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An Empirical Study of Second Order Rotatable Designs under Tri-Diagonal Correlated Structure of Errors Using a Pair of Partially Balanced Incomplete Block Type Designs

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Abstract

In this paper, following the works of Das (1997, 1999 and 2003), Rajyalakshmi and Victorbabu (2014, 2016 and 2018), second order rotatable designs under tri-diagonal correlated structure of errors using a pair of partially balanced incomplete block type designs (PBIBD) is suggested. Here we study the variance function of estimated response for different values of tri-diagonal correlated coefficient (ρ) and distance from the center (d) for various factors. This method gives designs with less number of design points than the corresponding second order rotatable designs (SORD) under tri-diagonal correlated structure of errors using pairwise balanced designs (PBD) and balanced incomplete block type designs (BIBD).

Keywords: Rotatability, correlated errors, incomplete block designs.

1. Introduction

The concept of rotatability was proposed by Box and Hunter (1957). Das and Narasimham (1962) constructed second order rotatable designs through balanced incomplete block designs. Narasimham et al. (1983) constructed second order rotatable designs through a pair of balanced incomplete block designs. Chowdhury and Gupta (1985) constructed second order rotatable designs associated with partially balanced incomplete block designs. Victorbabu (2004) suggested second order rotatable designs using a pair of partially balanced incomplete block designs. Panda and Das (1994) studied first order rotatable designs with correlated errors. Das (1997, 1999, 2003 and 2014) introduced and studied robust second order rotatable designs. Rajyalakshmi and Victorbabu (2014) suggested an empirical study of second order rotatable

designs under tri-diagonal correlated structures of errors using central composite designs. Rajyalakshmi and Victorbabu (2016) suggested an empirical study of second order rotatable designs under tri-diagonal correlated structures of errors using incomplete block designs like pairwise balanced design and symmetrical unequal block arrangements (SUBA) with two unequal block sizes and observed that it gives less number of design points than the corresponding SORD under tri-diagonal correlated structure of errors using balanced incomplete block designs in some cases. Rajyalakshmi and Victorbabu(2018) suggested an empirical study of second order rotatable designs under tri-diagonal correlated structures of errors using balanced incomplete block designs and observed that it gives less number of design points than the corresponding SORD under tri-diagonal correlated structure of errors using central composite designs in some cases.

In this paper, following the works of Das (1997, 1999 and 2003), Rajyalakshmi and Victorbabu (2014, 2016 and 2018), second order rotatable designs under tri-diagonal correlated structure of errors using a pair of partially balanced incomplete block designs (PBIBD) is suggested. Here we study the variance function of estimated response for the different values of tri-diagonal correlated coefficient (ρ) and distance from the center (d) for various factors. This method gives designs with less number of design points than the corresponding SORD under tri-diagonal correlated structure of errors using central composite designs (CCD), BIBD, pairwise balanced designs (PBD) and symmetrical unequal block arrangements (SUBA) with two unequal block sizes in some cases.

2. Conditions for Second Order Rotatable Designs (cf. Box and Hunter 1957)

A second order response surface as $D = ((x_{iu}))$ for fitting

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i=1}^{v} \sum_{j=1}^{v} b_{ij} x_{iu} x_{ju} + e_u, \quad (i \le j), \tag{1}$$

where x_{iu} denotes the level of the i^{th} factor $(i = 1, 2, ..., v)$ in the u^{th} run $(u = 1, 2, ..., N)$ of the experiment, e_u 's are uncorrelated random errors. A second order response surface design D is said to be a SORD, if the variance of the estimated response of $\hat{y}_u$ from the fitted surface is only a function of the distance $\left(d^2 = \sum_{i=1}^{v} x_i^2\right)$ from the origin (center) of the design i.e., $(V(\hat{y}_u) = g(d^2))$. Such a spherical variance function for estimation of responses in the second order response surface is achieved if the design points satisfy the following conditions (cf. Box and Hunter 1957):

$$\sum_{u=1}^{N} \prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0 \text{ if any } \alpha_i \text{ is odd for } \sum \alpha_i \le 4, \tag{2}$$

$$\sum_{u=1}^{N} x_{iu}^2 = \text{constant} = N\lambda_2, \tag{3}$$

$$\sum_{u=1}^{N} x_{iu}^4 = \text{constant} = 3N\lambda_4 \text{ for all } i, \tag{4}$$

$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \text{constant} = N\lambda_4 \text{ for all } i \neq j, \tag{5}$$

$$\sum_{u=1}^{N} x_{iu}^4 = 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2, \tag{6}$$

$$\frac{\lambda_4}{\lambda_2^2} > \frac{v}{(v+2)} \text{ (non-singularity condition)}, \tag{7}$$

where λ_2 and λ_4 are constants.

