Thailand Statistician
April 2020; 18(2): 95-107
http//:statassoc.or.th
Contributed paper

Characterization of General Class of Distributions by Truncated Moment

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Received: 21 September 2018
Revised: 17 January 2019
Accepted: 16 March 2019

Abstract

In this paper characterization studies are based on truncated moments for two general classes of continuous distributions. Characterization properties are discussed using a continuous function based on truncation from the right for a class of distributions, then the result is applied to the right truncated k^{th} moment for some distributions belonging to this class. A similar study is also carried out when truncation is from the left for another form of general class of distributions, and result is applied to the left truncated k^{th} inverse moment for some distributions belonging to this class. The results are obtained in simple and explicit manner which also unifies the earlier results obtained by several authors.

Keywords: Right truncation, left truncation, continuous probability distributions, Pareto distribution, power function distribution, exponential distribution, exponentiated exponential distribution.

1. Introduction

The characterization of probability distribution is very essential and plays an important role in statistical studies. It is used to know if the designed models fit the requirements of a specific underlying probability distribution or not. It has always been the topic of interest among researchers. There are several methods of characterization available in literature. Some of the characterization studies are based on moment properties, conditional expectation and dependence properties of order statistics, record values, generalized order statistics and its reverse case dual generalized order statistics. These various studies of characterization have been introduced by many authors, see for example Khan and Ali (1987), Nagaraja (1988), Khan and Abu-Salih (1989), Balasubramanian and Beg (1992), Franco and Ruiz (1995, 1997), Balasubramanian and Dey (1997), López-Blázquez and Moreno-Rebello (1997), Wesolowski and Ahsanullah (1997), Dembińska and Wesolowski (1998), Su and Huang (2000), Khan and Abouammoh (2000), Athar et al. (2003), Khan and Alzaid (2004), Khan and Athar (2004), Gupta and Ahsanullah (2006), Ahsanullah and Hamdani (2007), Huang and Su (2012),

Ahsanullah and Shakil (2013), Ahsanullah et al. (2014), Ahsanullah et al. (2015), Athar and Akhter (2015) and references therein.

In the recent time, there has been a great interest in the characterizations of probability distributions by truncated moments. The development of the general theory of the characterizations of probability distributions by truncated moment began with the work of Galambos and Kotz (1978). Further development on the characterizations of probability distributions by truncated moments continued with the contributions of many authors, among which Kotz and Shanbhag (1980), Glänzel (1987, 1990), Glänzel et al. (1984) and Ahsanullah et al. (2016) are notable.

In this paper the characterization studies are based on truncated moments for the two general classes of continuous distributions. First, we considered expectation of a continuous function based on truncation from the right for the class of distributions in its general case, then apply the result to the k^{th} moment for power function, Pareto, exponential and exponentiated exponential distributions. Secondly, we carried out a similar study for a continuous function when truncation is from left for the class of distributions, then apply the result to the k^{th} inverse moment for power function, Pareto and standard exponential distributions.

2. Characterization

Proposition 1 *Suppose an absolutely continuous (w.r.t Lebesgue measure) random variable X has cdf $F(x)$ and pdf $f(x)$ with $F(\alpha)=0$ and $F(\beta)=1$. Assume that $f'(x)$ and $E[\xi(X)\,|\,X\le x]$ exists, where $\xi(x)$ is a continuous function of x. Then*

$$E[\xi(X)\,|\,X\le x]=g(x)\eta(x), \tag{1}$$

where $g(x)$ and $\xi(x)$ are differentiable functions of $x\in(\alpha,\beta)$ and $\eta(x)=\dfrac{f(x)}{F(x)}$ with

$$f(x)=C\exp\left\{\int_{\alpha}^{x}\frac{\xi(u)-g'(u)}{g(u)}du\right\},$$

where C is determined such that

$$\int_{\alpha}^{\beta}f(x)dx=1.$$

Proof: We know that

$$E[\xi(X)\,|\,X\le x]=\frac{1}{F(x)}\int_{\alpha}^{x}\xi(u)f(u)du.$$

Therefore,

$$\frac{\int_{\alpha}^{x}\xi(u)f(u)du}{F(x)}=\frac{g(x)f(x)}{F(x)},$$

which implies

$$\int_{\alpha}^{x}\xi(u)f(u)du=g(x)f(x).$$

Differentiating both sides with respect to x, we get

$$\xi(x)f(x)=g'(x)f(x)+g(x)f'(x),$$

which on simplification yields

$$\frac{f'(x)}{f(x)} = \frac{\xi(x) - g'(x)}{g(x)}.$$

Integrating above expression, gives

$$f(x) = C \exp\left\{\int_{\alpha}^{x} \frac{\xi(u) - g'(u)}{g(u)} du\right\}.$$

