).

The variances of estimated slopes of these SOSRD under tri-diagonal correlated structure of errors using BIBD for $-0.9 \le \rho \le 0.9$ and for " v factors 3 to 8"are given in Appendix.

We may point out here that this SOSRD under tri-diagonal correlated structure of errors using BIBD with parameters $(v = 7, b = 7, r = 3, k = 3, \lambda = 1)$ has only 140 design points for 7 factors, whereas the corresponding SOSRD under tri-diagonal correlated structure of errors using CCD needs 156 design points. Thus the method leads to a 7-factor SOSRD under tri-diagonal correlated structure of errors using BIBD in less number of design points than the corresponding SOSRD under tri-diagonal correlated structure of errors using CCD.

5. Conclusions

From Appendix Table 1 we observed that,

1. When the values of ' ρ ' increases slope rotatability value of " a " is decreases for all the factors 3 to 8.
2. We observed that the slope rotatability value of " a " at $\rho = -1$ which is equal to the SOSRD with errors are uncorrelated and homoscedastic estimated value at "0" central points.
3. At $\rho = 0$ estimated value and slope rotatability derivative of SOSRD under tri-diagonal correlated structure is equal to the SOSRD uncorrelated errors case.
4. We may point out here that this SOSRD under tri-diagonal correlated structure of errors using BIBD in some cases leads to designs with less number of design points compared to designs constructed with the help of CCD.
5. We may also pointed out that for some factors up to some values of ' ρ ' only there exists at least one positive real root for other factors it does not. So, we are providing the estimated responses for the existed ρ only. For other factors we put a symbol 'dash'.

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APPENDIX

Table 1

Table 1.1 The variance of estimated derivatives (slopes) for the factor $3 \le v \le 8$ factors using BIBD

ρ	Balanced incomplete block designs					
	$v=3, b=3, r=2, k=2, \lambda=1$ $n=18, N=2n=36$		$v=4, b=6, r=3, k=2, \lambda=1$ $n=32, N=2n=64$		$v=4, b=4, r=3, k=3, \lambda=1$ $n=40, N=2n=80$	
	$\hat{a}$	$v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$	$\hat{a}$	$v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$	$\hat{a}$	$v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$
-0.9	2.0038	$0.0119\sigma^2+0.0475\sigma^2d^2$	1.8992	$0.0099\sigma^2+0.0475\sigma^2d^2$	2.6120	$0.0050\sigma^2+0.0119\sigma^2d^2$
