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A Size Biased Gamma Lindley Distribution

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Abstract

The present paper offers a new extension to the gamma Lindley distribution called size-biased gamma Lindley distribution (SBGaL). Several properties of this distribution such as moment method, maximum likelihood estimation, and limiting distribution of extreme order statistics are established. A simulation study is carried out to examine the bias and mean square error of the maximum likelihood estimators of the parameters. Finally, an application of the model to a real data set is presented and compared with the fit attained by some other well-known two (three)-parameter distributions.

Keywords: Size-biased distributions, Lindley distribution, maximum-likelihood estimation, simulation, goodness of fit.

1. Introduction

The size-biased distributions arise when the observations generated from a random process do not have equal probability of being recorded and are recorded according to some weight function. When the sampling mechanism is such that the sample units are selected with probability proportional to some measure of the unit size, the resulting distribution is called 'size-biased distribution'. Fisher (1934) first introduced such distributions to model ascertainment bias. Let X be a random variable with probability density function (PDF) with unknown parameter, then the corresponding weighted distribution function is given by

$$f^{\alpha}(x;\theta)=\frac{w(x)f(x;\theta)}{E(w(X))},$$

where $w(x)$ is a non-negative weight function such that $E[w(x)]$ exists. A special case of the weighted distributions, size-biased distributions is proposed by Rao (1965) when the weighted function has the form $w(x)=x^{\beta}$ which is called as size-biased distributions of order β, when $\beta=1$ or $\beta=2$, which are called length-biased and area-biased, respectively. Therefore, the PDF of the length biased distribution is defined by

$$f_L(x;\theta)=\frac{xf(x;\theta)}{E(X)}, \quad -\infty<x<\infty. \tag{1}$$

Patil and Ord (1975) studied the size-biased sampling and the related form-invariant weighted distribution whereas Van Deusen (1986) arrived at size-biased distribution theory independently and applied it to fitting distributions of diameter at breast height (DBH) data arising from horizontal point sampling (HPS). Later, Lappi and Bailey (1987) analyzed HPS diameter increment data using size-biased distribution. Patil and Rao (1977, 1978) examined some general models leading to size-biased distributions. The results were applied to the analysis of data relating to human populations and wild life management. Gove (2003) reviewed some of the recent results on size-biased distributions pertaining to parameter estimation in forestry with special emphasis on Weibull distribution. Simoj and Maya (2006) introduced some fundamental relationships between weighted and unique variables in the context of maintain ability function and inverted repair rate. Mir and Ahmad (2009), Das and Roy (2011) and Ducey and Gove (2015) have also studied the various aspects of size-biased distributions.

Recently, Zeghdoudi and Nedjar (2016a, 2016b) introduced a new distribution, named gamma Lindley distribution, based on mixtures of Gamma$(2,\theta)$ and one-parameter Lindley distributions. The idea using a mixture of two known distribution to generate a new distribution is not new. However, it is the first time in the literature to use a mixture of Gamma$(2,\theta)$ and one-parameter Lindley distribution to generate gamma Lindley distribution, where the density function of the random variable X is given by

$$f_{GaL}(x;\theta,\beta)=\begin{cases}\dfrac{\theta^2((\beta+\beta\theta-\theta)x+1)e^{-\theta x}}{\beta(1+\theta)}, & x,\theta>0 \text{ and } \beta>\dfrac{\theta}{1+\theta}\\ 0, \text{ otherwise.}\end{cases}$$

Using the $f(x;\theta,\beta)$ in (1), we find the new sized-based distribution called size-biased gamma Lindley distribution (SBGaL) where we devoted some properties and simulation.

The SBGaL distribution is motivated by the following: the SBGaL distribution use may be restricted to the tail of a distribution, but it is easy to apply. The formulas of the mean, variance, mean deviation, entropy and the quantile function are simple in form and may be used as quick approximations in many cases. Also, the SBGaL distribution can be viewed as a special case of size-biased Lindley distribution introduced by Ayesha (2017).

