Thailand Statistician
July 2019; 17(2): 223-234
http://statassoc.or.th
Contributed paper

Wrapped Length Biased Weighted Exponential Distribution

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Received: 2 November 2018
Revised: 28 December 2018
Accepted: 17 January 2019

Abstract

A new circular distribution to be called as wrapped length biased weighted exponential distribution with two parameters is proposed in this paper. The expressions for the characteristic function, the trigonometric moments and other related descriptive measures of the distribution are obtained. Estimation of the parameters is carried out using the maximum likelihood method and the estimators are shown to be consistent through a simulation study. The proposed model is fitted to a real-life data set on orientations and the goodness-of-fit of the distribution is assessed. The proposed distribution is found to be more appropriate in modelling the situations where the directions possessing lower magnitude have highest likelihood of occurrence and the likelihood of occurrence gradually decreases with an increase in magnitude of directions.

Keywords Weighted distribution, circular data, wrapped distribution, trigonometric moments, simulation study.

1. Introduction

Circular data is referred to as the directional data which arise in two dimensions. Circular data can be represented either as a point on a circle of unit radius, cantered at the origin or as a unit vector in the plane, connecting the origin to the corresponding point (Rao and Sengupta 2001). Examples of circular data are found in earth science, meteorology (wind direction analysis), biology (study of direction of animal movement), physics and various other scientific domains where the study of data recorded in degrees or radius in a circle is required. (Ferrari 2009).

The wrapping approach is one of the commonly used techniques of generating circular probability model from the distribution on real line. Under this approach, a linear random variable (r.v.) X is transformed to a circular r.v., say θ by reducing its modulo $2\pi,$ i.e. $\theta = X \left(\bmod 2\pi\right).$ It can also be thought of as the distribution on the line being wrapped around the circumference of the unit circle (Bhattacharjee 2017). Wrapped distributions were introduced by Lévy (1939). The wrapped versions of the generalized Gompertz distribution, weighted exponential distribution and the Lindley distribution were derived and their several properties were studied by Roy and Adnan (2012a), Roy and Adnan (2012b) and Joshi and Jose (2018), respectively. The applications of these wrapped distributions were seen in modelling the data on orientation of turtle after laying eggs and the orientation of the nest of noisy scrub birds.

Weighted distribution theory finds application in observational studies where biased data occur, for instance, where not all the observations to be collected have equal chance of being recorded (McDonald 2010). A length biased distribution arises as a particular case of weighted distribution, when the weight $w(x)=x$ and these distributions have found numerous applications in modelling data arising in reliability and survival studies. The length biased version of the weighted exponential distribution viz. the length biased weighted exponential (LBWE) distribution was proposed by Das and Kundu (2016). The LBWE distribution is a two-parameter distribution having an increasing hazard rate function and a decreasing mean residual life function for all values of the shape parameter. This distribution was applied to a data set pertaining to the number of million revolutions of each of 23 ball bearings before failure and it was found to provide a good fit to the data. The LBWE distribution possesses several other desirable reliability properties.

In order to explore its utility as a circular probability model, we propose the wrapped length biased weighted exponential distribution in this paper, which is obtained from the LBWE density through the classical wrapping approach and then investigate its several properties. Section 1 of the paper introduces the distribution and Section 2 contains the derivation of the density. Section 3 comprises of the expressions for characteristic function, trigonometric moments and other related measures of the proposed distribution. The maximum likelihood estimation of the parameters of the distribution is dealt with in Section 4. Section 5 displays the simulation study to illustrate the consistency of the estimators. In Section 6, the proposed model is applied to a data set on long-axis orientation measurements of 164 feldspar laths in basalt. The findings of the paper are summarized in Section 7.