The variance and covariance of the estimated parameters are

$$V(\hat{b}_0) = \frac{\lambda_4(v+2)\sigma^2}{N\Delta},$$

$$V(\hat{b}_i) = \frac{\sigma^2}{N\lambda_2},$$

$$V(\hat{b}_{ij}) = \frac{\sigma^2}{N\lambda_4},$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2(\lambda_4(v+1)-(v-1)\lambda_2^2)}{2N\lambda_4\Delta},$$

$$Cov(\hat{b}_0, \hat{b}_{ii}) = \frac{-\lambda_2\sigma^2}{N\Delta}, \sqrt{b^2-4ac}$$

$$Cov(\hat{b}_{ii}, \hat{b}_{ij}) = \frac{\sigma^2(\lambda_2^2-\lambda_4)}{2N\lambda_4\Delta}, \tag{8}$$

where $\Delta = [\lambda_4(v+2) - v\lambda_2^2]$ and other co-variances vanish.

Further, $$V(\hat{Y}) = V(\hat{b}_0) + [V(\hat{b}_i) + 2Cov(\hat{b}_0, \hat{b}_{ii})]d^2 + V(\hat{b}_{ii})d^4. \tag{9}$$

3. Conditions for SORD under Tri-diagonal Correlated Structure of Errors (cf. Das 2003)

A second order response surface design as given in Equation (1) with e_u's are correlated random errors is said to be a SORD under tri-diagonal correlated structure of errors, if the variance of the estimated response of $\hat{y}_u$ from the fitted surface is only a function of the distance $\left(d^2 = \sum_{i=1}^{v} x_i^2\right)$ from the origin (center) of the design i.e., $(V(\hat{y}_u) = g(d^2))$. Such a spherical variance function for estimation of responses in the second order response surface is achieved if the design points satisfy the following conditions (cf. Das 2003):

$$\sum_{u=1}^{2n}\prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0 \text{ if any } \alpha_i \text{ is odd for } \sum\alpha_i \leq 4, \tag{10}$$

$$\sum_{u=1}^{2n} x_{iu}^2 = \text{constant} = 2n\lambda_2, \tag{11}$$

$$\sum_{u=1}^{2n} x_{iu}^4 = \text{constant} = 3(2n)\lambda_4 \text{ for all } i, \tag{12}$$

$$\sum_{u=1}^{2n} x_{iu}^2 x_{ju}^2 = \text{constant} = 2n\lambda_4 \text{ for all } i \neq j, \tag{13}$$

$$\sum_{u=1}^{2n} x_{iu}^4 = 3\sum_{u=1}^{2n} x_{iu}^2 x_{ju}^2, \tag{14}$$

$$\frac{\lambda_4}{\lambda_2^2} > \frac{v(1-\rho)}{(v+2)} \quad \text{(non-singularity condition)}, \tag{15}$$

where λ_2 and λ_4 are constants.

The variances and covariances of the estimated parameters are

$$V(\hat{b}_0) = \frac{\lambda_4(v+2)(1+\rho)\sigma^2}{2n\Delta},$$

$$V(\hat{b}_i) = \frac{\sigma^2(1-\rho^2)}{2n\lambda_2},$$

$$V(\hat{b}_{ij}) = \frac{\sigma^2(1-\rho^2)}{2n\lambda_4},$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2(1-\rho^2)(\lambda_4(v+1)-(v-1)\lambda_2^2(1-\rho))}{2(2n)\lambda_4\Delta},$$

$$Cov(\hat{b}_0, \hat{b}_{ii}) = \frac{-\lambda_2\sigma^2(1-\rho^2)}{2n\Delta},$$

$$Cov(\hat{b}_{ii}, \hat{b}_{ij}) = \frac{\sigma^2(1-\rho^2)(\lambda_2^2(1-\rho)-\lambda_4)}{2(2n)\lambda_4\Delta}, \tag{16}$$

where $\Delta = (\lambda_4(v+2) - v\lambda_2^2(1-\rho))$ and other covariances are zero.