The main advantage of using sized-based distributions appears when the sample is recorded with unequal probabilities. Accordingly, the superiority of the SBGaL distribution is illustrated to ball bearings data. It is shown that the SBGaL is the most appropriate model for this data set as compared to others distributions. We believe that the SBGaL is an alternative distribution to lifetime data analysis.

The paper is organized as follows. In Section 2, we introduce the SBGaL distribution, and give immediate properties as the mode, cumulative, survival and hazard rate functions, plots of the density and cumulative functions for some parameter values. Section 3 deals on the moments, Lorenz curve, and extreme order statistics. In Section 4, we are interested in parameters estimation using both the maximum likelihood and the moment method. In this last section, simulation studies are reported and as well, are provided data driven applications allowing comparisons between others distributions. We finish the paper with a conclusive section.

2. Size-Biased Gamma Lindley Distribution (SBGaL) and Some Properties

In this section, we give the size-biased gamma Lindley distribution and study its properties. Let X be a random variable with PDF and CDF given by

$$f_{SBGaL}(x;\theta,\beta)=\frac{xf_{GaL}(x;\theta,\beta)}{E_{GaL}(X)},\ \text{where}\ E_{GaL}(X)=\frac{2\beta(1+\theta)-\theta}{\theta\beta(1+\theta)}.$$

We have

$$f_{SBGaL}(x;\theta,\beta)=\begin{cases}\dfrac{\theta^3((\beta+\beta\theta-\theta)x^2+x)e^{-\theta x}}{2\beta(1+\theta)-\theta}; x,\ \theta>0 \text{ and } \beta>\dfrac{\theta}{1+\theta}\\ 0, \text{ otherwise.}\end{cases} \tag{2}$$

And the cumulative distribution function of the SBGaL is

$$F_{SBGaL}(x)=1-\left(\frac{\theta^2(\beta+\theta\beta-\theta)}{2\beta(1+\theta)-\theta}x^2+\theta x+1\right)e^{-\theta x};\quad x,\theta>0\ \text{ and }\ \beta>\frac{\theta}{1+\theta}. \tag{3}$$

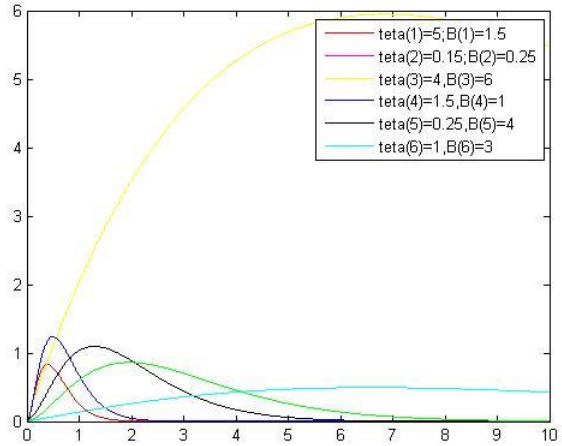

Figure 1 Plots of the density function for some parameter values

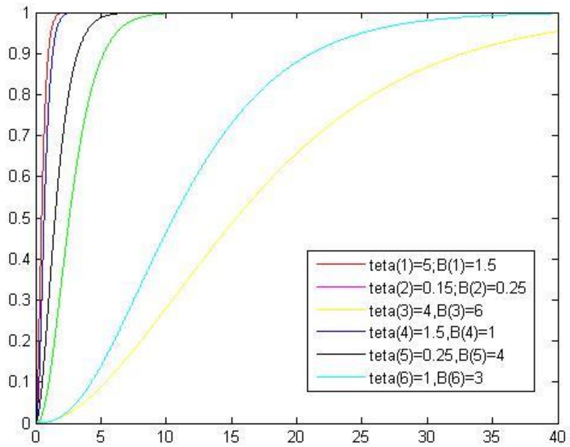

Figure 2 Plots of the distribution function for some parameter values

Therefore, the mode of SBGaL is given by

$$Mode(X)=\begin{cases}\dfrac{(\beta+\theta\beta-\frac{3}{2}\theta)+\sqrt{(\beta+\theta\beta-\theta)^2+\frac{\theta^2}{4}}}{\theta(\beta+\theta\beta-\theta)}, \text{ for } \beta\in\left[\dfrac{\theta}{1+\theta},\infty\right) \\ 0, \text{ otherwise.}\end{cases} \quad (4)$$