Theorem 1 *Suppose an absolutely continuous (w.r.t Lebesgue measure) random variable X has cdf $F(x)$ and pdf $f(x)$ with $F(\alpha) = 0$ and $F(\beta) = 1$. Further, if $f'(x)$ and $E[X^k \mid X \le x]$ exists for all x, $\alpha < x < \beta$ and $k = 1, 2, \ldots$. Then*

$$E[X^k \mid X \le x] = g(x)\eta(x), \tag{2}$$

where

$$\eta(x) = \frac{f(x)}{F(x)}$$

and

$$g(x) = \frac{x^k h(x) + \frac{\alpha^k b}{a} - k\int_{\alpha}^{x} u^{k-1} h(u) du}{h'(x)} \tag{3}$$

if and only if

$$F(x) = ah(x) + b,\ a \ne 0, x \in (\alpha, \beta). \tag{4}$$

Proof: First we shall prove the necessary part.

We have

$$\int_{\alpha}^{x} \xi(u) f(u) du = g(x) f(x).$$

For the cdf given in (4), the pdf is $f(x) = ah'(x)$.

Let $\xi(u) = u^k$, then

$$g(x) = \frac{\int_{\alpha}^{x} u^k h'(u) du}{h'(x)} = \frac{x^k h(x) + \frac{\alpha^k b}{a} - k\int_{\alpha}^{x} u^{k-1} h(u) du}{h'(x)}.$$

To prove sufficiency part, we have

$$\frac{f'(x)}{f(x)} = \frac{x^k - g'(x)}{g(x)}. \tag{5}$$

Let $g(x) = A(x) + B(x) - C(x)$, where $A(x) = \frac{x^k h(x)}{h'(x)}$, implies

$$A'(x) = x^k \left\{1 + \frac{k\, h(x)}{x\, h'(x)} - \frac{h(x)\, h''(x)}{\left(h'(x)\right)^2}\right\}.$$

$B(x) = \frac{\alpha^k b}{a\, h'(x)}$, implies

$$B'(x) = -\frac{\alpha^k b}{a}\left(h'(x)\right)^{-2} h''(x),$$

and $C(x) = \frac{k}{h'(x)}\int_{\alpha}^{x} u^{k-1}h(u)du,$ implies

$$C'(x) = k\left[\frac{1}{h'(x)}x^{k-1}h(x) - \frac{h''(x)}{\left(h'(x)\right)^2}\int_{\alpha}^{x} u^{k-1}h(u)du\right].$$

Therefore,

$$g'(x) = x^k + \lambda(x)g(x),$$

where

$$\lambda(x) = -\frac{h''(x)}{h'(x)}.$$

Thus from (5), we get

$$\frac{f'(x)}{f(x)} = -\lambda(x) = \frac{h''(x)}{h'(x)}. \tag{6}$$

This implies

$$F(x) = ah(x) + b.$$

This completes the proof.

2.1. Examples

In this subsection, the characterization of some well-known distributions based on Theorem 1 is demonstrated.

2.2.1 Power function

The characterization of power function distribution is presented in the following corollary.

