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Construction of Bivariate Copulas on the Hotelling's T^2 Control Chart

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Abstract

In this paper, five types of copulas, which are Gumbel, Clayton, Farlie-Gumbel-Morgenstern (FGM), Frank and Ali-Mikail-Haq (AMH) copulas are presented via construction of bivariate copulas on the Hotelling's T^2 control chart. The observations are generated from the exponential distribution and the dependent observations are measured by Kendall's tau (τ) values as weak, moderate and strongly positive dependences where τ are 0.1, 0.2, 0.5, 0.6, 0.8 and 0.9, respectively. Monte Carlo simulation was used to compare the performance of the control chart with the Average Run Length (ARL) as performance metric. The results indicate that the bivariate copulas approach can be fitted to the Hotelling's T^2 control chart.

Keywords: marginal distribution, joint distribution, multivariate control chart, Monte Carlo simulation.

1. Introduction

At the present time, the statistical process control (SPC) is one of most rapidly developed methods that has been applied in many areas such as economics, engineering, finance and other fields. SPC's purpose is to detect, monitor, control and improve production quality of many fields, especially in the industrial sector. SPC is able to detect process changes as quickly as possible after they occur. A control chart as one of the seven major tools in the statistical process control is one of the primary techniques of SPC for change detection in mean or variations of the process. Control charts are designed and evaluated under an assumption that observations on the process are independent and identically distributed (i.i.d.) having the normal distributions. In most cases, several characteristics are involved in the supervision of the system (X is the vector), and these characteristics are often correlated. It is, therefore, necessary to consider multivariate control charts (Montgomery 2009; Bersimis et al. 2007). The most familiar multivariate process monitoring and control procedure of a multivariate process is the Hotelling's T^2 control chart for detecting the mean vector of the process, which was proposed by Shewhart (1931). The Hotelling's T^2 control chart consist of two versions: subgroup data and individual observations. This paper will focus on Hotelling's T^2 control chart for individual observations and on Phase II control chart. Most multivariate detection procedures are evaluated under observations assumption, which is similar to univariate detection procedures in which observations are independent and identically distributed (i.i.d.) having the multivariate normal distribution. However, many processes assume non-normality and correlation so many multivariate control charts lack related joint distribution. Nidsunkid et al. (2018) explored the performance of MCUSUM control chart when the multivariate normality assumption is violated. Sukparungsee et al. (2018) were introduced the copula can specify this property.

A copula is a joint distribution of random variables $U_1, U_2, ..., U_p$ each of which is marginally standard uniformly distributed as $U(0,1)$. Copula modeling is based on a representative of Sklar's Theorem (1973). Nelson (2006) refered to copulas as "functions that joint or couple multivariate distribution functions to their one-dimensional marginal distribution function" and as "distribution functions whose one-dimensional margins are uniform". In the past, researchers studied copulas on control charts. For example, Sukparungsee et al. (2017) examined multivariate copulas on the MCUSUM control chart when the observations came from exponential distributions. Kuvattana et al. (2015) studied performance comparison of bivariate copulas on the Cumulative Sum (CUSUM) and Exponentially Weighted Moving Average (EWMA) control charts when the observations came from exponential distributions. Kuvattana et al. (2015) examined the efficiency of bivariate copulas on the CUSUM chart when the observations came from exponential distributions under four types of copulas namely Normal, Clayton, Frank and Gumbel. Kuvattana et al. (2016) examined the bivariate copulas on the EWMA control chart when the observations came from exponential distributions under four types of copulas namely Normal, Clayton, Frank and Gumbel. Sukparungsee et al. (2018) also studied bivariate copulas on the Hotelling's T^2 control chart when the observations came from exponential distributions under five types of copulas namely Normal, Clayton, Frank, Gumbel and Joe.

In addition, Kuvattana et al. (2015) compared efficiency between multivariate Shewhart and multivariate CUSUM control chart for bivariate copulas when the observations came from exponential distribution under two types of copulas namely Normal and Frank. Research on the construction of the copulas is as follows. Lieebscher (2008) introduced two methods for the construction of asymmetric multivariate copulas and explored the properties of the proposed families of copulas such as the dependence of two components (Kendall's tau, tail dependence), marginal distributions and the generation of random variates. Durante et al. (2007) provided a method for constructing a class of multivariate copulas depending on univariate functions and studied some properties of such class, a new family of n-quasi-copulas. Vadier (2013) also presented a new approach for the non-normal multivariate case and the control chart proposed is based on copula modelling. Copula modelling is a very effective tool for multivariate modelling and is used in a large number of application domains. Fahati et al. (2011) discovered a bivariate control chart based on copula functions, Fahati and other researchers achieved the joint distribution of two correlated Zero-Inflated Poisson (ZIP) distributions using the copula function approach and develop a bivariate control chart, which can be used for monitoring correlated events.