2. Definition and Derivation of the Wrapped Length Biased Weighted Exponential Distribution

Let X follow length biased weighted exponential distribution. Then the probability density function of X is given by

$$f\left(x\right)=\frac{\left\{\lambda\left(\alpha+1\right)\right\}}{\alpha\left(\alpha+2\right)}xe^{-\lambda x}\left\{1-e^{-\lambda\alpha x}\right\},\quad x>0, \tag{1}$$

where $\alpha>0$ is the shape parameter and $\lambda>0$ is the scale parameter and we write $X\sim$ LBWE (α,λ).

The cumulative distribution function of X is

$$F\left(x\right)=1-\frac{\left(1+\alpha\right)^{2}}{\alpha\left(\alpha+2\right)}\left(1+x\right)e^{-x}+\frac{1}{\alpha\left(\alpha+2\right)}\left[1+\left(1+\alpha\right)x\right]e^{-(1+\alpha)x},\ x>0. \tag{2}$$

Using the wrapping approach of obtaining circular probability density from real line, the density of the wrapped circular r.v. β corresponding to the linear r.v. X is found as shown below (Mardia and Jupp 2000)

$$f_w(\beta)=\sum_{k=-\infty}^{\infty} f(\beta+2k\pi). \tag{3}$$

Again, the cumulative distribution function of β is given by

$$F_w(\beta)=\sum_{k=-\infty}^{\infty}\left\{F(\beta+2k\pi)-F(2k\pi)\right\}, \quad \beta\in[0,2\pi). \tag{4}$$

Therefore, the p.d.f. of the wrapped length biased weighted exponential r.v. is obtained using (1) and (3) as

$$f_w(\beta)=\frac{\left\{\lambda(\alpha+1)\right\}^2}{\alpha(\alpha+2)}e^{-\lambda\beta}\left[\begin{array}{l}\dfrac{1}{1-e^{-2\pi\lambda}}\left\{\beta+\dfrac{2\pi e^{-2\pi\lambda}}{1-e^{-2\pi\lambda}}\right\}\\ -\dfrac{e^{-\lambda\beta\alpha}}{1-e^{-2\pi\lambda(1+\alpha)}}\left\{\beta+\dfrac{2\pi e^{-2\pi\lambda(1+\alpha)}}{1-e^{-2\pi\lambda(1+\alpha)}}\right\}\end{array}\right], \quad \alpha>0,\lambda>0,\beta\in(0,2\pi]. \tag{5}$$

The r.v. β conforming to the wrapped length biased weighted exponential probability law with parameters is denoted by β ~ WLBWE(α,λ).

The c.d.f of the wrapped length biased weighted exponential distribution is obtained using (2) and (4) as

$$F_w(\beta)=\frac{1}{\alpha(\alpha+2)}\left[\begin{array}{l}\dfrac{(1+\alpha)^2\left\{1-(1+\beta)e^{-\beta}\right\}}{1-e^{-2\pi}}+\dfrac{(1+\alpha)^2\,2\pi\left\{1-e^{-\beta}\right\}e^{-2\pi}}{\left\{1-e^{-2\pi}\right\}^2}\\ +\dfrac{e^{\{-\beta(1+\alpha)\}}\left\{1+\beta(1+\alpha)\right\}-1}{1-e^{\{-2\pi(1+\alpha)\}}}+\dfrac{2\pi(1+\alpha)e^{\{-2\pi(1+\alpha)\}}\left[e^{-\beta(1+\alpha)}-1\right]}{\left[1-e^{\{-2\pi(1+\alpha)\}}\right]^2}\end{array}\right]. \tag{6}$$

The behavior of the p.d.f. of the WLBWE(α,λ) for $\alpha<1$, $\alpha=1$ and $\alpha>1$ for different values of λ is shown in Figures 1 to 3, respectively. It is seen that for $\alpha\geq1$, $\lambda\geq1$; $\alpha<1$, $\lambda\geq1$; $\alpha\geq1$, $\lambda<1$ and $\alpha<1$, $\lambda<1$, the area under the probability curve decreases with an increase in the value of β. Further, the probability of occurrence of directions of lower magnitude is higher, which gradually decreases as we move towards directions of higher magnitude.