Here ρ is the correlation between errors of any two observations each has the same variance σ^2 and n -denotes non-central design points.

Further, $V(\hat{Y}) = V(\hat{b}_0) + (V(\hat{b}_i) + 2Cov(\hat{b}_0, \hat{b}_{ii}))d^2 + V(\hat{b}_{ii})d^4,$

$$\begin{aligned} V(\hat{Y}) = {} & \frac{(v+2)(1+\rho)\lambda_4}{2n((v+2)\lambda_4 - v\lambda_2^2(1-\rho))}\sigma^2 + \left[\frac{(1-\rho^2)}{2n\lambda_2}\sigma^2\right. \\ & \left. +2\left\{-\frac{\lambda_2(1-\rho^2)}{2n[(v+2)\lambda_4 - v\lambda_2^2(1-\rho)]}\sigma^2\right\}\right]d^2 \\ & + \frac{(1-\rho^2)}{2(2n)\lambda_4}\left[\frac{(v+1)\lambda_4 - (v-1)\lambda_2^2(1-\rho)}{(v+2)\lambda_4 - v\lambda_2^2(1-\rho)}\right]\sigma^2 d^4. \end{aligned} \tag{17}$$

4. An Empirical Study on SORD under Tri-diagonal Correlated Structure of Errors Using a Pair of Partially Balanced Incomplete Block Type Designs

Following Narasimham et al. (1983), Chowdhury and Gupta (1985), Victorbabu and Narasimham (1996), Victorbabu (2004), an empirical study on SORD under tri-diagonal

correlated structure of errors using a pair of partially balanced incomplete block type designs is studied.

Consider an incomplete block arrangement with constant block size and replication in which some pair of treatments occur λ_{11} times each $(\lambda_{11} \neq 0)$ and some other pairs do not occur at all $(\lambda_{12} = 0)$ (the design need not be PBIBD). Take this as the first design. For the second design take the incomplete block design with all missing pairs (in the first design) once each with $k_2 = 2, \lambda_{21} = 0, \lambda_{22} = 1.$ Such pairs of designs can be construct in a straight forward manner using existing two-associate PBIB designs with one of the λ 's equal to zero.

Let $D_1 = (v, b_1, r_1, k_1, \lambda_{11} \neq 0, \lambda_{12} = 0)$ be an incomplete block design with constant replication in which only some pair of treatments occur a constant number of times λ_{11} $(\lambda_{12} = 0)$, $2^{t(k_1)}$ denote a fractional replicate of 2^{k_1} in $+1$ and -1 levels, in which no interaction with less than five factors is confounded. Let $[1-(v, b_1, r_1, k_1, \lambda_{11}, \lambda_{12} = 0)]$ denote the design points generated from the transpose of the incidence matrix of incomplete block design D_1. $[1-(v, b_1, r_1, k_1, \lambda_{11}, \lambda_{12} = 0)]2^{(k_1)}$ are the $b_1 2^{t(k_1)}$ design points generate from D_1 by "multiplication" (see Raghavarao 1971, pp.298-300). Let $D_2 = (v, b_2, r_2, k_2, \lambda_{21} = 0, \lambda_{22} = 1)$ be the associated second design containing only the missing pairs of treatments of above design D_1. Let $[a-(v, b_2, r_2, k_2, \lambda_{21} = 0, \lambda_{22} = 1)]2^2$ are the $b_2 2^2$ design points generated from D_2 by "multiplication", with levels $+a$ and $-a$.