From the above-mentioned statements, the copulas can be utilized to find the average run length (ARL) such as Monte Carlo simulation (MC) or other methods. Nevertheless, modern processes often detect and control more than one characteristic so the copulas, which are combined by constructions, have become popular tools for modelling nonlinearity and tail dependence in many areas. They can be used in the study of the dependence between random variables more than one copula. The construction of bivariate copulas can be applied to the multivariate control chart. For this reason, we have to study related literature, several methods of constructing copulas are as follows: Behan and Cox (2007) described a simple way to construct bivariate copulas with specified marginal and partially specified dependence and provide an Excel workbook to demonstrate the method, using Monte Carlo simulation. Dolati and Ubeda-Flores (2009) introduced two transformations on a given copula to construct new and retrieve already-existent families. This method is based on the choice of order statistics of the marginal distributions. Properties of the transformations and their effects on the dependence and symmetry structures of the copulas are also studied. Fernandez-Sanchez and Ubeda-Flores (2014) studied the construction of copulas that can be considered as a patchwork-like association of arbitrary copulas, with non-overlapping rectangles as patches. Their paper aims to provide a characterization of such constructions in order to be a copula. Yu and Voit (2006) studied bivariate distributions whose marginal is S-distributions. They are copulas to construct bivariate S-distributions because of copulas advantages. That is, correlation structure between random variables can be separated from the marginal. In addition, they discovered that distributions permit the full range of dependence $[-1,1]$ as well as varying degrees of tail-dependence, such as stronger dependence in the upper tail, lower tail or both. Ghosh (2017) introduced properties and applications of bivariate Kumaraswamy models via modified FGM copulas. Ghosh considered a new modified class of Farlie-Gumbel-Morgenstern (FGM) bivariate copula for constructing several different bivariate Kumaraswamy type copulas and discussed their structural properties, including dependence structures.

Other related literature on copulas is as follows. Biller and Corlu (2012) studied the copula-based input models which are appropriate to provide multivariate input-modelling support for stochastic simulations with dependent inputs by focusing on the tail dependence. They found that the bivariate copulas with tail-dependence power have been well studied, but they did not rapidly extend to multiple dimensions. Mazo et al. (2015) introduced a class of multivariate, called PBC copulas, which is based on a product of arbitrary bivariate copulas and use a graphical structure, which helps to visualize the dependencies and to compute the full joint likelihood, even in high dimension. Tang et al. (2013) investigated the impacts of copulas for modelling bivariate distributions on system reliability under incomplete probability information. They discovered that the probability of system failure of the parallel system under incomplete probability could not be uniquely determined.

In this paper, we discuss a method to construct a new class of bivariate copulas based on products of two symmetric copulas (e.g. FGM and AMH) and three asymmetric copulas (e.g. Clayton, Gumbel and Frank) with power arguments in order to determine if the proposed construction can offer the an added value for modeling asymmetric bivariate data (i.e. data from exponential distribution). Furthermore, Mukherjee et al. (2018) proposed construction of bivariate asymmetric copulas, these methods can be useful in fields such as finance, climate and social science. In addition, we approximate average run length (ARL) by using Monte Carlo simulation method based on construction of bivariate copulas on the Hotelling's T^2 control chart when observations are from exponential distribution and dependence among random variables, and we used nonparametric measures of association is Kendall's tau to estimate the parameter of constructing copulas. The paper is divided into 8 sections. Section 1 is the introduction. In Section 2, Hotelling's T^2 control chart is described. Section 3 presents copulas function and constructing bivariate copulas, and Section 4 describes copula parameters estimation for constructing bivariate copulas. Section 5 introduces a dependence measure of data. Section 6 describes the ARL and performances. In Section 7, we compare analytical results obtained from Monte Carlo simulation. Finally, the discussions and conclusions of the study are provided in Section 8.

2. Hotelling's T^2 Control Chart

Hotelling's T^2 control chart is the most familiar monitoring and control procedure of a multivariate process for monitoring the mean vector of the process. The Hotelling's T^2 control chart have two versions such as subgroup data and individual observations. However, this paper will focus on individual observations and on Phase II control charts and their performance.

Regarding individual observations, suppose that m samples, and p is the number of quality characteristics observed in each sample. Let $\bar{\mathbf{x}}$ and $\mathbf{S}$ be the mean vector and variance-covariance matrix, respectively, of these observations. The Hotelling's T^2 statistic is defined as

$$T^2 = \left(\mathbf{x}-\bar{\mathbf{x}}\right)' \mathbf{S}^{-1}\left(\mathbf{x}-\bar{\mathbf{x}}\right), \tag{1}$$

where $\bar{\mathbf{x}} = \frac{1}{m}\sum_{i=1}^{m}\mathbf{x}_i$ and $\mathbf{S} = \frac{1}{m-1}\sum_{i=1}^{m}\left(\mathbf{x}_i-\bar{\mathbf{x}}\right)'\left(\mathbf{x}_i-\bar{\mathbf{x}}\right).$

There are two distinct phases of the Hotelling's T^2 control chart usage. Phase I is the use of the charts to test whether the process is in control. Phase I's purpose is to obtain an in-control set of observations, so control limits can be established for Phase II.

Tracy et al. (1992) and Bersimis et al. (2007) showed that Phase I control limits for the T^2 control chart are given by

$$UCL = \frac{(m-1)^2}{m}\beta_{\alpha,\frac{p}{2},\frac{(m-p-1)}{2}} \quad \text{and} \quad LCL = 0, \tag{2}$$

where $\beta_{\alpha,\frac{p}{2},\frac{(m-p-1)}{2}}$ is upper α percentage point of a beta distribution with parameters $p/2$ and $(m-p-1)/2.$

The phase II control limits for this statistic are

$$UCL = \frac{p(m+1)(m-1)}{m^2-mp}F_{\alpha,p,m-p} \quad \text{and} \quad LCL = 0. \tag{3}$$

When the number of preliminary samples m is large $(m > 100)$ many practitioners use an approximate control chart limit, either

$$UCL = \frac{p(m-1)}{m-p}F_{\alpha,p,m-p}, \tag{4}$$

or

$$UCL = \chi^2_{\alpha,p}. \tag{5}$$

For $m > 100,$ (4) is a reasonable approximation. The chi-square limit in (5) is appropriate only if the variance-covariance matrix is known, but it is widely used as an approximate estimate.