2.1. Survival and hazard rate function

Let

$$S_{SBGaL}(x)=1-F_{SBGaL}(x)=\left(\frac{\theta^2(\beta+\theta\beta-\theta)}{2\beta(1+\theta)-\theta}x^2+\theta\left(x+\frac{1}{\theta}\right)\right)e^{-\theta x} \quad (5)$$

and

$$h(x)=\frac{f_{SBGaL}(x)}{1-F_{SBGaL}(x)}=\frac{\theta^2((\beta+\theta\beta-\theta)x^2+x)}{\theta(\beta+\theta\beta-\theta)x^2+\left(2\beta(1+\theta)\left(x+\frac{1}{\theta}\right)\right)} \quad (6)$$

be the survival and hazard rate function, respectively.

3. Moments, Lorenz Curve and Extreme Order Statistics

3.1. Moments and related measures

The k^{th} moment about the origin of the size-biased gamma Lindley distribution is

$$m_k=E(X^k)=3\frac{6^{k-1}}{\theta^k}-\frac{6^{k-1}}{\theta^{k-1}(2\beta(1+\theta)-\theta)}-\frac{6}{25}\frac{(-5k+6^k-1)}{\theta^k}. \quad (7)$$

Proposition 1 *Let* $X_1,X_2,...,X_n$ *be* n *independent random variables from* $SBGaL(\theta,\beta)$ *distribution. Then the moment generating function (mgf) of* $S=\sum_{i=1}^{n}X_i$, *is given by*

$$M_X(t)=E(e^{tX})=\theta^3\left(\frac{-1}{(t-\theta)^3}-\frac{t}{(2\beta(1+\theta\beta)-\theta)(t-\theta)^3}\right)$$

and

$$M_S(t)=\theta^{3n}\left(\frac{-1}{(t-\theta)^3}-\frac{t}{(2\beta(1+\theta\beta)-\theta)(t-\theta)^3}\right)^n. \quad (8)$$

Proof: Let $X_1,X_2,...,X_n$ be n independent random variables, we have

$$M_S(t)=E(e^{tS})=M_X(t)^n=\theta^{3n}\left(\frac{-1}{(t-\theta)^3}-\frac{t}{(2\beta(1+\theta\beta)-\theta)(t-\theta)^3}\right)^n.$$

Remark 1 The moment generating function for X and S exists $(E(e^{tX})<\infty)$ only if $t<\theta$.

Corollary 1 *Let* $X\sim SBGaL(\theta,\beta)$, *then the mean and variance of* X *are*

$$E(X)=\frac{3}{\theta}-\frac{1}{2\beta(1+\theta)-\theta},\quad E(X^2)=\frac{6\beta(1+\theta)-4\theta}{\theta(2\beta(1+\theta)-\theta)} \quad (9)$$

$$Var(X) = \frac{3}{\theta^2} - \frac{1}{(2\beta(1+\theta)-\theta)^2}. \tag{10}$$

Proof: We have

$$E(X) = M'(t=0) = \frac{2}{\theta} - \frac{1}{2\beta(1+\theta)-\theta}, E(X^2) = M''(t=0) = \frac{6\beta(1+\theta)-4\theta}{\theta(2\beta(1+\theta)-\theta)}, \text{ and}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{3}{\theta^2} - \frac{1}{(2\beta(1+\theta)-\theta)^2}.$$

Remark 2 The relation between the moments is

$$\frac{\theta}{6} m_k = m_{k-1} - \frac{k-1}{\theta^k}, k = 0,1,2,... \tag{11}$$

This equality has been verified with demonstration by recurrence.