Corollary 1 *Let X be continuous random variable with cdf $F(x)$ and pdf $f(x)$ for $0 < x < 1,$ and that $f'(x)$ and $E(X^k \mid X \le x)$ exist for all x. Then for $k = 1, 2, \ldots$*

$$E[X^k \mid X \le x] = g(x)\eta(x), \tag{7}$$

where

$$g(x) = \frac{x^{k+1}}{k+p}$$

and

$$\eta(x) = \frac{f(x)}{F(x)} = \frac{p}{x}$$

if and only if

$$F(x) = x^p, 0 < x < 1, p > 0. \tag{8}$$

Proof: To prove if part of the corollary, compare (8) with (4) to get

$$a = 1,\ h(x) = x^p \text{ and } b = 0.$$

Therefore, in view of (3) with $\alpha = 0,$ it can be easily seen that

$$g(x) = \frac{x^{k+1}}{k+p}.$$

Hence, if part of the corollary is proved.
To prove only if part, in view of (6), we have

$$\frac{f'(x)}{f(x)} = \frac{h''(x)}{h'(x)} = \frac{p-1}{x}.$$

On integrating both the sides of the above expression, we get

$$f(x) = C\,x^{p-1},$$

where C is a constant. Now after using the condition $\int_{-\infty}^{\infty} f(x)dx = 1,$ we get

$$f(x) = px^{p-1}, 0 < x < 1.$$

Further,

$$\int_0^x f(t)dt = p\int_0^x t^{p-1}dt,$$

which gives

$$F(x) = x^p, 0 < x < 1.$$

2.1.2 Pareto distribution

In the following corollary characterization of Pareto distribution is presented.

Corollary 2 *Suppose a random variable X has an absolutely continuous (w.r.t. Lebesgue measure) cdf $F(x)$ and pdf $f(x)$. Assume that $F(1) = 0$ and $F(\infty) = 1$ for all $x > 1,$ and $E(X^k \mid X \le x)$ exist for all x. Then, for $k = 1, 2, \ldots$*

$$E[X^k \mid X \le x] = g(x)\eta(x), \tag{9}$$

where

$$g(x) = \frac{x^{p+1} - x^{k+1}}{p-k}$$

and

$$\eta(x) = \frac{f(x)}{F(x)} = \frac{px^{-(p+1)}}{1-x^{-p}}$$

if and only if

$$F(x) = 1 - x^{-p};\ p > 1, 1 < x < \infty. \tag{10}$$

Proof: First, we shall prove necessary part. On comparison of (10) with (4), we have

$$a = -1,\ h(x) = x^{-p},\ b = 1 \text{ and } \alpha = 1.$$

Thus, $g(x)$ can be obtained using (3) as

$$g(x) = \frac{x^{p+1} - x^{k+1}}{p-k}.$$

Thus, the necessary part is proved.
To prove sufficiency part, on using (6), we get

$$\frac{f'(x)}{f(x)} = \frac{h''(x)}{h'(x)} = -\frac{p+1}{x}.$$

This gives

$$f(x) = p\,x^{-(p+1)},\ p > 1,\ 1 < x < \infty.$$

Now integerating both sides,

$$\int_1^x f(t)dt = p\int_1^x t^{-(p+1)}dt,$$

which gives

$$F(x) = 1 - x^{-p};\ p > 1,\ 1 < x < \infty.$$

2.1.3 Exponential distribution

The characterization of exponential distribution is provided in the following corollary.

Corollary 3 *Let* X *be continuous random variable with cdf* $F(x)$ *and pdf* $f(x)$ *for* $x > 0$, *and that* $f'(x)$ *and* $E(X^k \mid X \le x)$ *exist for all* x. *For* $k = 1, 2, \ldots$

$$E[X^k \mid X \le x] = g(x)\eta(x), \tag{11}$$

where

$$g(x) = -\frac{x^k}{\theta} + \frac{k\gamma(k,x;\theta)}{\theta e^{-\theta x}},$$

$\gamma(k,x;\theta) = \int_0^x u^{k-1}e^{-\theta u}du$ *is the lower incomplete gamma and* $\eta(x) = \dfrac{f(x)}{F(x)} = \dfrac{\theta e^{-\theta x}}{1-e^{-\theta x}}$

if and only if

$$F(x) = 1 - e^{-\theta x};\ x > 0, \theta > 0. \tag{12}$$

Proof: First, we shall use (12) to get (11). After comparing (12) with (4), we get

$$a = -1,\ h(x) = e^{-\theta x},\ b = 1 \text{ and } \alpha = 0.$$

Thus, in view of (3),

$$g(x) = -\frac{x^k}{\theta} + \frac{k\gamma(k,x;\theta)}{\theta e^{-\theta x}},$$

which gives the equation (11). Now, to prove (11) implies (12), in view of (6), we have

$$\frac{f'(x)}{f(x)} = \frac{h''(x)}{h'(x)} = -\frac{\theta^2 e^{-\theta x}}{\theta e^{-\theta x}} = -\theta.$$