3. Copulas Function and Construction of Bivariate Copulas

In accordance with copula function by Sklar's theorem, if G is an n-dimensional joint distribution function with 1-dimensional margin $F_1,\ldots,F_n,$ then there exists a function C (called an "n-copula") from the unit n-cube to the interval such that

$$G\left(x_1,x_2,\ldots,x_n\right) = C\left(F_1\left(x_1\right),F_2\left(x_2\right),\ldots,F_n\left(x_n\right)\right).$$

Theorem presented by Sklar (1959) is as follows:

Theorem 1 *Let H be a joint distribution function with margins F and G. Then there exists a copula C such that for all x, y in $\overline{\mathbb{R}}$,*

$$H(x, y) = C(F(x), G(y)). \tag{6}$$

If F and G are continuous, then C is unique; otherwise, C is uniquely determined on $\text{Ran}F \times \text{Ran}G$. Conversely, if C is a copula and F and G are distribution functions, then the function H defined by (6) is a joint distribution function with margins F and G.

From the above-mentioned illustration, we know about copulas. Next, we present general procedures to simulate bivariate as well as multivariate variables.

Therefore, we exploit the Lemma 1, Corollary 1, Theorem 2 by Nelson (2006) and Propositions 1 and 2 by Salvadori et al. (2007) to construct copulas, as follows:

Lemma 1 *Let H be a joint distribution function with margins F and G. Then there exists a unique subcopula C' such that*

1) *$Dom\ C' = RanF \times RanG$*
2) *For all x, y in $\overline{\mathbb{R}}$, $H(x, y) = C'(F(x), G(y))$.*

Corollary 1 *Let H, F, G and C' be as in Lemma 1, and let $F^{(-1)}$ and $G^{(-1)}$ be quasi-inverse of F and G, respectively. Then for any (u, v) in $DomC'$,*

$$C'(u, v) = H\left(F^{(-1)}(u), G^{(-1)}(v)\right).$$

Note: (1) Let C' be a subcopula. Then there exists copula C such that $C(u,v) = C'(u,v)$ for all (u,v) in Dom C'; i.e., any subcopula can be extended to a copula. The extension is generally non-unique.

(2) Let F be a distribution function. Then a *quasi-inverse* of F is any function $F^{(-1)}$ with domain $\mathbb{I}$ such that

2.1) if t is in $RanF$, then $F^{(-1)}(t)$ is any number x in $\overline{\mathbb{R}}$ such that $F(x) = t$, i.e., for all t in $RanF$,

$$F\left(F^{(-1)}(t)\right) = t.$$

2.2) if t is not in $RanF$, then

$$F^{(-1)}(t) = \inf\{x \mid F(x) \ge t\} = \sup\{x \mid F(x) \le t\}.$$

If F is strictly increasing, then it has but a single quasi-inverse, which is of course the ordinary inverse, for which we use the customary notation F^{-1}. In this paper, we concentrate the 2-dimensional case.

A general algorithm for generating observations (x, y) from a pair of random variable (r.v.)'s (X, Y) with marginals F_X, F_Y, joint distribution F_{XY}, and 2-copula C is as follows. By virtue of Sklar's Theorem, we need only to generate a pair (u, v) of observations of r.v.'s (U, V), Uniform on $\mathbb{I}$ and having the 2-copula C. Then using the probability integral transform, we transform (u, v) into (x, y), i.e.,

$$\begin{cases} x = F_X^{(-1)}(u) \\ y = F_Y^{(-1)}(v). \end{cases}$$

In order to generate the pair (u,v), we consider the conditional distributions of V given the event $\{U = u\}$ (Durante 2007):

$$c_u(v) = \mathbb{P}\{V \le v \mid U = u\} = \frac{\partial}{\partial u}\mathbb{C}(u,v).$$

A possible algorithm is as follows:

1. Generate independent variables u,t uniform on $\mathbb{I}$.
2. Set $v = c_u^{(-1)}(t)$.

The desired pair is then (u,v).

Theorem 2 *Let X and Y be continuous random variables with copula C_{XY}. If α and β are strictly increasing on $RanX$ and $RanY$, respectively, then $C_{\alpha(X)\beta(Y)} = C_{XY}$. Thus C_{XY} is invariant under strictly increasing transformations of X and Y.*

Let us denote by Θ the set of continuous and strictly increasing functions $h:\mathbb{I}\to\mathbb{I}$, with $h(0)=0$ and $h(1)=1$. Let $H:\mathbb{I}^2\to\mathbb{I}$ be increasing in each variable, with $H(0,0)=0$ and $H(1,1)=1$. Given f_1, f_2, g_1 and g_2 in Θ, and $\boldsymbol{A}$ and $\boldsymbol{B}$ 2-copulas. The function

$$F_{\mathbf{A},\mathbf{B}}(u,v) = H\left(\mathbf{A}(f_1(u), g_1(v)), \mathbf{B}(f_2(u), g_2(v))\right)$$

is called the composition of $\boldsymbol{A}$ and $\boldsymbol{B}$. In general, $F_{\mathbf{A},\mathbf{B}}$ is not a copula. Some conditions on H, f_1, f_2, g_1 and g_2 are given in order to ensure that $F_{\mathbf{A},\mathbf{B}}$ is a copula.