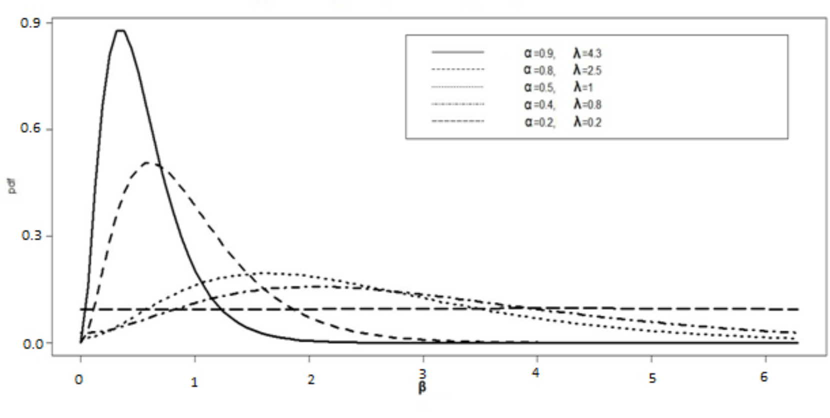

Figure 1 Density plot of the WLBWE(α,λ) for $\alpha<1$

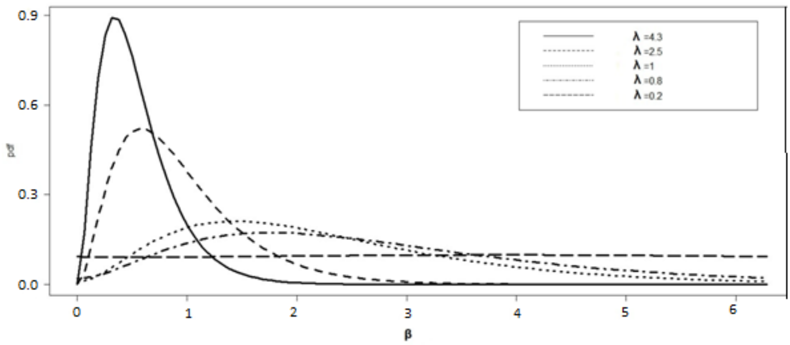

Figure 2 Density plot of the WLBWE(α,λ) for $\alpha=1$

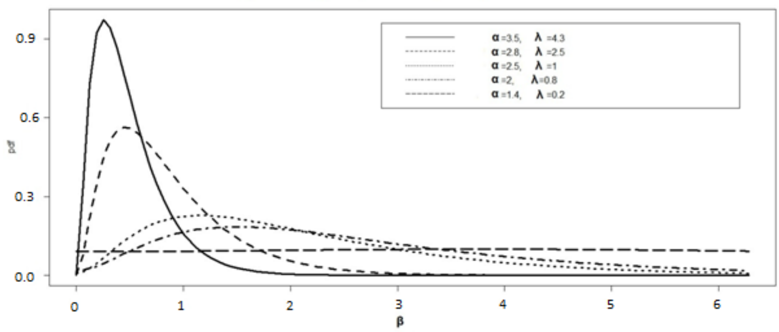

Figure 3 Density plot of the WLBWE(α,λ) for $\alpha>1$

3. Properties of WLBWE(α, λ)

In this section, the expressions for the characteristic function, trigonometric moments, coefficient of skewness and kurtosis of WLBWE(α,λ) are derived and their behavior for different values of the parameters are studied.