Consider a SORD using a pair of PBIBD having 'n' non-central design points. The set of 'n'-non central design points are extended to $2n$ design points by adding 'n' $(n_0 = n)$ central points just below or above the 'n'-non-central design points. Hence the total number of design points of SORD under tri-diagonal correlated structure of errors using a pair of PBIBD is $N = 2n,$ where $n = b_1 2^{t(k_1)} + b_2 2^2.$

For the design points generated from the pair of PBIBD, simple symmetry conditions (10) to (14) are true. Further, from (11) to (13), we have,

$$\sum_{u=1}^{2n} x_{iu}^2 = r_1 2^{t(k_1)} + r_2 2^2 a^2 = \text{constant} = 2n\lambda_2 \text{ for all } i, \tag{18}$$

$$\sum_{u=1}^{2n} x_{iu}^4 = r_1 2^{t(k_1)} + r_2 2^2 a^4 = \text{constant} = 3(2n)\lambda_4 \text{ for all } i, \tag{19}$$

$$\sum_{u=1}^{2n} x_{iu}^2 x_{ju}^2 = \lambda_{11} 2^{t(k_1)} + \lambda_{21} 2^2 a^4 = \text{constant} = 2n\lambda_4 \text{ for all } i \neq j, \tag{20}$$

From (19) and (20), we get, $a^4 = \dfrac{3(\lambda_{11} - r_1)2^{t(k_1)}}{r_2 2^2}.$

Substituting a^4 in (19) and (20) also a^2 in (18), we get λ_2 and $\lambda_4,$ respectively.

The variance of estimated response for a various factors is tabulated for $(-0.9 \leq \rho \leq 0.9)$. The numerical calculations are appended in the Appendix in Table 1.

Example: An empirical study on SORD under tri-diagonal correlated structure of errors using a pair of partially balanced incomplete block designs with

$D_1 = (v = 6, b_1 = 4, r_1 = 2, k_1 = 3, \lambda_{11} = 1, \lambda_{12} = 0)$,

$D_2 = (v = 6, b_2 = 3, r_2 = 1, k_2 = 2, \lambda_{21} = 0, \lambda_{22} = 1)$ is given below.

From (18), (19) and (20), we have

$$\sum_{u=1}^{2n} x_{iu}^2 = 16 + 4a^2 = \text{constant} = 2n\lambda_2, \tag{21}$$

$$\sum_{u=1}^{2n} x_{iu}^4 = 16 + 4a^4 = \text{constant} = 3(2n)\lambda_4, \tag{22}$$

$$\sum_{u=1}^{2n} x_{iu}^2 x_{ju}^2 = 8 = \text{constant} = 2n\lambda_4. \tag{23}$$

From (22) and (23), we get $a^4 = 2$. The non-singularity condition (15) is also satisfied.

5. A Study of Dependence of the Variance Function of the Response at Different Design Points

Here, the dependence of variance function of response at different design points for SORD under tri-diagonal correlated structure of errors using a pair of PBIBD;
$D_1 = (v = 6, b_1 = 4, r_1 = 2, k_1 = 3, \lambda_{11} = 1, \lambda_{12} = 0)$, $D_2 = (v = 6, b_2 = 3, r_2 = 1, k_2 = 2, \lambda_{21} = 0, \lambda_{22} = 1)$ with tri-diagonal correlation coefficient ' ρ ' and distance from the center ' d ' is studied.

The variance function is given by

$$V(\hat{Y}) = V(\hat{b}_0) + (V(\hat{b}_i) + 2Cov(\hat{b}_0, \hat{b}_{ii}))d^2 + V(\hat{b}_{ii})d^4,$$

$$V(\hat{Y}) = \frac{(v+2)(1+\rho)\lambda_4}{2n((v+2)\lambda_4 - v\lambda_2^2(1-\rho))}\sigma^2 + \left[\frac{(1-\rho^2)}{2n\lambda_2}\sigma^2 + 2\left\{-\frac{\lambda_2(1-\rho^2)}{2n[(v+2)\lambda_4 - v\lambda_2^2(1-\rho)]}\sigma^2\right\}\right]d^2 + \frac{(1-\rho^2)}{2(2n)\lambda_4}\left[\frac{(v+1)\lambda_4 - (v-1)\lambda_2^2(1-\rho)}{(v+2)\lambda_4 - v\lambda_2^2(1-\rho)}\right]\sigma^2 d^4.$$

Substituting $v, \lambda_4, \lambda_2^2$ and $\rho = -0.5$ and on simplification, we get,

$$V(\hat{Y}) = 0.00075\sigma^2 + 0.00133\sigma^2 d^2 + 0.09954\sigma^2 d^4,$$

$$= 0.0073 \text{ by taking } d = 0.5 \text{ and } \sigma^2 = 1.$$

The study of variance function of response at different design points for SORD under tri-diagonal correlated structure of errors using a pair of PBIBD and distance from center d for $0.1 \le d \le 1$ are tabulated. The numerical calculations are appended in the Appendix in Table 2.