Proposition 1 *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two copulas. Then*

$$C_{\alpha,\beta}(u,v) = \mathbf{A}(u^{\alpha}, v^{\beta})\mathbf{B}\left(u^{1-\alpha}, v^{1-\beta}\right) \tag{7}$$

defines a family of copulas $C_{\alpha,\beta}$ with parameters $\alpha,\beta\in\mathbb{I}$.

In particular, if $\alpha=\beta=1$, then $C_{1,1}=\mathbf{A}$, and if $\alpha=\beta=0$, then $C_{0,0}=\mathbf{B}$. For $\alpha\ne\beta$, the copula C in (7) is, in general, asymmetric, that is $C(u,v)\ne C(v,u)$ for some $(u,v)\in\mathbb{I}^2$.

An interesting statistical interpretation can be given for this family. Let U_1, U_2, V_1 and V_2 be random variables uniform on $\mathbb{I}$. If $\boldsymbol{A}$ is the copula of (U_1,U_2) and $\boldsymbol{B}$ is the copula of (V_1,V_2), and the pairs (U_1,U_2) and (V_1,V_2) are independent, then $C_{\alpha,\beta}$ is the joint distribution function of

$$U = \max\left\{U_1^{1/\alpha}, V_1^{1/(1-\alpha)}\right\} \text{ and } V = \max\left\{U_2^{1/\beta}, V_2^{1/(1-\beta)}\right\}.$$

This probabilistic interpretation can be easily simulated by copula (7), (Salvadori et al. 2007).

Proposition 2 *Salvadori et al. (2007) shown the proposition (Khoudraji). Let C can be a symmetric copulas, $C\ne H_2$. A family of asymmetric copulas $C_{\alpha,\beta}$ with parameters $0<\alpha,\beta<1, \alpha\ne\beta$, that includes C as a limiting case, is given by*

$$C_{\alpha,\beta}(u,v) = u^{\alpha}, v^{\beta} C\left(u^{1-\alpha}, v^{1-\beta}\right). \tag{8}$$

4. Copula Parameters Estimation for Constructing Bivariate

In this section, we present copula parameters estimation for constructing bivariate copulas. In the first step, we generate 2 variables, which have the exponential distribution with positive correlation 0.1, 0.2, 0.5, 0.6, 0.8 and 0.9. In the second step, we transform data to continuous uniform distribution on interval (0,1). Finally, we find optimization of parameters for constructing bivariate copulas (α,β) as shown in (8). We used maximum pseudo-likelihood estimator and the Nelder and Mead (1965) method for optimization and which this paper implements, uses only functional values and is robust but relatively slow. The method will work reasonably well for non-differentiable functions.

Busababodhin and Amphanthong (2016) reviewed the copula modelling for multivariate statistical process control. The results of three control charts like multivariate Shewhart, MCUSUM and MEWMA, and use of copulas modelling for a bivariate case are considered.

This paper focuses on the combination of bivariate copulas consisting of the single copula as follows: Gumbel, Clayton, Farlie-Gumbel-Morgenstern (FGM), Frank and Ali-Mikail-Haq (AMH) copulas because Gumbel, Clayton and Frank are asymmetric copulas and space of parameter are different, moreover we would like to compare the single copula (Sukparungsee et al. 2018) with constructing bivariate copulas method. In addition, we choose FGM and AMH because there are symmetric copulas; furthermore, the one of the most important parametric family of copulas is the FGM family (Ghosh 2017). Mazo et al. (2015) examined a class of multivariate based on products of bivariate copulas based on single copula e.g. AMH and FGM etc.

Therefore, this study focuses on the combination of bivariate copulas consisting of the single copula as follows: Gumbel, Clayton, FGM, Frank and AMH copulas whose functions and properties are shown in Table 1. For different combinations of copulas, 10 combined copulas are considered as weak, moderate and strong dependence as Kendall's tau values at 0.1, 0.2, 0.5, 0.6, 0.8 and 0.9, respectively.

Table 1 Five different copula functions and properties

Copula	Type	Copula function	Kendall's tau	Parameter space of θ
Gumbel	Asymmetric	$\exp\left(-\left[\left(-\ln u\right)^{\theta} + \left(-\ln v\right)^{\theta}\right]^{1/\theta}\right)$	$\frac{\theta-1}{\theta}$	$[1,\infty)$
Clayton	Asymmetric	$\left[\max\left(u^{-\theta}+v^{-\theta}-1,0\right)\right]^{-1/\theta}$	$\frac{\theta}{\theta+2}$	$(0,\infty)$
FGM	Symmetric	$uv+\theta uv\left(1-u\right)\left(1-v\right)$	$\frac{2\theta}{3-\theta}$	$[-1,1]$
Frank	Asymmetric	$-\frac{1}{\theta}\log\left[1+\frac{\left(e^{-\theta u}-1\right)\left(e^{-\theta v}-1\right)}{e^{-\theta}-1}\right]$	$1-\frac{4\left(D_1\left(\theta\right)-1\right)}{\theta}$	$(-\infty,\infty)\setminus\{0\}$
AMH	Symmetric	$\frac{uv}{1-\theta\left(1-u\right)\left(1-v\right)}$	$\frac{\arcsin\left(\theta\right)}{\pi/2}$	$[-1,1]$