-0.8	1.9330	$0.0233\sigma^2+0.0900\sigma^2d^2$	1.8101	$0.0194\sigma^2+0.0900\sigma^2d^2$	2.5492	$0.0097\sigma^2+0.0225\sigma^2d^2$
-0.7	1.8674	$0.0341\sigma^2+0.1275\sigma^2d^2$	1.7262	$0.0284\sigma^2+0.1275\sigma^2d^2$	2.4943	$0.0140\sigma^2+0.0319\sigma^2d^2$
-0.6	1.8083	$0.0440\sigma^2+0.1600\sigma^2d^2$	1.6516	$0.0367\sigma^2+0.1600\sigma^2d^2$	2.4474	$0.0178\sigma^2+0.0400\sigma^2d^2$
-0.5	1.7563	$0.0529\sigma^2+0.1875\sigma^2d^2$	1.5886	$0.0440\sigma^2+0.1875\sigma^2d^2$	2.4079	$0.0211\sigma^2+0.0469\sigma^2d^2$
-0.4	1.7115	$0.0606\sigma^2+0.2100\sigma^2d^2$	1.5374	$0.0502\sigma^2+0.2100\sigma^2d^2$	2.3750	$0.0238\sigma^2+0.0525\sigma^2d^2$
-0.3	1.6734	$0.0669\sigma^2+0.2275\sigma^2d^2$	1.4963	$0.0552\sigma^2+0.2275\sigma^2d^2$	2.3476	$0.0260\sigma^2+0.0569\sigma^2d^2$
-0.2	1.6411	$0.0717\sigma^2+0.2400\sigma^2d^2$	1.4633	$0.0590\sigma^2+0.2400\sigma^2d^2$	2.3247	$0.0276\sigma^2+0.0600\sigma^2d^2$
-0.1	1.6139	$0.0749\sigma^2+0.2475\sigma^2d^2$	1.4364	$0.0614\sigma^2+0.2475\sigma^2d^2$	2.3055	$0.0286\sigma^2+0.0619\sigma^2d^2$
0	1.5909	$0.0766\sigma^2+0.2500\sigma^2d^2$	1.4142	$0.0625\sigma^2+0.2500\sigma^2d^2$	2.2892	$0.0290\sigma^2+0.0625\sigma^2d^2$
0.1	1.5712	$0.0765\sigma^2+0.2475\sigma^2d^2$	1.3957	$0.0623\sigma^2+0.2475\sigma^2d^2$	2.2752	$0.0288\sigma^2+0.0619\sigma^2d^2$
0.2	1.5543	$0.0748\sigma^2+0.2400\sigma^2d^2$	1.3800	$0.0607\sigma^2+0.2400\sigma^2d^2$	2.2632	$0.0280\sigma^2+0.0600\sigma^2d^2$
0.3	1.5397	$0.0714\sigma^2+0.2275\sigma^2d^2$	1.3665	$0.0578\sigma^2+0.2275\sigma^2d^2$	2.2528	$0.0266\sigma^2+0.0569\sigma^2d^2$
0.4	1.5270	$0.0663\sigma^2+0.2100\sigma^2d^2$	1.3548	$0.0536\sigma^2+0.2100\sigma^2d^2$	2.2437	$0.0247\sigma^2+0.0525\sigma^2d^2$
0.5	1.5158	$0.0595\sigma^2+0.1875\sigma^2d^2$	1.3446	$0.0480\sigma^2+0.1875\sigma^2d^2$	2.2357	$0.0221\sigma^2+0.0469\sigma^2d^2$
0.6	1.5060	$0.0511\sigma^2+0.1600\sigma^2d^2$	1.3356	$0.0411\sigma^2+0.1600\sigma^2d^2$	2.2286	$0.0189\sigma^2+0.0400\sigma^2d^2$
0.7	1.4972	$0.0409\sigma^2+0.1275\sigma^2d^2$	1.3276	$0.0329\sigma^2+0.1275\sigma^2d^2$	2.2222	$0.0151\sigma^2+0.0319\sigma^2d^2$
0.8	1.4893	$0.0289\sigma^2+0.0900\sigma^2d^2$	1.3204	$0.0232\sigma^2+0.0900\sigma^2d^2$	2.2165	$0.0106\sigma^2+0.0225\sigma^2d^2$
0.9	1.4823	$0.0153\sigma^2+0.0475\sigma^2d^2$	1.3140	$0.0123\sigma^2+0.0475\sigma^2d^2$	2.2114	$0.0056\sigma^2+0.0119\sigma^2d^2$