6. Conclusions

From the Table 2 given in the Appendix, we observed that, $V(\hat{Y})$ is robustly increasing when $d = 0.1(0.1)1.0$, for given v, ρ.

Further, we may point out here that the SORD under tri-diagonal correlated structure of errors using a pair of PBIBD has 88 design points for 6-factors whereas the corresponding SORD under tri-diagonal correlated structure of errors by Rajyalakshmi and Victorbabu (2016 and

2018) using PBD with parameters $(v=6, b=7, r=3, k_1=3, k_2=2, \lambda=1)$ and BIBD with parameters $(v=6, b=10, r=5, k=3, \lambda=2)$ needs 112 and 184 design points, respectively. Therefore this method gives less number of design points than the corresponding SORD under tri-diagonal correlated structure of errors using PBD and BIBD.

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Appendix

Table 1 Variance of the estimated response under tri-diagonal correlated structure of errors using a pair of PBIBD $D_1 = (v = 6, b_1 = 4, r_1 = 2, k_1 = 3, \lambda_{11} = 1, \lambda_{12} = 0)$, $D_2 = (v = 6, b_2 = 3, r_2 = 1, k_2 = 2, \lambda_{21} = 0, \lambda_{22} = 1); N = 2n = 88$

ρ	$V(\hat{Y})$
-0.9	$0.00003\sigma^2 + 0.00049\sigma^2 d^2 + 0.00171\sigma^2 d^4$
-0.8	$0.00012\sigma^2 + 0.00086\sigma^2 d^2 + 0.00962\sigma^2 d^4$
-0.7	$0.00027\sigma^2 + 0.00111\sigma^2 d^2 + 0.02715\sigma^2 d^4$
-0.6	$0.00048\sigma^2 + 0.00127\sigma^2 d^2 + 0.05669\sigma^2 d^4$
-0.5	$0.00075\sigma^2 + 0.00133\sigma^2 d^2 + 0.09954\sigma^2 d^4$
-0.4	$0.00108\sigma^2 + 0.00132\sigma^2 d^2 + 0.15596\sigma^2 d^4$
-0.3	$0.00147\sigma^2 + 0.00125\sigma^2 d^2 + 0.22514\sigma^2 d^4$
-0.2	$0.00193\sigma^2 + 0.00112\sigma^2 d^2 + 0.30524\sigma^2 d^4$
-0.1	$0.00244\sigma^2 + 0.00095\sigma^2 d^2 + 0.39334\sigma^2 d^4$
0.0	$0.00301\sigma^2 + 0.00076\sigma^2 d^2 + 0.48547\sigma^2 d^4$
0.1	$0.00364\sigma^2 + 0.00055\sigma^2 d^2 + 0.57660\sigma^2 d^4$
0.2	$0.00433\sigma^2 + 0.00034\sigma^2 d^2 + 0.66067\sigma^2 d^4$
0.3	$0.00508\sigma^2 + 0.00014\sigma^2 d^2 + 0.73051\sigma^2 d^4$
0.4	$0.00589\sigma^2 + 0.00004\sigma^2 d^2 + 0.77795\sigma^2 d^4$
0.5	$0.00676\sigma^2 - 0.00019\sigma^2 d^2 + 0.79373\sigma^2 d^4$
0.6	$0.00769\sigma^2 - 0.00029\sigma^2 d^2 + 0.76754\sigma^2 d^4$
0.7	$0.00896\sigma^2 - 0.00034\sigma^2 d^2 + 0.68803\sigma^2 d^4$
0.8	$0.00974\sigma^2 - 0.00031\sigma^2 d^2 + 0.54276\sigma^2 d^4$
0.9	$0.01085\sigma^2 - 0.00020\sigma^2 d^2 + 0.31826\sigma^2 d^4$