Theorem 1 *Let* $X \sim SBGaL(\theta,\beta)$, $M = mode(X)$, $me = medain(X)$ *and* $\mu = E(X)$. *Then* $M < me < \mu.$

Proof:

$$F_{SBGaL}(M) = 1 - \left(\frac{\left[(\beta+\theta\beta-\frac{3}{2}\theta) + \sqrt{(\beta+\theta\beta-\theta)^2 - \frac{\theta^2}{4}} \right]^2}{(2\beta(1+\theta)-\theta)(\beta+\theta\beta-\theta)} + \frac{(\beta+\theta\beta-\frac{3}{2}\theta) + \sqrt{(\beta+\theta\beta-\theta)^2 - \frac{\theta^2}{4}}}{(\beta+\theta\beta-\theta)} + 1 \right) e^{-\frac{(\beta+\theta\beta-\frac{3}{2}\theta)+\sqrt{(\beta+\theta\beta-\theta)^2-\frac{\theta^2}{4}}}{(\beta+\theta\beta-\theta)}};$$

$\theta > 0$, and $\beta > \frac{\theta}{1+\theta}.$

$$F_{SBGaL}(x) = 1 - \left(\frac{(\beta+\theta\beta-\theta)(6\beta(1+\theta)-4\theta)^2}{(2\beta(1+\theta)-\theta)^3} + \frac{6\beta(1+\theta)-4\theta}{2\beta(1+\theta)-\theta} + 1 \right) e^{-\frac{6\beta(1+\theta)-4\theta}{2\beta(1+\theta)-\theta}}; \theta > 0, \text{ and } \beta > \frac{\theta}{1+\theta}.$$

We have

$$\frac{6\beta(1+\theta)-4\theta}{2\beta(1+\theta)-\theta} > 1, \quad \frac{(\beta+\theta\beta-\frac{3}{2}\theta) + \sqrt{(\beta+\theta\beta-\theta)^2 - \frac{\theta^2}{4}}}{(\beta+\theta\beta-\theta)} < 2 \text{ for } \beta > \frac{\theta}{1+\theta}.$$

It simple to check that

$$F(M) < F(me) < F(m).$$

To this end, we have $M < me < m.$

3.2. Lorenz curve

The Lorenz curve for a positive random variable X is defined as the graph of the ratio

$$L(F(x)) = \frac{E(X \mid X \le x)}{E(X)} \tag{12}$$

against $F(x)$ with the properties $L(p) = p, L(1) = 1.$ If X represents annual income, $L(p)$ is the proportion of total income that accrues to individuals having the $100p\%$ lowest incomes. If all individuals earn the same income then $L(p) = p$ for all $p.$ The area between the line $L(p) = p$ and the Lorenz curve may be regarded as a measure of inequality of income, or more generally, of the variability of the variability of X see Gail and Gastwirth (1978) and Dagum (1985) for extensive discussion of Lorenz curves. For the exponential distribution, it is well known that the Lorenz curve is given by

$$L(p)=p\left\{p+(1-p)\log(1-p)\right\}.$$

For the SBGaL distribution in (3),

$$E(X\mid X\le x)F(x)=\frac{6\beta(1+\theta)-4\theta}{\theta(2\beta(1+\theta)-\theta)}-\frac{e^{-\theta x}}{2\beta(1+\theta)-\theta}\left[(\beta+\theta\beta-\theta)(\frac{6}{\theta}+6x+3\theta x^2+\theta^2x^3)+2+2\theta x+\theta^2x^2\right].$$

Thus, from (3) we obtain the Lorenz curve for the Size Biased Gamma Lindley distribution as

$$L(p)=1-\frac{(2\beta(1+\theta)-\theta)(1-p)\left[(\beta+\theta\beta-\theta)\left(\dfrac{6}{\theta}+6x+3\theta x^2+\theta^2x^3\right)+2+2\theta x+\theta^2x^2\right]}{(6\beta(1+\theta)-4\theta)\left[\dfrac{\theta(\beta+\theta\beta-\theta)}{2\beta(1+\theta)-\theta}x^2+x+\dfrac{1}{\theta}\right]},$$

where $x=F^{-1}(p)$ with $F(.)$ given by (3).