5. Dependence Measure of Data

According to theorem for bivariate case of copulas, a copula can be used in the study of dependence of association between random variables. In this paper, we use Kendall's tau to measure dependence of observations. If X and Y are two continuous random variables with copula C, then the Kendall's tau is given

$$\tau = 4\iint_{\mathbb{I}^2} C(u,v)dC(u,v) - 1.$$

Kendall's tau is a non-parametric measurement of the associations, which considers a copula based on dependence measurement. Genest and Mackay (1986) considered Kendall's tau of Archimedean copula C as

$$\tau_{Arch} = 4\int_0^1 \frac{\phi(t)}{\phi'(t)}dt + 1.$$

6. Average Run Length

Average Run Length (ARL) is the basic characteristic that describes the performance of control charts and is the average number of point plotted within the limits of control charts when the process behavior for a manufacturing operation or other fields such as business is evaluated.

For in-control process ARL_0 is the average of the observations are taken before the first point signal out of control and out-of-control process ARL_1, the expected of observations taken from change-point time until the control chart signals that the process is out-of-control. The expected of ARL_0 and ARL_1 as shown in (9) and (10)

$$\text{ARL}_0 \approx E_\omega(\kappa) \geq K; \quad \text{where } \omega = \infty \tag{9}$$

and

$$\text{ARL}_1 \approx E_\omega(\kappa \mid \kappa \geq 1); \quad \text{where } \omega = 1, \tag{10}$$

where ω is the change point time,

κ is the stopping time,

$E_\omega(.)$ is the expectation under the assumption that the change point occurs at time.

Theoretically, an acceptable ARL in a control condition should be adequate and is called ARL_0. However, ARL should be small when the process is out-of-control, in which case it is called the average of delay time (ARL_1).

In the section concerning simulation, we carry out to access the access to the performance of the Hotelling's T^2 control chart. Dependence of random variables via Kendall's tau is based on five type of copulas via constructing bivariate copulas. These copulas are implemented in the R Foundation for Statistical Computing (R Core Team 2013) with the number of simulations runs at 50,000 rounds. The observations are generated from copula based on exponential distribution with rate (λ) parameter equal 1, for an in-control process $(\lambda = 1)$ and shift of the process level (δ) by $\mu_1 = \mu_0 + (\sigma \cdot \delta)$ where $\mu_0 = \lambda$ and $\sigma^2 = \lambda$ (level of shift at 0.01, 0.05, 0.1, 0.5, 1 and 5) and the sample sizes at 6,000 for each type of copulas. For all copula models, setting θ corresponds with Kendall's tau as weak (τ = 0.1, 0.2), moderate (τ = 0.5, 0.6) and strong (τ = 0.8, 0.9) dependence, respectively.

7. Analytical Results

In this section, we show the results obtained from the empiric Monte Carlo simulation with 50,000 rounds via constructing bivariate copulas which is used to compare the ARL on Hotelling's T^2 control chart when the observations are from exponential $(\lambda = 1)$ for an in-control and shift of the process level (δ) by $\mu_0 + (\sigma \cdot \delta)$ where $\mu_0 = \lambda$ and $\sigma^2 = \lambda$ ($\delta =$ 0.01, 0.05, 0.1, 0.5, 1 and 5) and the sample sizes at 6,000 for each type of copula combinations.

Tables 2-7 show the simulation comparison results of ARL_1 of Hotelling's T^2 control chart obtained from Monte Carlo simulation (MC), which constructs bivariate copulas in (7) using Khoudraji's device in (8) and we obtain optimization parameter alpha and beta to constructing bivariate copulas by using maximum pseudo-likelihood estimation method. An analysis of performance is compared to the ARL_1 of copula's combinations when given ARL_0 = 370 and in control parameter of exponential distribution as $\lambda = 1$ when we shifts the exponential parameters, at weak, moderate and strong dependence (τ = 0.1, 0.2, 0.5, 0.6, 0.8, 0.9) as shown in Tables 2-7, respectively. UCL in Tables 2-7 correspond with ARL_0 = 370 by simulation studies. Tables 2 and 3 show weak dependence (τ = 0.1, 0.2) the performance of Frank×AMH is mostly superior to other combinations of copulas with minimum ARL_1 at all shifts. With moderate dependence (τ = 0.5, 0.6) the performance of Clayton×AMH is superior to other combinations of copula with all shifts as shown in Tables 4 and 5. Tables 6 and 7 show the strong dependence (τ = 0.8, 0.9) the performance of FGM×Frank gives the minimum ARL_1 value at small shift (δ = 0.01, 0.05), otherwise Frank×AMH, FGM×AMH, Clayton×AMH and Gumbel×AMH are superior to other copulas with the minimum ARL_1 at moderate and large shift, respectively.