3.1. Characteristic function

The characteristic function of a wrapped circular variable, say ϕ_p at an integer value p can be obtained from the characteristic function of the corresponding unwrapped linear r.v. say $\phi_X(t)$ via the following relation (Rao and Sengupta 2001):

$$\phi_p=\phi_X(t).$$

The characteristic function of the LBWE(α,λ) is given by

$$\phi_X(t)=\left(1-\frac{2it}{2+\alpha}\right)(1-it)^{-2}\left(1-\frac{it}{1+\alpha}\right)^{-2},i=\sqrt{-1}. \tag{7}$$

Therefore, the characteristic function of the LBWE(α,λ) distribution is given by

$$\phi_p = \frac{(1-ip)^{-2}\left(1-\frac{ip}{1+\alpha}\right)^{-2}}{\left\{1-\left(\frac{2p}{2+\alpha}\right)i\right\}^{-1}} \quad , p = \pm 1, \pm 2, \ldots \tag{8}$$

Using the result of Roy and Adnan (2012b) which gives for all for all $a, b, r \in R^+$,

$$(a-ib)^{-r} = \left(a^2+b^2\right)^{-\frac{r}{2}} e^{-i\arctan\left(\frac{b}{a}\right)},$$

we obtain

$$(1-ip)^{-2} = \left(1+p^2\right)^{-1} e^{i\arctan p},$$

$$\left(1-\frac{ip}{1+\alpha}\right)^{-2} = \left\{1+\left(\frac{p}{1+\alpha}\right)^2\right\}^{-1} e^{i\arctan\left(\frac{p}{1+\alpha}\right)},$$

$$\left\{1-\left(\frac{2p}{2+\alpha}\right)i\right\}^{-1} = \left\{1+\left(\frac{2p}{2+\alpha}\right)^2\right\}^{-\frac{1}{2}} e^{-i\arctan\left(\frac{2p}{2+\alpha}\right)}.$$

Hence, we finally have

$$\phi_p = \frac{\left\{\left\{1+\left(\frac{2p}{2+\alpha}\right)^2\right\}^{-\frac{1}{2}}\right\}}{\left(1+p^2\right)\left\{1+\left(\frac{p}{1+\alpha}\right)^2\right\}} e^{i\arctan p + i\arctan\left(\frac{p}{1+\alpha}\right) - i\arctan\left(\frac{2p}{2+\alpha}\right)}. \tag{9}$$

Also, an alternative expression for ϕ_p is

$$\phi_p = \rho_p e^{i\mu_p}.$$

Hence, we have

$$\rho_p = \frac{\left\{\left\{1+\left(\frac{2p}{2+\alpha}\right)^2\right\}^{-\frac{1}{2}}\right\}}{\left(1+p^2\right)\left\{1+\left(\frac{p}{1+\alpha}\right)^2\right\}}, \tag{10}$$

$$\mu_p = \arctan(p) + \arctan\left(\frac{p}{1+\alpha}\right) - \arctan\left(\frac{2p}{2+\alpha}\right). \tag{11}$$

3.2. Trigonometric moments and related descriptive measures

Let $\beta \sim$ WLBWE(α, λ). Then the p^{th} non-central trigonometric moment of β is given by (Fisher 1993)

$$\phi_p = \alpha_p + i\beta_p,$$

where $\alpha_p = \rho_p \cos\mu_p$, $\beta_p = \rho_p \sin\mu_p$.

So we have

$$\alpha_p = \frac{\left\{\left\{1+\left(\frac{2p}{2+\alpha}\right)^2\right\}^{-\frac{1}{2}}\right\}}{\left(1+p^2\right)\left\{1+\left(\frac{p}{1+\alpha}\right)^2\right\}} \cos\left[\arctan(p)+\arctan\left(\frac{p}{1+\alpha}\right)-\arctan\left(\frac{2p}{2+\alpha}\right)\right], \quad (12)$$

$$\beta_p = \frac{\left\{\left\{1+\left(\frac{2p}{2+\alpha}\right)^2\right\}^{-\frac{1}{2}}\right\}}{\left(1+p^2\right)\left\{1+\left(\frac{p}{1+\alpha}\right)^2\right\}} \sin\left[\arctan(p)+\arctan\left(\frac{p}{1+\alpha}\right)-\arctan\left(\frac{2p}{2+\alpha}\right)\right]. \quad (13)$$