3.3. Extreme order statistics

If $X_1,X_2,...,X_n$ is a random sample from (1) and if $\bar{X}=\dfrac{X_1+\cdots+X_n}{n}$ denotes the sample mean then by the usual central limit theorem $\sqrt{n}\dfrac{(\bar{X}-E(X))}{\sqrt{Var(X)}}$ approaches the standard normal distribution as $n\to\infty$. Sometimes one would be interested in the asymptotics of the extreme value $M_n=\max(X_1,.....X_n)$, and $m_n=\min(X_1,.....,X_n)$. For the CDF in (4), it can be seen that

$$\lim_{t\to\infty}\frac{1-F(t+x)}{1-F(t)}=\exp(-\theta x)$$

and

$$\lim_{t\to 0}\frac{F(tx)}{F(t)}=x.$$

Thus, it follows from Theorem 1.6.2 in Leadbetter, Lindgren, Rootzén (1985), that there must be norming constants $a_n>0, b_n, c_n>0$ and d_n such that

$$\Pr\left\{a_n(M_n-b_n)\le x\right\}\to\exp(-\theta x) \tag{13}$$

and

$$\Pr\left\{c_n(m_n-d_n)\le x\right\}\to 1-\exp(-\theta x) \tag{14}$$

as $n\to\infty$. The form of the norming constants can also be determined. For instance, using Corollary 1.6.3 in Leadbetter, Lindgren, Rootzén (1985), one can see that $a_n=1$ and $b_n=F^{-1}\left(1-\dfrac{1}{n}\right)$ with $F(.)$ given by (3).

4. Estimation and Simulation

4.1. Maximum likelihood estimates

In this section we shall discuss the point estimation on the parameters that index the $SBGaL(\theta,\beta)$. Let the log-likelihood function of single observation (say x_i) for the vector of parameter (θ,β) can be written as

$$\ln l(x_i;\theta,\beta) = 3\ln\theta + \ln((\beta+\theta\beta-\theta)x_i+1) + \ln x_i - \theta x_i - \ln(2\beta(1+\theta)-\theta).$$

$$\frac{\delta \ln l(x_i;\theta,\beta)}{\delta\theta} = \frac{3}{\theta} - \frac{2\beta-1}{2\beta(1+\theta)-\theta} - x_i - \frac{(\beta-1)x_i}{(\beta+\theta\beta-\theta)x_i+1} \tag{15}$$

$$\frac{\delta \ln l(x_i;\theta,\beta)}{\delta\beta} = -\frac{2(1+\theta)}{2\beta(1+\theta)-\theta} + \frac{x_i(1+\theta)}{(\beta+\theta\beta-\theta)x_i+1}. \tag{16}$$

The maximum likelihood estimator θ of θ and β of β is obtained by solving non linear equations (15) and (16) we give

$$\begin{cases} \theta = \dfrac{2}{x} \\ \beta = \dfrac{2}{x+2} \end{cases} \tag{17}$$

and

$$\begin{cases} E(\theta) = \dfrac{2\theta\beta(1+\theta)}{2\beta(1+\theta)-\theta} \\ E(\beta) = \dfrac{2\theta((\beta+\theta\beta-\theta)(1-2\theta)+\theta)+2\theta^3(-4(\beta+\theta\beta-\theta)+2)e^{2\theta}E_i(-2\theta)}{2\beta(1+\theta)-\theta}. \end{cases} \tag{18}$$

4.2. Moments estimates

Using the first moment m and m_2 second moment about SBGaL distribution, we have

$$\begin{cases} m = \dfrac{3}{\theta} - \dfrac{1}{2\beta(1+\theta)-\theta} \\ m_2 = \dfrac{12}{\theta^2} - \dfrac{6}{\theta(2\beta(1+\theta)-\theta)} \end{cases} \tag{19}$$

where $m_2 = s^2 + m^2$ and s^2 is the variance. We solve this non linear system and we find the couple (θ,β), where $(\theta,\beta) > 0$ also $s > 0, m > 0$, the solving of non linear system (19) gives

$$m_2\theta^2 - 6m\theta + 6 = 0 \text{ and } \beta = \frac{4\theta - \theta^2 m}{2(1+\theta)(3-\theta m)}.$$

It is easy to check that the solution of

$$\theta = \frac{3m + \sqrt{3}\sqrt{m^2 - 2s^2}}{m_2} \text{ and } \beta = \frac{4\theta - \theta^2 m}{2(1+\theta)(3-\theta m)}.$$

4.3. Simulation

In this subsection, we investigate the behavior of the ML estimators for a finite sample size (n). A simulation study consisting of the following steps is being carried out for each triplet(n,θ,β), where θ = 0.1, 0.5, 1, 3, β = 0.1, 0.5, 0.75, 1, 6 and n =10, 30, 50.