Table 1.2 The variance of estimated derivatives (slopes) for the factor $3 \le v \le 8$ factors using BIBD

ρ	Balanced incomplete block designs					
	$v=5, b=10, r=4, k=2, \lambda=1$ $n=50, N=2n=100$		$v=5, b=5, r=4, k=4, \lambda=3$ $n=90, N=2n=180$		$v=5, b=10, r=6, k=3, \lambda=3$ $n=40, N=2n=80$	
	$\hat{a}$	$v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$	$\hat{a}$	$v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$	$\hat{a}$	$v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$
-0.9	1.8116	$0.0084\sigma^2+0.0475\sigma^2d^2$	3.2792	$0.0022\sigma^2+0.0040\sigma^2d^2$	2.7192	$0.0030\sigma^2+0.0079\sigma^2d^2$
-0.8	1.6875	$0.0166\sigma^2+0.0900\sigma^2d^2$	3.2263	$0.0042\sigma^2+0.0075\sigma^2d^2$	2.6566	$0.0058\sigma^2+0.0150\sigma^2d^2$
-0.7	1.5641	$0.0244\sigma^2+0.1275\sigma^2d^2$	3.1820	$0.0061\sigma^2+0.0106\sigma^2d^2$	2.6034	$0.0083\sigma^2+0.0213\sigma^2d^2$
-0.6	1.4553	$0.0316\sigma^2+0.1600\sigma^2d^2$	3.1455	$0.0076\sigma^2+0.0133\sigma^2d^2$	2.5593	$0.0105\sigma^2+0.0267\sigma^2d^2$
-0.5	1.3692	$0.0380\sigma^2+0.1875\sigma^2d^2$	3.1154	$0.0090\sigma^2+0.0156\sigma^2d^2$	2.5232	$0.0123\sigma^2+0.0313\sigma^2d^2$
-0.4	1.3035	$0.0433\sigma^2+0.2100\sigma^2d^2$	3.0905	$0.0101\sigma^2+0.0175\sigma^2d^2$	2.4937	$0.0139\sigma^2+0.0350\sigma^2d^2$
-0.3	1.2530	$0.0475\sigma^2+0.2275\sigma^2d^2$	3.0699	$0.0110\sigma^2+0.0190\sigma^2d^2$	2.4695	$0.0151\sigma^2+0.0379\sigma^2d^2$
-0.2	1.2131	$0.0507\sigma^2+0.2400\sigma^2d^2$	3.0525	$0.0116\sigma^2+0.0200\sigma^2d^2$	2.4495	$0.0160\sigma^2+0.0400\sigma^2d^2$
-0.1	1.1807	$0.0527\sigma^2+0.2475\sigma^2d^2$	3.0379	$0.0120\sigma^2+0.0206\sigma^2d^2$	2.4328	$0.0165\sigma^2+0.0413\sigma^2d^2$
0	1.1537	$0.05360\sigma^2+0.2500\sigma^2d^2$	3.0253	$0.0121\sigma^2+0.0208\sigma^2d^2$	2.4187	$0.0168\sigma^2+0.0417\sigma^2d^2$
0.1	1.1309	$0.0533\sigma^2+0.2475\sigma^2d^2$	3.0146	$0.0120\sigma^2+0.0206\sigma^2d^2$	2.4067	$0.0166\sigma^2+0.0413\sigma^2d^2$
0.2	1.1113	$0.0520\sigma^2+0.2400\sigma^2d^2$	3.0052	$0.0117\sigma^2+0.0200\sigma^2d^2$	2.3963	$0.0161\sigma^2+0.0400\sigma^2d^2$
0.3	1.0941	$0.0495\sigma^2+0.2275\sigma^2d^2$	2.9970	$0.0111\sigma^2+0.0190\sigma^2d^2$	2.3874	$0.0153\sigma^2+0.0379\sigma^2d^2$
0.4	1.0790	$0.0458\sigma^2+0.2100\sigma^2d^2$	2.9898	$0.0103\sigma^2+0.0175\sigma^2d^2$	2.3795	$0.0142\sigma^2+0.0350\sigma^2d^2$
0.5	1.0656	$0.0410\sigma^2+0.1875\sigma^2d^2$	2.9834	$0.0092\sigma^2+0.0156\sigma^2d^2$	2.3726	$0.0127\sigma^2+0.0313\sigma^2d^2$
0.6	1.0535	$0.0351\sigma^2+0.1600\sigma^2d^2$	2.9776	$0.0078\sigma^2+0.0133\sigma^2d^2$	2.3665	$0.0108\sigma^2+0.0267\sigma^2d^2$
0.7	1.0426	$0.0281\sigma^2+0.1275\sigma^2d^2$	2.9725	$0.0062\sigma^2+0.0106\sigma^2d^2$	2.3610	$0.0086\sigma^2+0.0213\sigma^2d^2$
0.8	1.0326	$0.0199\sigma^2+0.0900\sigma^2d^2$	2.9679	$0.0044\sigma^2+0.0075\sigma^2d^2$	2.3561	$0.0061\sigma^2+0.0150\sigma^2d^2$
0.9	1.0236	$0.0105\sigma^2+0.0475\sigma^2d^2$	2.9636	$0.0023\sigma^2+0.0040\sigma^2d^2$	2.3516	$0.0032\sigma^2+0.0079\sigma^2d^2$

Table 1.3 The variance of estimated derivatives (slopes) for the factor $3 \le v \le 8$ factors using BIBD