We have, in particular, the mean resultant length of β as

$$\rho_1 = \rho = \frac{\left\{1+\left(\frac{2}{2+\alpha}\right)^2\right\}^{\frac{1}{2}}}{2\left\{1+\left(\frac{1}{1+\alpha}\right)^2\right\}}, \quad (14)$$

and the mean direction of β as

$$\mu_1 = \mu = \arctan(1)+\arctan\left(\frac{1}{1+\alpha}\right)-\arctan\left(\frac{2}{2+\alpha}\right). \quad (15)$$

ρ indicates the extent of concentration of β towards the mean μ and it lies between 0 and 1. The closer it is to 1, the higher is the concentration towards μ. The circular variance of β is defined by

$$V = 1-\rho = 1-\frac{\left\{1+\left(\frac{2}{2+\alpha}\right)^2\right\}^{\frac{1}{2}}}{2\left\{1+\left(\frac{1}{1+\alpha}\right)^2\right\}}. \quad (16)$$

The p^{th} central trigonometric moments of β are given by

$$\bar{\alpha}_p = \rho_p \cos\left(\mu_p - p\mu\right),\ \bar{\beta}_p = \rho_p \sin\left(\mu_p - p\mu\right).$$

Therefore,

$$\bar{\alpha}_p = \frac{\left\{\left\{1+\left(\frac{2p}{2+\alpha}\right)^2\right\}^{-\frac{1}{2}}\right\}}{\left(1+p^2\right)\left\{1+\left(\frac{p}{1+\alpha}\right)^2\right\}}\cos\left[\arctan(p)+\arctan\left(\frac{p}{1+\alpha}\right)-\arctan\left(\frac{2p}{2+\alpha}\right)\right]$$
$$-p\left\{\arctan(1)+\arctan\left(\frac{1}{1+\alpha}\right)-\arctan\left(\frac{2}{2+\alpha}\right)\right\}, \tag{17}$$
$$\bar{\beta}_p = \frac{\left\{\left\{1+\left(\frac{2p}{2+\alpha}\right)^2\right\}^{-\frac{1}{2}}\right\}}{\left(1+p^2\right)\left\{1+\left(\frac{p}{1+\alpha}\right)^2\right\}}\sin\left[\arctan(p)+\arctan\left(\frac{p}{1+\alpha}\right)-\arctan\left(\frac{2p}{2+\alpha}\right)\right]$$
$$-p\left\{\arctan(1)+\arctan\left(\frac{1}{1+\alpha}\right)-\arctan\left(\frac{2}{2+\alpha}\right)\right\}.$$

The measures of skewness and kurtosis are denoted by ξ_1 and ξ_2, respectively and are defined as

$$\xi_1 = \frac{\bar{\beta}_2}{V^{\frac{3}{2}}},\ \xi_2 = \frac{\bar{\alpha}_2 - \rho^4}{V^2}.$$

Thus, ξ_1 and ξ_2 for WLBWE(α, λ) is given by

$$\xi_1 = \frac{\bar{\beta}_2}{\left\{1-\frac{\left\{1+\left(\frac{2}{2+\alpha}\right)^2\right\}^{\frac{1}{2}}}{2\left\{1+\left(\frac{1}{1+\alpha}\right)^2\right\}}\right\}^{\frac{3}{2}}}, \tag{18}$$

$$\xi_2 = \frac{\bar{\alpha}_2 - \rho^4}{\left\{1-\frac{\left\{1+\left(\frac{2}{2+\alpha}\right)^2\right\}^{\frac{1}{2}}}{2\left\{1+\left(\frac{1}{1+\alpha}\right)^2\right\}}\right\}^{\frac{3}{2}}}. \tag{19}$$

For unimodal symmetric data sets, ξ_1 is close to zero and for the data sets which are single peaked, ξ_2 is close to zero. In this case, wrapped normal distribution provides a good fit (Mardia and Jupp 2000).