– choose the initial values of θ_0, β_0 for the corresponding elements of the parameter vector $\Theta = (\theta, \beta)$ to specify SBGaL distribution;

– choose sample size n;

– generate N independent samples of size n from SBGaL(θ, β);

– compute the ML estimate Θ_n of Θ for each of the N samples;

– compute the mean of the obtained estimators over all N samples,

Average bias$(\theta) = \frac{1}{N}\sum_{i=1}^{N}(\Theta_i - \Theta_0)$ and the average square error MSE$(\theta) = \frac{1}{N}\sum_{i=1}^{N}(\Theta_i - \Theta_0)^2$.

Table 1 Average bias of the simulated estimates

	$\theta=1, \beta=6$		$\theta=1, \beta=0.1$		$\theta=1, \beta=0.75$	
	bias(θ)	bias(β)	bias(θ)	bias(β)	bias(θ)	bias(β)
n=10	0.0043	-0.5549	0.1666	0.0162	0.0500	-0.0250
n=30	0.0014	-0.1849	5.5556×10^{-2}	0.0054	1.6666×10^{-2}	-0.0020
n=50	0.0008	-0.1109	3.3334×10^{-2}	0.0032	0.0100	-0.0012
	$\theta=0.1, \beta=1$		$\theta=0.5, \beta=1$		$\theta=3, \beta=1$	
	bias(θ)	bias(β)	bias(θ)	bias(β)	bias(θ)	bias(β)
n=10	0.0004	-0.0911	0.0100	-0.0680	0.1800	-0.0858
n=30	1.5800×10^{-4}	-0.0303	3.3333×10^{-2}	-0.0226	0.0600	-0.0286
n=50	0.0096×10^{-2}	-0.0182	0.0020	-0.0136	0.0360	-0.0171
	$\theta=3, \beta=0.5$		$\theta=0.5, \beta=0.5$		$\theta=0.1, \beta=0.5$	
	bias(θ)	bias(β)	bias(θ)	bias(β)	bias(θ)	bias(β)
n=10	0.9000	0.8286	0.0250	-0.0149	0.0010	-0.0480
n=30	0.3000	0.2762	8.3333×10^{-3}	-0.0049	3.3333×10^{-4}	-0.0136
n=50	0.1800	0.1657	0.0050	-0.0029	0.0002	-0.0081

Table 2 Average MSE of the simulated estimates

	$\theta=1, \beta=6$		$\theta=1, \beta=0.1$		$\theta=1, \beta=0.75$	
	MSE(θ)	MSE(β)	MSE(θ)	MSE(β)	MSE(θ)	MSE(β)
n=10	1.8922×10^{-4}	3.0800	0.2777	0.0026	0.0250	0.0062
n=30	6.3075×10^{-5}	1.0267	0.0925	8.8551×10^{-4}	8.3333×10^{-3}	0.0020
n=50	3.7845×10^{-5}	0.6160	0.0555	0.0005	0.0050	0.0012
	$\theta=0.1, \beta=1$		$\theta=0.5, \beta=1$		$\theta=3, \beta=1$	
	MSE(θ)	MSE(β)	MSE(θ)	MSE(β)	MSE(θ)	MSE(β)
n=10	2.3040×10^{-6}	0.0830	0.0010	0.0463	0.3240	0.0736
n=30	7.6800×10^{-7}	0.0276	3.3333×10^{-4}	0.0154	0.1080	0.0245
n=50	4.6080×10^{-7}	0.0166	0.0002	0.0092	0.0648	0.0147
	$\theta=3, \beta=0.5$		$\theta=0.5, \beta=0.5$		$\theta=0.1, \beta=0.5$	
	MSE(θ)	MSE(β)	MSE(θ)	MSE(β)	MSE(θ)	MSE(β)
n=10	8.1000	6.8669	6.2500×10^{-3}	0.0022	0.1000×10^{-4}	0.0166
n=30	2.7000	2.2889	2.0833×10^{-3}	0.0007	3.3333×10^{-6}	0.0055
n=50	1.6200	1.3733	0.0012	0.0004	0.2000×10^{-5}	0.0033