This implies

$$f(x) = \theta e^{-\theta x};\ x > 0,\ \theta > 0$$

and subsequently,

$$F(x) = 1 - e^{-\theta x};\ x > 0, \theta > 0.$$

2.1.4 Exponentiated exponential distribution

The characterization of exponentiated exponential distribution is presented in the following corollary.

Corollary 4 *Suppose a random variable X has an absolutely continuous (w.r.t. Lebesgue measure) cdf $F(x)$ and pdf $f(x)$. Assume that $F(0)=0$ and $F(\infty)=1$ for all $x>0$, and $E(X \mid X \le x)$ exist for all x, then*

$$E[X \mid X \le x] = g(x)\eta(x), \tag{13}$$

where

$$g(x) = \frac{x(1-e^{-\lambda x})^{\delta}}{\delta\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\delta-1}} - \frac{\int_0^x (1-e^{-\lambda t})^{\delta}\,dt}{\delta\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\delta-1}}$$

and $\eta(x) = \dfrac{f(x)}{F(x)} = \dfrac{\lambda\delta e^{-\lambda x}}{1-e^{-\lambda x}}$ *if and only if*

$$F(x) = (1-e^{-\lambda x})^{\delta};\ \delta>0, \lambda>0, x>0. \tag{14}$$

Proof: First, we shall prove the necessary part. When we compare (14) with (4), then we have

$$a=1,\ h(x)=(1-e^{-\lambda x})^{\delta},\ b=0 \text{ and } \alpha=0.$$

Therefore in view of (3), it can be seen easily that

$$g(x) = \frac{x(1-e^{-\lambda x})^{\delta}}{\delta\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\delta-1}} - \frac{\int_0^x (1-e^{-\lambda t})^{\delta}\,dt}{\delta\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\delta-1}}.$$

Hence, necessary part of the corollary is proved.

To prove sufficiency part, on application of (6), it is noted that

$$\frac{f'(x)}{f(x)} = \frac{h''(x)}{h'(x)} = -\lambda + \frac{(\delta-1)\lambda e^{-\lambda x}}{1-e^{-\lambda x}}.$$

After integrating the above expression, we get

$$f(x) = k\, e^{\int_0^x S(t)dt}.$$

Since,

$$\int_0^x S(t)\,dt = \int_0^x \left(-\lambda + \frac{(\delta-1)\lambda e^{-\lambda t}}{1-e^{-\lambda t}}\right) dt = -\lambda x + (\delta-1)\ln(1-e^{-\lambda x}).$$

Therefore,

$$f(x) = ke^{-\lambda x}(1-e^{-\lambda x})^{\delta-1}.$$

Now $\int_0^{\infty} f(x)dx = 1$ gives $k = \delta\lambda$. Thus,

$$F(x) = (1-e^{-\lambda x})^{\delta};\ \delta>0,\ x>0,$$

which completes the proof.

Similarly with proper choice of a, b and $h(x)$ several distributions may be characterized using Theorem 1. For more distributions belonging to this general class, one may refer to Khan and Abu-Salih (1989) and Athar and Akhter (2015).

Proposition 2 *For an absolutely continuous (w.r.t Lebesgue measure) random variable $X \in (\alpha, \beta)$ with cdf $F(x)$ and pdf $f(x)$. Assume that $f'(x)$ and $E[\xi(X) \mid X \ge x]$ exists, where $\xi(x)$ is a continuous function of x. Then*

$$E[\xi(X) \mid X \geq x] = g(x)\tau(x), \tag{15}$$

where $g(x)$ *and* $\xi(x)$ *are differentiable functions of* x *and* $\tau(x) = \dfrac{f(x)}{1-F(x)}$ *with*