The values of the above measures for different values of α are listed in Table 1. It can be observed from the expressions (12) to (19) that the trigonometric moments and related measures of this distribution are independent of λ.

Table 1 Values of the trigonometric moments and related measures of WLBWE(α,λ) for the different values of α

Measure	$\alpha<1$			
	$\alpha=0.3$	$\alpha=0.5$	$\alpha=0.7$	$\alpha=0.9$
μ	0.725	0.698	0.679	0.666
ρ	0.416	0.443	0.462	0.475
V	0.583	0.556	0.537	0.524
ξ_1	-0.103	-0.119	-0.134	-0.148
ξ_2	0.234	0.283	0.326	0.364
Measure	$\alpha>1$			
	$\alpha=1.2$	$\alpha=1.8$	$\alpha=2.2$	$\alpha=2.6$
μ	0.653	0.644	0.643	0.636
ρ	0.488	0.501	0.504	0.506
V	0.511	0.498	0.495	0.493
ξ_1	-0.167	-0.198	-0.212	-0.223
ξ_2	0.409	0.466	0.486	0.497

Table 1 shows that for both $\alpha<1$ and $\alpha>1$, skewness is decreasing with increasing value of α and the value of kurtosis is increasing in the positive direction with an increase in the value of α.

4. Maximum Likelihood Estimation of the Parameters

In this section, the estimation of the parameters of WLBWE(α,λ) through the maximum likelihood method is discussed.

Let $\beta_1,\beta_2,...,\beta_n$ be a random sample of size n from WLBWE(α,λ). Then the log-likelihood function is given by

$$\log L = 2n\log\left\{\lambda\left(\alpha+1\right)\right\} - n\log\left\{\alpha\left(\alpha+2\right)\right\} - \lambda\sum_{i=1}^{n}\beta_i$$
$$+\sum_{i=1}^{n}\log\left[\frac{1}{1-e^{-2\pi\lambda}}\left\{\beta_i+\frac{2\pi e^{-2\pi\lambda}}{1-e^{-2\pi\lambda(1+\alpha)}}\right\}-\frac{e^{-\alpha\lambda\beta_i}}{1-e^{-2\pi\lambda(1+\alpha)}}\left\{\beta_i+\frac{2\pi e^{-2\pi\lambda(1+\alpha)}}{1-e^{-2\pi\lambda(1+\alpha)}}\right\}\right]. \tag{20}$$

The maximum likelihood estimation (m.l.e.) of the parameters are computed by solving the maximum likelihood equations, which are given by

$$\frac{\partial}{\partial\alpha}\log L = 0, \quad \frac{\partial}{\partial\lambda}\log L = 0.$$

Since the maximum likelihood equations are non-linear in nature and difficult to be solved analytically, we use suitable numerical technique to solve the above equations for α and λ.

5. Simulation Study

We carry out a simulation study to generate random variables from WLBWE(α,λ) and then obtain m.l.e. of the parameters α and λ. For different values of λ and α, we generate samples of size 100, 250, 500 and 800. The program is replicated N = 1,000 times to get the m.l.e. of λ and α. Steps of the simulation algorithm to obtain the m.l.e. of the parameters are as given below:

Step 1: A r.v. is generated from the $U(0,1)$ distribution and we name it u.

Step 2: The expression of c.d.f. given in equation (6) is equated with u and is solved for β, which is a r.v. from WLBWE(α,λ). Steps 1 and 2 are repeated to get a sample of the desired size from WLBWE(α,λ).

Step 3: The m.l.e. of α and λ is obtained by substituting the valuesof β generated in Step 2 in (20) and maximizing this with respect to α and λ, respectively.