Table 1 shows that θ is positively biased and the bias(θ) → 0, θ → 0, and β is negatively biased if $\beta > 0.1$ and the bias(β) → 0, β →0. Table 2 shows that MSE(θ) and MSE(β) → 0 where θ → 0 and $n \to \infty$.

4.4. Application to real data sets

In this section, we illustrate the applicability of SBGaL distribution by considering two different data sets used by different researchers. We also fit generalized Lindley by Zakerzadah and Dolati (2010), quasi Lindley by Shanker, and Mishra (2013), two-parameter Lindley by Shanker, and Sharma (2013), Weibull and lognormal distributions.

In each of these distributions, the parameters are estimated by using the moment method because it is simple, easy to handle, exact, and for comparison we use negative log-likelihood values $(-LL)$, the Akaike information criterion (AIC) and Bayesian information criterion (BIC) which are defined by $-LL+2q$ and $-LL+q\log(n)$, respectively, where q is the number of parameters estimated and n is the sample size. Further K-S (Kolmogorov-Smirnov) test statistic defined as $\text{K-S} = \sup_x \left|F_n(x) - F(x)\right|$ where $F_n(x)$ is empirical distribution function and $F(x)$ is cumulative distribution function is calculated and shown for all the data sets.

Example 1

We consider from Lawless (2003), pp.204 and 263 two series of real data. The first one, represents the failure times (mm) for a sample of fifteen electronic components in an acceleration life test: 1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2. The second set of data, are the number of cycles to failure for 25 100-cm specimens of yarn, tested at a particular strain level: 15, 20, 38, 42, 61, 76, 86, 98, 121, 146, 149, 157, 175, 176, 180, 180, 198, 220, 224, 251, 264, 282, 321, 325, 653.

According to Table 3, we can observe that size-biased gamma-Lindley distribution provide smallest $-LL$, K-S, AIC and BIC values as compared to generalized Lindley, gamma Lindley, quasi-Lindley, two parameter Lindley, gamma, Weibull, lognormal, size-biased Lindley distributions, and hence best fits the data among all the models considered.

5. Conclusions

In this paper, a size-biased gamma-Lindley distribution (SBGaL), of which the size-biased Lindley distribution (SBLD) is a particular case, has been introduced to model count data which structurally excludes zero counts. The estimation of its parameters has been discussed using the method of maximum likelihood and the method of moments. A simulation study is carried out to examine the bias and mean square error of the maximum likelihood estimators of the parameters. Several applications of the model to a real data set are presented finally and compared with the fit attained by some other well-known two and three parameters. The adequacy of fits was assessed in terms AIC values, BIC values and density plots. We can show that the size-biased gamma-Lindley distribution can be used quite effectively in analyzing real lifetime data and actuarial science.