ρ	Balanced incomplete block designs					
	$v=6, b=15, r=5, k=2, \lambda=1$ $n=72, N=2n=144$		$v=6, b=10, r=5, k=3, \lambda=2$ $n=92, N=2n=184$		$v=6, b=6, r=5, k=5, \lambda=4$ $n=108, N=2n=216$	
	$\hat{a}$	$v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$	$\hat{a}$	$v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$	$\hat{a}$	$v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$
-0.9	1.7262	$0.0073\sigma^2+0.0475\sigma^2 d^2$	2.4443	$0.0037\sigma^2+0.0119\sigma^2 d^2$	3.5157	$0.0018\sigma^2+0.0030\sigma^2 d^2$
-0.8	1.5305	$0.0146\sigma^2+0.0900\sigma^2 d^2$	2.3554	$0.0070\sigma^2+0.0225\sigma^2 d^2$	3.4568	$0.0035\sigma^2+0.0056\sigma^2 d^2$
-0.7	1.2910	$0.0219\sigma^2+0.1275\sigma^2 d^2$	2.2833	$0.0101\sigma^2+0.0319\sigma^2 d^2$	3.4103	$0.0049\sigma^2+0.0080\sigma^2 d^2$
-0.6	1.0586	$0.0288\sigma^2+0.1600\sigma^2 d^2$	2.2278	$0.0128\sigma^2+0.0400\sigma^2 d^2$	3.374	$0.0062\sigma^2+0.0100\sigma^2 d^2$
-0.5	0.8438	$0.0350\sigma^2+0.1875\sigma^2 d^2$	2.1858	$0.0151\sigma^2+0.0469\sigma^2 d^2$	3.3455	$0.0073\sigma^2+0.0117\sigma^2 d^2$
-0.4	-	-	2.1538	$0.0170\sigma^2+0.0525\sigma^2 d^2$	3.3228	$0.0082\sigma^2+0.0131\sigma^2 d^2$
-0.3	-	-	2.1291	$0.0185\sigma^2+0.0569\sigma^2 d^2$	3.3045	$0.0089\sigma^2+0.0142\sigma^2 d^2$
-0.2	-	-	2.1095	$0.0196\sigma^2+0.0600\sigma^2 d^2$	3.2894	$0.0094\sigma^2+0.0150\sigma^2 d^2$
-0.1	-	-	2.0938	$0.0203\sigma^2+0.0619\sigma^2 d^2$	3.2770	$0.0098\sigma^2+0.0155\sigma^2 d^2$
0	-	-	2.0810	$0.0206\sigma^2+0.0625\sigma^2 d^2$	3.2665	$0.0099\sigma^2+0.0156\sigma^2 d^2$
0.1	-	-	2.0702	$0.0204\sigma^2+0.0619\sigma^2 d^2$	3.2576	$0.0098\sigma^2+0.0155\sigma^2 d^2$
0.2	-	-	2.0612	$0.0198\sigma^2+0.0600\sigma^2 d^2$	3.2499	$0.0095\sigma^2+0.0150\sigma^2 d^2$
0.3	-	-	2.0535	$0.0188\sigma^2+0.0569\sigma^2 d^2$	3.2432	$0.0090\sigma^2+0.0142\sigma^2 d^2$
0.4	-	-	2.0468	$0.0174\sigma^2+0.0525\sigma^2 d^2$	3.2374	$0.0083\sigma^2+0.0131\sigma^2 d^2$
0.5	-	-	2.0410	$0.0155\sigma^2+0.0469\sigma^2 d^2$	3.2322	$0.0074\sigma^2+0.0117\sigma^2 d^2$
0.6	-	-	2.0358	$0.0133\sigma^2+0.0400\sigma^2 d^2$	3.2277	$0.0063\sigma^2+0.0100\sigma^2 d^2$
0.7	-	-	2.0313	$0.0106\sigma^2+0.0319\sigma^2 d^2$	3.2236	$0.0051\sigma^2+0.0080\sigma^2 d^2$
0.8	-	-	2.0272	$0.0075\sigma^2+0.0225\sigma^2 d^2$	3.2199	$0.0036\sigma^2+0.0056\sigma^2 d^2$
0.9	-	-	2.0236	$0.0039\sigma^2+0.0119\sigma^2 d^2$	3.2166	$0.0019\sigma^2+0.0030\sigma^2 d^2$

Table 1.4 The variance of estimated derivatives (slopes) for the factor $3 \le v \le 8$ factors using BIBD