$$f(x) = C \exp\left\{-\int_x^\beta \frac{\xi(t)+g'(t)}{g(t)}\right\} dt,$$

where C *is determined such that*

$$\int_\alpha^\beta f(x)dx = 1.$$

Proof: We know that

$$E[\xi(X) \mid X \geq x] = \frac{1}{1-F(x)} \int_x^\beta \xi(t) f(t) dt,$$

which implies

$$\int_x^\beta \xi(t) f(t) dt = g(x) f(x).$$

Differentiating both sides with respect to x, we get

$$-\xi(x) f(x) = g'(x) f(x) + g(x) f'(x),$$

which on simplification gives

$$\frac{f'(x)}{f(x)} = -\frac{\xi(x)+g'(x)}{g(x)}. \tag{16}$$

Integrating above expression (16), gives

$$f(x) = C \exp\left\{-\int_x^\beta \frac{\xi(t)+g'(t)}{g(t)}\right\} dt.$$

Theorem 2 *For an absolutely continuous (w.r.t. Lebesgue measure) random variable* X *having cdf* $F(x)$ *and pdf* $f(x)$ *with* $F(\alpha) = 0$ *and* $F(\beta) = 1$. *Further, if* $f'(x)$ *and* $E(X^{-k} \mid X \geq x)$ *exists for all* x, $\alpha < x < \beta$ *and* $k = 1, 2, \ldots$ *Then*

$$E[X^{-k} \mid X \geq x] = g(x)\tau(x), \tag{17}$$

where

$$\tau(x) = \frac{f(x)}{1-F(x)}$$

and

$$g(x) = \frac{-\dfrac{\beta^{-k} b}{a} - x^{-k} h(x) + k\displaystyle\int_x^\beta t^{-(k+1)} h(t) dt}{h'(x)} \tag{18}$$

if and only if

$$1 - F(x) = ah(x) + b,\ x \in (\alpha, \beta) \tag{19}$$

with $a \neq 0$ *and* $b \neq 0$.

Proof: Necessary part can be proved on the lines of Theorem 1.
To prove sufficiency part, we have

$$g(x)=\frac{-\frac{\beta^{-k}b}{a}-x^{-k}h(x)+k\int_x^{\beta}t^{-(k+1)}h(t)dt}{h'(x)},$$

which implies

$$g'(x)=-x^{-k}-\frac{h''(x)}{h'(x)}g(x).$$

Therefore, in view of (16) with $\xi(x)=x^{-k}$, we get

$$\frac{f'(x)}{f(x)}=-\frac{x^{-k}+g'(x)}{g(x)}=\frac{h''(x)}{h'(x)}.$$

On integrating above equation gives

$$f(x)=-ah'(x),$$

which implies

$$F(x)=1-[ah(x)+b].$$

2.2. Examples

In this subsection, the characterization of some well known distributions based on Theorem 2 is presented.

2.2.1 Power function

In the following corollary, characterization of power function distribution is presented.

Corollary 5 *Let* X *be continuous random variable with cdf* $F(x)$ *and pdf* $f(x)$ *for* $x>0$, *and that* $f'(x)$ *and* $E(X^{-k}\mid X\geq x)$ *exist for all* x *and* $k=1,2,\ldots$ *Then*

$$E[X^{-k}\mid X\geq x]=g(x)\tau(x), \tag{20}$$

where

$$g(x)=-\frac{1-x^{-(k-p)}}{(k-p)x^{p-1}}$$

and

$$\tau(x)=\frac{f(x)}{1-F(x)}=\frac{px^{p-1}}{1-x^{p}}$$

if and only if

$$F(x)=x^{p};\ p>1, 0<x<1. \tag{21}$$

Proof: Necessary Part: On comparison of (21) with (19), we get

$$a=-1, h(x)=x^{p}, b=1 \text{ and } \beta=1.$$

Therefore, in view of (18), it can be seen that

$$g(x)=-\frac{1-x^{-(k-p)}}{(k-p)x^{p-1}}.$$

Thus, necessary part is proved. Sufficiency Part: We have

$$\frac{f'(x)}{f(x)} = \frac{h''(x)}{h'(x)} = \frac{p-1}{x}.$$

This implies

$$f(x) = p\,x^{p-1},\ 0 < x < 1,$$

and hence

$$F(x) = x^p;\ p > 1, 0 < x < 1.$$

2.2.2 Pareto distribution

In the following corollary characterization of Pareto distribution based on Theorem 2 is demonstrated.