Table 3 Comparison between distributions

Data	Distribution	β	θ	γ	$-LL$	K-S	AIC	BIC
Serie 1	Generalized Lindley	1.203	0.064	0.083	64.080	0.095	134.16	136.28
$n = 15$	GaL	1.129	0.684		64.015	0.094	132.03	133.45
$m = 27.546$	QLD	4.016	-0.99		1504	0.93	3012	3013.4
$s = 20.059$	TwoPLD	0.0704	1.110			0.196		
	Gamma	1.442	0.052		64.197	0.102	132.39	133.81
	Weibull	1.306	0.034		64.026	0.450	132.05	133.47
	Lognormal	1.061	2.931		64.626	0.163	135.25	136.67
	SBLD		1.84		64.107	0.106	133.23	135.55
	SBGaL	1.295	0.84		63.52	0.089	132.02	133.33
Serie 2	Generalized Lindley	1.505	0.012	0.018	152.369	0.137	310.74	314.39
$n = 25$	GaL	1.086	0.010		152.132	0.129	308.26	310.7
$m = 178.32$	QLD	0.0107	8.514		1045.9	0.94	2131.8	2156.2
$s = 131.097$	TwoPLD	0.0107	0.125			0.232		
	Gamma	1.794	0.010		152.371	0.135	308.74	311.18
	Weibull	1.414	0.005		152.440	0.697	308.88	310.7
	Lognormal	0.891	4.880		154.092	0.155	312.18	314.62
	SBLD		0.080		155.12	0.159	313.56	315.78
	SBGaL	1.112	0.069		151.35	0.125	307.67	310.7

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References

Ayesha A. Size biased Lindley distribution and its properties a special case of weighted distribution. Appl. Math. 2017; 8(6): 808-819.

Dagum C. Lorenz curve. In: Kotz S, Johnson NL, Read CB, editers. Encyclopedia of statistical sciences. New York: Wiley; 1985. p.156-161.

Das KK, Roy TD. On some length biased weighted Weibull distribution. Adv Appl Sci Res. 2011; 2(5): 465-475.

Ducey MJ, Gove JH. Size-biased distributions in the generalized beta distribution family with applications to forestry. Int. J. Forest Res. 2015; 88: 143-151.

Fisher RA. The effects of methods of ascertainment upon the estimation of frequency. Ann. Eugenics. 1937; 6(1): 13-25.

Gail MH, Gastwirth JL. A scale-free goodness-of-fit test for the exponential distribution based on the Lorenz curve. J. Am. Stat. Assoc. 1978; 73: 787-793.

Gove HJ. Estimation and application of size-biased distributions in forestry. In: Amaro A, Reed D, Soares P, editers. Modeling forest systems. Wallingford: CAB International; 2003. p.201-212.

Lappi LJ, Bailey RL. (1987) Estimation of diameter increment function or other tree relations. Using angle-count samples. Forest Sci. 1987; 33: 725-739.

Lawless J. Statistical models and methods for lifetime data. New York: Wiley; 2003.

Leadbetter MR, Lindgren G, Rootzén H. Extremes and related properties of random sequences and processes. New York: Springer Verlag; 1987.

Mir KA, Ahmad M. Size-biased distributions and their applications. Pakistan J. Stat. 2009; 25(3): 283-294.

Patil GP, Rao CR. The weighted distributions: a survey and their applications. In: Krishnaiah PR, editor. Applications of statistics. Amsterdam: North Holland; 1977. p.383-405.

Patil GP, Rao CR. Weighted distributions and size-biased sampling with applications to wild life populations and human families. Biometrics. 1978; 34(2): 179-189.

Rao CR. On discrete distributions arising out of methods of ascertainment. In: Patil GP, editor. Classical and contagious discrete distributions. Calcutta: Statistical Publishing Society; 1965. p.320-332.

Shanker R, Mishra A. A quasi Lindley distribution. Afr. J. Math. Comput. Sci. Res. 2013; 6(4): 64-71.

Shanker R, Sharma S, Shanker R. A two-parameter Lindley distribution for modeling waiting and survival times data. Appl. Math. 2013; 4(2): 363-368.

Simoj SM, Maya SS. Some properties of weighted distributions in the context of reparable systems, Commun. Stat.-Theory Methods. 2006; 35, 223-228.

Van Deusen PC. Fitting assumed distributions to horizontal point sample diameters. Forest Sci. 1986; 32: 146-148.

Zakerzadah H, Dolati A. Generalized Lindley distribution. J. Math. Ext. 2010; 3(2): 13-25.

Zeghdoudi H, Nedjar S. Gamma Lindley distribution and its application. J. Appl. Prob. Stat. 2016; 11(1): 129-138.

Zeghdoudi H, Nedjar S. On gamma Lindley distribution: properties and simulations. J. Comp. Appl. Math. 2016; 298: 167-174.