ρ	Balanced incomplete block designs					
	$v=6, b=15, r=10, k=4, \lambda=6$ $n=252, N=2n=504$		$v=7, b=7, r=3, k=3, \lambda=1$ $n=70, N=2n=140$		$v=7, b=7, r=4, k=4, \lambda=2$ $n=126, N=2n=252$	
	$\hat{a}$	$v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$	$\hat{a}$	$v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$	$\hat{a}$	$v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$
-0.9	3.7187	$0.0010\sigma^2+0.0020\sigma^2 d^2$	2.1167	$0.0058\sigma^2+0.0238\sigma^2 d^2$	2.8848	$0.0024\sigma^2+0.0059\sigma^2 d^2$
-0.8	3.6718	$0.0019\sigma^2+0.0038\sigma^2 d^2$	1.9687	$0.0113\sigma^2+0.0450\sigma^2 d^2$	2.8046	$0.0045\sigma^2+0.0113\sigma^2 d^2$
-0.7	3.6333	$0.0027\sigma^2+0.0053\sigma^2 d^2$	1.8585	$0.0165\sigma^2+0.0638\sigma^2 d^2$	2.7448	$0.0065\sigma^2+0.0159\sigma^2 d^2$
-0.6	3.6019	$0.0034\sigma^2+0.0067\sigma^2 d^2$	1.7873	$0.0211\sigma^2+0.0800\sigma^2 d^2$	2.7014	$0.0081\sigma^2+0.0200\sigma^2 d^2$
-0.5	3.5761	$0.0040\sigma^2+0.0078\sigma^2 d^2$	1.7412	$0.0249\sigma^2+0.0938\sigma^2 d^2$	2.6695	$0.0096\sigma^2+0.0234\sigma^2 d^2$
-0.4	3.5548	$0.0045\sigma^2+0.0088\sigma^2 d^2$	1.7096	$0.0281\sigma^2+0.1050\sigma^2 d^2$	2.6455	$0.0108\sigma^2+0.0263\sigma^2 d^2$
-0.3	3.5370	$0.0049\sigma^2+0.0095\sigma^2 d^2$	1.6868	$0.0306\sigma^2+0.1138\sigma^2 d^2$	2.6270	$0.0117\sigma^2+0.0284\sigma^2 d^2$
-0.2	3.5220	$0.0052\sigma^2+0.0100\sigma^2 d^2$	1.6696	$0.0325\sigma^2+0.1200\sigma^2 d^2$	2.6125	$0.0124\sigma^2+0.0300\sigma^2 d^2$
-0.1	3.5092	$0.0054\sigma^2+0.0103\sigma^2 d^2$	1.6562	$0.0336\sigma^2+0.1238\sigma^2 d^2$	2.6007	$0.0128\sigma^2+0.0309\sigma^2 d^2$
0	3.4983	$0.0054\sigma^2+0.0104\sigma^2 d^2$	1.6454	$0.0340\sigma^2+0.1250\sigma^2 d^2$	2.5911	$0.0129\sigma^2+0.0313\sigma^2 d^2$
0.1	3.4888	$0.0054\sigma^2+0.0103\sigma^2 d^2$	1.6365	$0.0337\sigma^2+0.1238\sigma^2 d^2$	2.5831	$0.0128\sigma^2+0.0309\sigma^2 d^2$
0.2	3.4805	$0.0052\sigma^2+0.0100\sigma^2 d^2$	1.6291	$0.0328\sigma^2+0.1200\sigma^2 d^2$	2.5763	$0.0124\sigma^2+0.0300\sigma^2 d^2$
0.3	3.4732	$0.0049\sigma^2+0.0095\sigma^2 d^2$	1.6228	$0.0311\sigma^2+0.1138\sigma^2 d^2$	2.5704	$0.0118\sigma^2+0.0284\sigma^2 d^2$
0.4	3.4667	$0.0046\sigma^2+0.0088\sigma^2 d^2$	1.6175	$0.0287\sigma^2+0.1050\sigma^2 d^2$	2.5654	$0.0109\sigma^2+0.0263\sigma^2 d^2$
0.5	3.4610	$0.0041\sigma^2+0.0078\sigma^2 d^2$	1.6128	$0.0257\sigma^2+0.0938\sigma^2 d^2$	2.5610	$0.0097\sigma^2+0.0234\sigma^2 d^2$
0.6	3.4558	$0.0035\sigma^2+0.0067\sigma^2 d^2$	1.6087	$0.0219\sigma^2+0.0800\sigma^2 d^2$	2.5571	$0.0083\sigma^2+0.0200\sigma^2 d^2$
0.7	3.4511	$0.0028\sigma^2+0.0053\sigma^2 d^2$	1.6051	$0.0175\sigma^2+0.0638\sigma^2 d^2$	2.5537	$0.0066\sigma^2+0.0159\sigma^2 d^2$
0.8	3.4469	$0.0020\sigma^2+0.0038\sigma^2 d^2$	1.6018	$0.0124\sigma^2+0.0450\sigma^2 d^2$	2.5506	$0.0047\sigma^2+0.0113\sigma^2 d^2$
0.9	3.4431	$0.0010\sigma^2+0.0020\sigma^2 d^2$	1.5989	$0.0065\sigma^2+0.0238\sigma^2 d^2$	2.5479	$0.0025\sigma^2+0.0059\sigma^2 d^2$