Table 2 Comparison ARL_1 of the Hotelling's T^2 control chart via constructing bivariate copulas when $\lambda = 1$ and ARL_0 = 370 with weak dependence (τ = 0.1)

Copula names	Parameters of construction		UCL	Shifts (δ)					
	α	β		0.01	0.05	0.1	0.5	1	5
[1]	0.263	0.156	37.510	334.333	251.448	212.153	25.316	6.140	1.053
[2]	0.816	0.359	34.157	**323.799**	**239.562**	200.491	22.436	5.559	1.051
[3]	0.195	0.996	32.750	334.436	242.643	201.288	**20.901**	**5.212**	**1.046**
[4]	0.862	0.499	33.900	**326.885**	240.611	200.538	22.073	5.495	1.050
[5]	0.401	0.994	32.790	331.266	241.943	**199.362**	**21.096**	**5.250**	**1.047**
[6]	0.899	0.976	33.220	331.546	**239.621**	**198.861**	21.346	5.359	1.048
[7]	0.661	0.989	32.900	330.299	239.412	199.853	21.351	5.220	1.243
[8]	0.976	0.662	33.010	330.935	241.148	200.541	21.170	5.297	**1.047**
[9]	0.360	0.063	32.970	330.595	242.376	202.068	21.164	5.280	**1.047**
[10]	0.330	0.494	32.820	**327.633**	**238.782**	**198.960**	**20.956**	5.263	1.049

Bold figures refer to the three minimum of ARL_1 by shifts
Copula names i.e., [1] Gumbel×Clayton, [2] Gumbel×FGM, [3] Gumbel×Frank, [4] Gumbel×AMH, [5] Clayton×FGM, [6] Clayton×Frank, [7] Clayton×AMH, [8] FGM×Frank, [9] FGM×AMH, [10] Frank×AMH

Table 3 Comparison ARL_1 of the Hotelling's T^2 control chart via constructing bivariate copulas when λ = 1 and ARL_0 = 370 with weak dependence (τ = 0.2)

Copula names	Parameters of construction		UCL	Shifts (δ)					
	α	β		0.01	0.05	0.1	0.5	1	5
[1]	0.816	0.184	37.100	336.414	**230.902**	**162.587**	23.548	8.049	**1.052**
[2]	0.729	0.444	38.700	334.941	**231.862**	**163.117**	23.883	8.361	1.055
[3]	0.837	0.418	37.350	334.105	248.576	209.903	24.271	6.095	1.054
[4]	0.846	0.557	36.600	336.105	249.886	208.290	23.703	5.969	1.056
[5]	0.860	0.906	34.240	330.881	240.896	199.262	**21.374**	5.466	1.053
[6]	0.886	0.727	34.085	330.448	238.558	197.513	21.453	**5.409**	1.053
[7]	0.988	0.801	33.700	**330.297**	238.463	198.940	21.402	**5.304**	1.262
[8]	0.181	0.103	34.200	331.906	240.923	200.287	**21.390**	5.439	1.253
[9]	0.094	0.249	33.950	**329.251**	238.629	199.724	21.428	5.411	**1.051**
[10]	0.298	0.106	34.000	**327.134**	**236.614**	**197.092**	**21.326**	**5.392**	**1.052**

Bold figures refer to the three minimum of ARL_1 by shifts
Copula names i.e., [1] Gumbel×Clayton, [2] Gumbel×FGM, [3] Gumbel×Frank, [4] Gumbel×AMH, [5] Clayton×FGM, [6] Clayton×Frank, [7] Clayton×AMH, [8] FGM×Frank, [9] FGM×AMH, [10] Frank×AMH

Table 4 Comparison ARL_1 of the Hotelling's T^2 control chart via constructing bivariate copulas when λ = 1 and ARL_0 = 370 with moderate dependence (τ = 0.5)

Copula names	Parameters of construction		UCL	Shifts (δ)					
	α	β		0.01	0.05	0.1	0.5	1	5
[1]	0.421	0.381	58.250	337.137	259.206	222.983	31.401	8.495	1.621
[2]	0.625	0.557	47.950	336.797	**234.106**	**157.878**	27.299	7.601	1.090
[3]	0.583	0.473	56.800	337.900	259.389	219.722	28.419	8.008	1.139
[4]	0.545	0.554	50.100	337.541	236.994	159.657	22.287	7.844	**1.025**
[5]	0.763	0.825	34.300	**327.134**	238.697	197.825	**21.360**	**5.334**	1.258
[6]	0.909	0.495	38.173	328.840	237.765	196.310	21.735	5.537	1.070
[7]	0.817	0.929	34.722	**326.559**	**214.825**	**135.058**	**20.184**	**4.856**	**1.014**
[8]	0.825	0.786	40.830	**324.399**	236.982	195.667	21.594	5.666	1.085
[9]	0.204	0.216	34.103	329.778	**215.535**	**135.371**	**20.295**	5.347	1.256
[10]	0.915	0.653	34.505	330.087	240.380	198.562	21.533	**4.963**	1.282

Bold figures refer to the three minimum of ARL_1 by shifts
Copula names i.e., [1] Gumbel×Clayton, [2] Gumbel×FGM, [3] Gumbel×Frank, [4] Gumbel×AMH, [5] Clayton×FGM, [6] Clayton×Frank, [7] Clayton×AMH, [8] FGM×Frank, [9] FGM×AMH, [10] Frank×AMH

Table 5 Comparison ARL_1 of the Hotelling's T^2 control chart via constructing bivariate copulas when λ = 1 and ARL_0 = 370 with moderate dependence (τ = 0.6)