Corollary 6 *Suppose a random variable* X *has an absolutely continuous (w.r.t Lebesgue measure) cdf* $F(x)$ *and pdf* $f(x)$. *Assume that* $F(1) = 0$ *and* $F(\infty) = 1$ *for all* $x > 1$, *and* $E(X^{-k} \mid X \geq x)$ *exists for all* x. *Then,* $k = 1, 2, \ldots$

$$E[X^{-k} \mid X \geq x] = g(x)\tau(x), \tag{22}$$

where

$$g(x) = \frac{1}{k+p}\frac{1}{x^{k-1}}$$

and

$$\tau(x) = \frac{f(x)}{1-F(x)} = \frac{p}{x}$$

if and only if

$$F(x) = 1 - x^{-p};\ p > 1, 1 < x < \infty. \tag{23}$$

Proof: First, we shall prove (23) implies (22).

Now on comparison of (23) with (19), we get

$$a = 1,\ h(x) = 1 + x^{-p},\ b = -1 \text{ and } \beta = \infty.$$

Thus, $g(x)$ can be obtained using (18).

To prove (22) implies (23), we have

$$\frac{f'(x)}{f(x)} = \frac{h''(x)}{h'(x)} = -\frac{p+1}{x}.$$

This gives

$$f(x) = p\,x^{-(p+1)};\ p > 1,\ 1 < x < \infty$$

and hence

$$F(x) = 1 - x^{-p};\ p > 1,\ 1 < x < \infty.$$

2.2.3 Standard exponential distribution

In following corollary characterization of staandard exponential distribution using Theorem 2 is presented.

Corollary 7 *Let* X *be continuous random variable with cdf* $F(x)$ *and pdf* $f(x)$ *for* $x > 0$, *and that* $f'(x)$ *and* $E(X^{-k} \mid X \geq x)$ *exist for all* x. *Then for* $k = 1, 2, \ldots$

$$E[X^{-k} \mid X \geq x] = g(x)\tau(x), \tag{24}$$

where

$$g(x) = x^{-k} - \frac{k\,\Gamma(-k,t)}{e^{-x}},$$

$\Gamma(a,x) = \int_x^{\infty} t^{a-1}e^{-t}dt$, $a \in \mathbb{R}$, *is upper incomplete gamma and*

$$\tau(x) = \frac{f(x)}{1-F(x)} = 1$$

if and only if

$$F(x) = 1 - e^{-x},\ x > 0. \tag{25}$$

Proof: Necessary part: Suppose (25) is true. Then on comparison of (25) with (19), we get

$$a = 1,\ h(x) = 1 + e^{-x},\ b = -1 \text{ and } \beta = \infty.$$

Now, in view of (18),

$$g(x) = x^{-k} - \frac{k\,\Gamma(-k,t)}{e^{-x}}.$$

Hence, necessary part is proved. Sufficiency part: We have

$$\frac{f'(x)}{f(x)} = \frac{h''(x)}{h'(x)} = \frac{e^{-x}}{-e^{-x}} = -1.$$

This implies

$$f(x) = e^{-x},\ x > 0,$$

or

$$F(x) = 1 - e^{-x},\ x > 0.$$

Similarly with proper choice of a, b and $h(x)$ several distributions may be characterized by using the Theorem 2. For more examples belonging to the given class one may refer to Khan and Abu-Salih (1989) and Noor and Athar (2014).

3. Conclusions

The above study demonstrates the important characterization properties of two general forms of distributions by truncated moments. Further, some basic continuous distributions like power function, exponential, Pareto, exponentiated exponential are considered as examples of this study. One can apply these theorems to other distributions. Since characterization results play an important role in the determination of distribution, therefore these results may be useful for the researchers who belong to applied and physical sciences.

Acknowledgements

The authors acknowledge with thanks to both the referees and the editor for their fruitful suggestions and comments, which led to the overall improvement in the manuscript.

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