To calculate the average bias and MSE of the m.l.e., we use the following formulae:

Let the true value of the parameter α be α^* and the m.l.e be $\hat{\alpha}$. Then the bias and mean square error (MSE) of $\hat{\alpha}$ in estimating α is given by,

$$Bias\left(\hat{\alpha}\right)=\frac{1}{N}\sum_{i=1}^{N}\left(\alpha_i-\alpha^*\right),\ MSE\left(\hat{\alpha}\right)=\frac{1}{N}\sum_{i=1}^{N}\left(\alpha_i-\alpha^*\right)^2,$$

where N is the number of replications and α_i is the m.l.e. of α obtained in the i^{th} replicate.

Similarly, the bias and MSE of the m.l.e. of λ are calculated. The m.l.e. is consistent if the bias and MSE decreases (approaches to zero) with an increase in the sample size. Table 2 shows the average values of the bias and MSE of the m.l.e. of α and λ for the different sample sizes and for different set of values of α and λ.

Table 2 shows that the bias and MSE of the m.l.e. of both α and λ decreases to zero with an increase in the sample size. Hence, the estimates of the parameters are consistent.

The calculation of the trigonometric moments, other related measures is carried out using the R software version 3.5.0, through the user contributed packages viz. CircStats (Lund and Agostinelli 2018) and circular (Lund and Agostinelli 2017) with the help of self-programmed codes. The maxLik package (Toomet and Henningsen 2015) is used to obtain the maximum likelihood estimates of the parameters and the rootSolve package (Soetaert 2016) is used to generate random variables from WLBWE(α,λ).

6. Application to Real Data Set

This section consists in applying the proposed distribution to a real-life data set. The data set considered the measurements of long-axis orientation of 164 feldspar laths in basalt, which is procured from Smith (1988) and published in Fisher (1993), Appendix B5.

We first plot the histogram of the data under consideration, which is presented in Figure 4 below:

Table 2 Average values of bias and MSE of the m.l.e of α and λ for different sample sizes and for different values of α and λ

n	$\alpha = 0.5, \lambda = 0.7$			
	Bias(α)	MSE(α)	Bias(λ)	MSE(λ)
100	0.9056	1.0442	0.5109	0.2744
250	0.642	0.5872	0.4904	0.2633
500	0.4612	0.3416	0.2558	0.0702
800	0.2023	0.159	0.2439	0.0598
n	$\alpha = 0.8, \lambda = 1.5$			
	Bias(α)	MSE(α)	Bias(λ)	MSE(λ)
100	0.7852	1.4352	-0.647	0.4192
250	0.5543	1.4141	-0.5054	0.2556
500	0.3656	1.0609	-0.4708	0.2256
800	0.1687	0.4981	-0.3071	0.1052

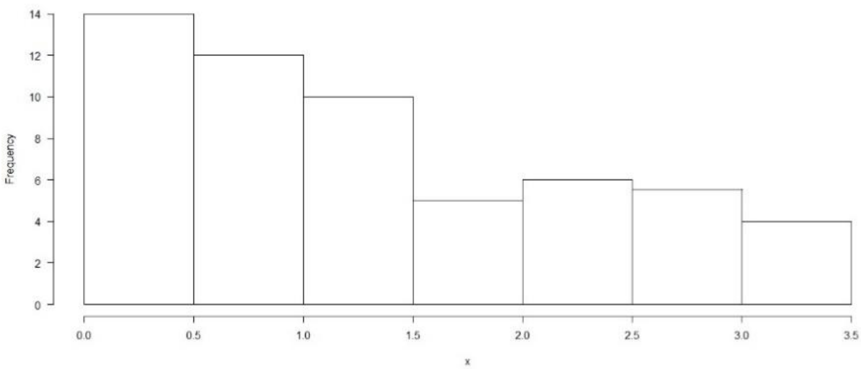

Figure 4 Histogram of the data on measurements of long-axis orientation of 64 feldspar laths in basalt

It is evident from the histogram of the data that the frequency of directions of lower magnitude is higher, which gradually decreases as we proceed towards directions of higher magnitude. In other words, the data set under consideration can be suitably modelled by a right skewed distribution. Again, it can be observed from the density plots of WLBWE(α,λ) presented in Figures 1 to 3 that the density curve of WLBWE(α,λ) is skewed towards the right. Thus, it may be appropriate to apply the WLBWE (α,λ) distribution to the data.