Table 1.5 The variance of estimated derivatives (slopes) for the factor $3 \le v \le 8$ factors using BIBD

ρ	Balanced incomplete block designs					
	$v=7, b=7, r=6, k=6, \lambda=5$ $n=238, N=2n=476$		$v=7, b=21, r=6, k=2, \lambda=1$ $n=98, N=2n=196$		$v=8, b=14, r=7, k=4, \lambda=3$ $n=240, N=2n=480$	
	$\hat{a}$	$v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$	$\hat{a}$	$v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$	$\hat{a}$	$v\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$
-0.9	4.3137	$0.0008\sigma^2+0.0012\sigma^2 d^2$	1.6312	$0.0065\sigma^2+0.0475\sigma^2 d^2$	3.0546	$0.0015\sigma^2+0.0040\sigma^2 d^2$
-0.8	4.2670	$0.0016\sigma^2+0.0023\sigma^2 d^2$	-	-	2.9763	$0.0028\sigma^2+0.0075\sigma^2 d^2$
-0.7	4.2307	$0.0022\sigma^2+0.0032\sigma^2 d^2$	-	-	2.9198	$0.0040\sigma^2+0.0106\sigma^2 d^2$
-0.6	4.2022	$0.0028\sigma^2+0.0040\sigma^2 d^2$	-	-	2.8792	$0.0050\sigma^2+0.0133\sigma^2 d^2$
-0.5	4.1796	$0.0033\sigma^2+0.0047\sigma^2 d^2$	-	-	2.8495	$0.0058\sigma^2+0.0156\sigma^2 d^2$
-0.4	4.1614	$0.0037\sigma^2+0.0053\sigma^2 d^2$	-	-	2.8272	$0.0066\sigma^2+0.0175\sigma^2 d^2$
-0.3	4.1465	$0.0040\sigma^2+0.0057\sigma^2 d^2$	-	-	2.8100	$0.0071\sigma^2+0.0190\sigma^2 d^2$
-0.2	4.1341	$0.0042\sigma^2+0.0060\sigma^2 d^2$	-	-	2.7963	$0.0075\sigma^2+0.0200\sigma^2 d^2$
-0.1	4.1236	$0.0044\sigma^2+0.0062\sigma^2 d^2$	-	-	2.7853	$0.0078\sigma^2+0.0206\sigma^2 d^2$
0	4.1147	$0.0044\sigma^2+0.0063\sigma^2 d^2$	-	-	2.7762	$0.0078\sigma^2+0.0208\sigma^2 d^2$
0.1	4.1071	$0.0044\sigma^2+0.0062\sigma^2 d^2$	-	-	2.7686	$0.0078\sigma^2+0.0206\sigma^2 d^2$
0.2	4.1005	$0.0043\sigma^2+0.0060\sigma^2 d^2$	-	-	2.7622	$0.0075\sigma^2+0.0200\sigma^2 d^2$
0.3	4.0946	$0.0040\sigma^2+0.0057\sigma^2 d^2$	-	-	2.7567	$0.0072\sigma^2+0.0190\sigma^2 d^2$
0.4	4.0895	$0.0037\sigma^2+0.0053\sigma^2 d^2$	-	-	2.7519	$0.0066\sigma^2+0.0175\sigma^2 d^2$
0.5	4.0849	$0.0033\sigma^2+0.0047\sigma^2 d^2$	-	-	2.7477	$0.0059\sigma^2+0.0156\sigma^2 d^2$
0.6	4.0808	$0.0028\sigma^2+0.0040\sigma^2 d^2$	-	-	2.7440	$0.0050\sigma^2+0.0133\sigma^2 d^2$
0.7	4.0772	$0.0023\sigma^2+0.0032\sigma^2 d^2$	-	-	2.7407	$0.0040\sigma^2+0.0106\sigma^2 d^2$
0.8	4.0738	$0.0016\sigma^2+0.0023\sigma^2 d^2$	-	-	2.7377	$0.0028\sigma^2+0.0075\sigma^2 d^2$
0.9	4.0708	$0.0008\sigma^2+0.0012\sigma^2 d^2$	-	-	2.7351	$0.0015\sigma^2+0.0040\sigma^2 d^2$