Copula names	Parameters of construction		UCL	Shifts (δ)					
	α	β		0.01	0.05	0.1	0.5	1	5
[1]	0.434	0.352	63.250	332.356	259.191	222.848	31.073	8.635	1.166
[2]	0.310	0.293	65.300	338.152	261.186	225.686	32.562	9.077	1.169
[3]	0.467	0.577	61.900	334.877	256.921	217.293	28.017	8.009	1.168
[4]	0.590	0.572	50.780	337.287	259.770	**159.185**	22.213	7.856	1.529
[5]	0.422	0.400	36.100	330.023	**237.078**	199.377	21.515	**4.961**	**1.067**
[6]	0.805	0.860	45.430	**324.540**	**233.572**	193.038	**21.478**	5.886	1.117
[7]	0.828	0.875	34.736	**329.323**	**213.844**	**134.581**	**16.064**	**4.786**	**1.014**
[8]	0.543	0.542	38.410	**328.787**	239.169	198.192	21.652	5.404	1.074
[9]	0.212	0.210	34.200	329.405	242.765	**135.722**	**15.982**	5.371	**1.011**
[10]	0.699	0.762	35.500	329.990	239.239	198.115	21.696	**4.830**	1.308

Bold figures refer to the three minimum of ARL_1 by shifts
Copula names i.e., [1] Gumbel×Clayton, [2] Gumbel×FGM, [3] Gumbel×Frank, [4] Gumbel×AMH, [5] Clayton×FGM, [6] Clayton×Frank, [7] Clayton×AMH, [8] FGM×Frank, [9] FGM×AMH, [10] Frank×AMH

Table 6 Comparison ARL_1 of the Hotelling's T^2 control chart via constructing bivariate copulas when λ = 1 and ARL_0 = 370 with strong dependence (τ = 0.8)

Copula names	Parameters of construction		UCL	Shifts (δ)					
	α	β		0.01	0.05	0.1	0.5	1	5
[1]	0.348	0.285	75.300	337.481	258.997	221.181	30.676	8.723	1.223
[2]	0.846	0.847	40.350	328.910	248.572	208.416	24.419	6.346	**1.076**
[3]	0.501	0.421	73.700	336.165	253.355	217.155	27.295	8.176	1.223
[4]	0.665	0.722	47.250	333.199	253.968	217.676	27.624	7.279	**1.028**
[5]	0.487	0.411	39.680	**328.827**	237.532	197.758	**21.653**	**5.334**	1.371
[6]	0.706	0.885	54.930	329.757	**232.216**	**189.714**	21.837	7.016	1.170
[7]	0.532	0.474	38.700	329.851	238.023	**197.183**	**21.677**	**5.013**	1.364
[8]	0.513	0.505	42.500	**327.128**	**234.057**	196.209	22.306	6.019	1.104
[9]	0.230	0.232	34.150	332.121	239.957	200.616	**21.363**	**5.313**	**1.013**
[10]	0.464	0.477	43.250	**326.862**	**234.461**	**132.291**	27.183	5.889	1.428

Bold figures refer to the three minimum of ARL_1 by shifts
Copula names i.e., [1] Gumbel×Clayton, [2] Gumbel×FGM, [3] Gumbel×Frank, [4] Gumbel×AMH, [5] Clayton×FGM, [6] Clayton×Frank, [7] Clayton×AMH, [8] FGM×Frank, [9] FGM×AMH, [10] Frank×AMH

Table 7 Comparison ARL_1 of the Hotelling's T^2 control chart via constructing bivariate copulas when λ = 1 and ARL_0 = 370 with strong dependence (τ = 0.9)

Copula names	Parameters of construction		UCL	Shifts (δ)					
	α	β		0.01	0.05	0.1	0.5	1	5
[1]	0.304	0.264	81.055	334.885	256.588	218.955	22.684	8.568	1.202
[2]	0.623	0.641	50.550	337.159	258.732	158.846	35.478	7.773	1.140
[3]	0.395	0.380	81.380	331.967	253.529	215.185	27.270	8.432	1.232
[4]	0.931	0.920	38.105	332.993	**220.462**	**141.298**	**17.308**	**5.183**	**1.019**
[5]	0.442	0.452	44.600	333.246	236.286	**132.372**	27.470	**6.214**	**1.136**
[6]	0.855	0.753	66.000	**326.021**	230.177	189.670	23.329	7.702	**1.085**
[7]	0.053	0.051	62.230	**328.928**	229.895	188.835	21.678	7.396	1.540
[8]	0.996	0.995	74.200	**327.745**	**229.528**	187.978	23.543	8.047	1.218
[9]	0.112	0.094	34.632	330.983	237.736	198.881	**21.416**	**5.445**	1.256
[10]	0.413	0.409	51.100	329.156	**214.039**	**134.085**	**18.762**	7.355	1.528

Bold figures refer to the three minimum of ARL_1 by shifts
Copula names i.e., [1] Gumbel×Clayton, [2] Gumbel×FGM, [3] Gumbel×Frank, [4] Gumbel×AMH, [5] Clayton×FGM, [6] Clayton×Frank, [7] Clayton×AMH, [8] FGM×Frank, [9] FGM×AMH, [10] Frank×AMH

Figure 1 clearly shows that the ARL_1 of the Hotelling's T^2 control chart via constructing bivariate copulas classified by dependence, at the weak dependence (τ = 0.1, 0.2) the performance of Frank×AMH is mostly superior to other combinations of copulas with minimum ARL_1 at all shifts. With moderate dependence (τ = 0.5, 0.6) the performance of Clayton×AMH is superior to other combinations of copula with all shifts. The strong dependence (τ = 0.8, 0.9) the performance of AMH×other copulas (i.e., Gumbel×AMH, Clayton×AMH, FGM×AMH and Frank×AMH) are superior to other combinations of copulas.