The wrapped length biased weighted exponential distribution, WLBWE(α,λ), is applied to the data set under consideration and the parameters are estimated. Table 3 summarizes the estimated values of the parameters. The goodness-of-fit of WLBWE(α,λ) to the data is checked using Watson's U^2 one sample test (Bhattacharjee and Das 2017).

Table 3 Estimated parameters of WLBWE(α, λ) fitted to the data on

Distribution	m.l.e of the parameters
WLBWE(α, λ)	$\hat{\lambda} = 0.00015, \hat{\alpha} = 17.97321$

The observed value of the Watson's U^2 one sample test statistic has come out to be 0.1228 whereas the critical value of the statistic at 5% level of significance 0.187. The test generates a p-value of 0.07 > 0.05, which shows that the Wrapped Length Biased Weighted Exponential distribution is a good fit to the given data. Figure 5 shows the distribution function plot of WLBWE(0.00015, 17.97321) fitted to the data.

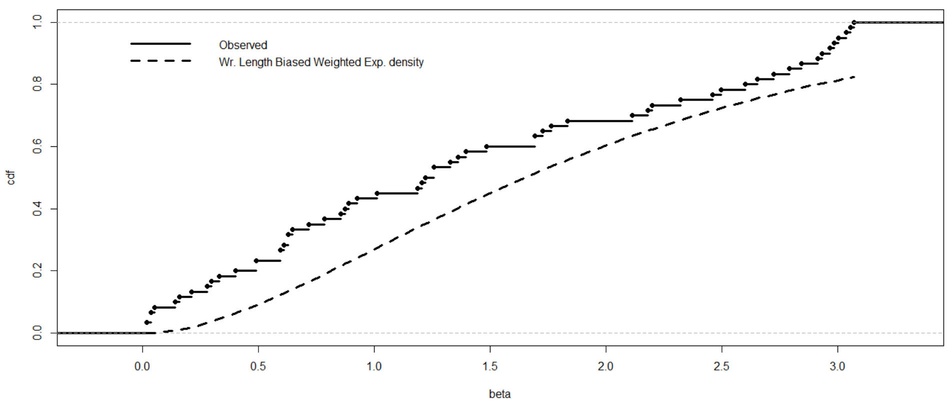

Figure 5 Distribution function plot of WLBWE (0.00015, 17.97321) fitted to the data on measurements of long-axis orientation of 64 feldspar laths in basalt

7. Conclusions

This paper introduces a new wrapped distribution namely wrapped length biased weighted exponential distribution with parameters α and λ is proposed. The expressions for the p.d.f. and c.d.f. of the proposed distribution are derived and their behavior is studied. Expressions for characteristic function, trigonometric moments and other measures are worked out. The maximum likelihood method of estimation is employed to estimate the parameters and a simulation study is performed to check the consistency of the m.l.e. of the parameters thus obtained. Lastly, to exhibit an application of the proposed model, the real data set on measurements of long-axis orientation of 164 feldspar laths in basalt is modelled using this distribution, as the histogram of the data suggests that this distribution might be appropriate. The goodness-of-fit test applied to the data showed that the wrapped length biased weighted exponential distribution is a good fit to the data set under consideration. This distribution is found to be more appropriate for modelling situations where directions of lower magnitude have high probability of occurrence and those of higher magnitude have low probability of occurrence.

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