8. Conclusions and Discussions

This paper is intended to be the approximations of ARL for Hotelling's T^2 control chart, which are proposed when observations from bivariate copulas via constructed copula by Khoudraji's device could be used in simulation where the marginal of the variables is exponential as λ = 1. The simulation results suggest no meaningful difference as weak, moderate and strong dependence in the performance of copula's combinations at very large shift ($\delta \geq 5$). In addition, the performance of constructed bivariate copulas method on the Hotelling's T^2 control chart is superior to single copula at very large shift ($\delta \geq 5$) (Sukparungsee et al. 2018).

The magnitudes of shift are very small to moderate (δ = 0.01, 0.05, 0.1, 0.5) as weak dependence the performance of Frank×AMH is superior to other copula's combinations and moderate dependence the performance of Clayton×AMH gives the minimum ARL_1 value. In the case of strong dependence the performance of FGM×Frank gives the minimum ARL_1 value at the magnitude of small shift and the performance of combinations versus AMH (i.e., Gumbel×AMH, Clayton×AMH, Frank×AMH, FGM×AMH) gives the minimum ARL_1 value at the magnitudes of shift are moderate to large ($\delta < 5$). In addition, advantage of constructed bivariate copulas are tool for constructing multivariate distribution and formalizing the dependence structure between random variables and the method of constructed bivariate copulas can be useful in fields such as finance, climate change and social science (Mukherjee et al. 2018). However, construction of bivariate copulas might be the over fitting problem due to an increase of the number of parameters.

In the future work, we will extend our constructing of bivariate copulas with real data application. Some of copula families can be useful for analyzing the dependence structure of asymmetric data such as financial return data, cancer data in Bioinformatics and climate change etc. (Mukherjee et al. 2018)

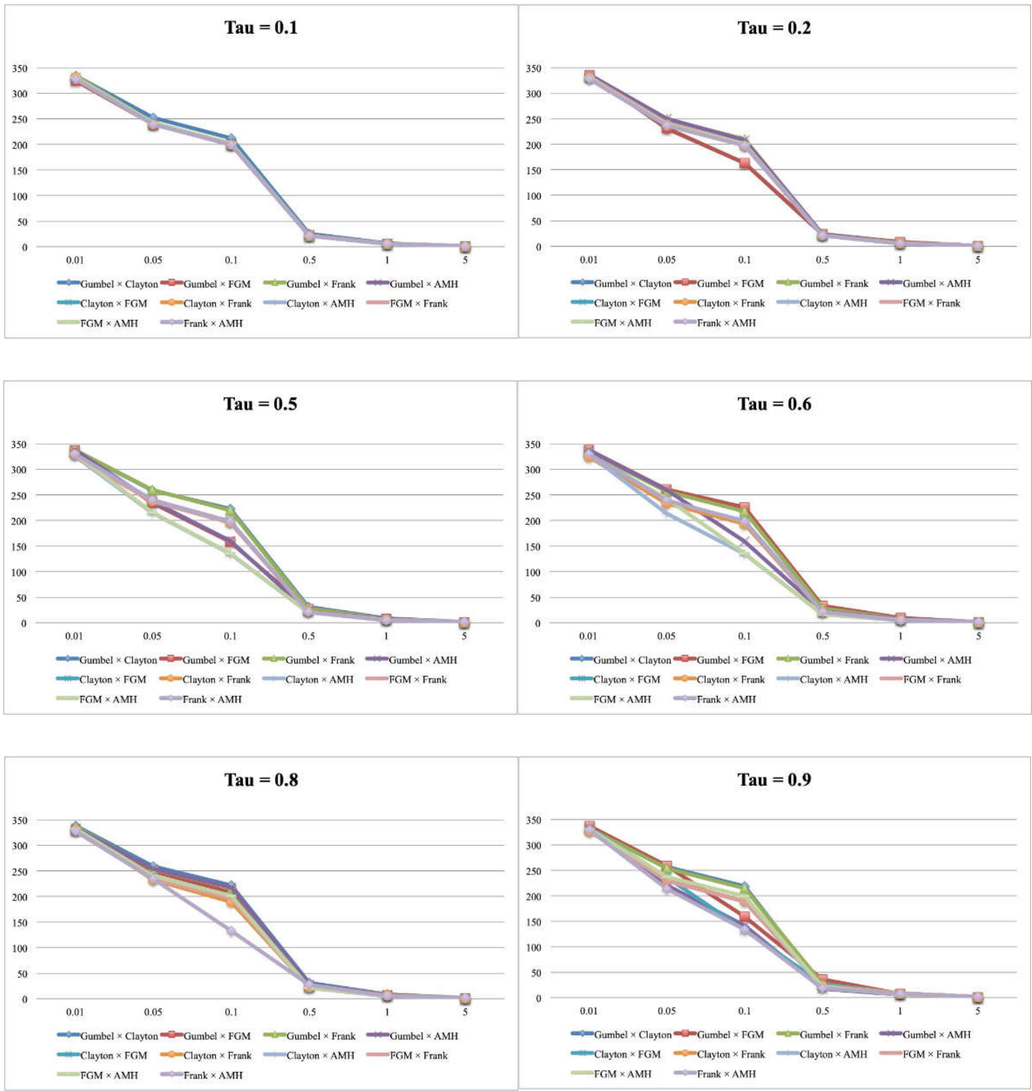

Figure 1 ARL_1 of the Hotelling's T^2 control chart via constructing bivariate copulas

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