Table 1 (Continued) $D_1 = (v = 10, b_1 = 8, r_1 = 4, k_1 = 5, \lambda_{11} = 2, \lambda_{12} = 0)$, $D_2 = (v = 10, b_2 = 5, r_2 = 1, k_2 = 2, \lambda_{21} = 0, \lambda_{22} = 1); N = 2n = 296$

ρ	$V(\bar{Y})$
-0.9	$0.00003\sigma^2 + 0.00014\sigma^2 d^2 + 0.01680\sigma^2 d^4$
-0.8	$0.00012\sigma^2 + 0.00023\sigma^2 d^2 + 0.10676\sigma^2 d^4$
-0.7	$0.00026\sigma^2 + 0.00027\sigma^2 d^2 + 0.31897\sigma^2 d^4$
-0.6	$0.00046\sigma^2 + 0.00026\sigma^2 d^2 + 0.68799\sigma^2 d^4$
-0.5	$0.00071\sigma^2 + 0.00022\sigma^2 d^2 + 1.23394\sigma^2 d^4$
-0.4	$0.00103\sigma^2 + 0.00016\sigma^2 d^2 + 1.96242\sigma^2 d^4$
-0.3	$0.00140\sigma^2 + 0.00007\sigma^2 d^2 + 2.86458\sigma^2 d^4$
-0.2	$0.00182\sigma^2 - 0.00003\sigma^2 d^2 + 3.91706\sigma^2 d^4$
-0.1	$0.00231\sigma^2 - 0.00014\sigma^2 d^2 + 5.08202\sigma^2 d^4$
0.0	$0.00285\sigma^2 - 0.00026\sigma^2 d^2 + 6.30714\sigma^2 d^4$
0.1	$0.00345\sigma^2 - 0.00037\sigma^2 d^2 + 7.52561\sigma^2 d^4$
0.2	$0.00410\sigma^2 - 0.00046\sigma^2 d^2 + 8.65615\sigma^2 d^4$
0.3	$0.00481\sigma^2 - 0.00054\sigma^2 d^2 + 9.60297\sigma^2 d^4$
0.4	$0.00558\sigma^2 - 0.00059\sigma^2 d^2 + 10.25583\sigma^2 d^4$
0.5	$0.00640\sigma^2 - 0.00061\sigma^2 d^2 + 10.48997\sigma^2 d^4$
0.6	$0.00728\sigma^2 - 0.00059\sigma^2 d^2 + 10.16616\sigma^2 d^4$
0.7	$0.00822\sigma^2 - 0.00053\sigma^2 d^2 + 9.13071\sigma^2 d^4$
0.8	$0.00921\sigma^2 - 0.00041\sigma^2 d^2 + 7.21539\sigma^2 d^4$
0.9	$0.01026\sigma^2 - 0.00024\sigma^2 d^2 + 4.23755\sigma^2 d^4$

Table 1 (Continued) $D_1 = (v = 12, b_1 = 8, r_1 = 4, k_1 = 6, \lambda_{11} = 2, \lambda_{12} = 0)$, $D_2 = (v = 12, b_2 = 6, r_2 = 1, k_2 = 2, \lambda_{21} = 0, \lambda_{22} = 1); N = 2n = 560$

ρ	$V(\hat{Y})$
-0.9	$0.00003\sigma^2 + 0.00007\sigma^2 d^2 + 0.05829\sigma^2 d^4$
-0.8	$0.00010\sigma^2 + 0.00011\sigma^2 d^2 + 0.34921\sigma^2 d^4$
-0.7	$0.00022\sigma^2 + 0.00012\sigma^2 d^2 + 1.02132\sigma^2 d^4$
-0.6	$0.00038\sigma^2 + 0.00010\sigma^2 d^2 + 2.17889\sigma^2 d^4$
-0.5	$0.00059\sigma^2 + 0.00006\sigma^2 d^2 + 3.88185\sigma^2 d^4$
-0.4	$0.00084\sigma^2 + 0.00001\sigma^2 d^2 + 6.14583\sigma^2 d^4$
-0.3	$0.00114\sigma^2 - 0.00006\sigma^2 d^2 + 8.94209\sigma^2 d^4$
-0.2	$0.00148\sigma^2 - 0.00013\sigma^2 d^2 + 12.19762\sigma^2 d^4$
-0.1	$0.00187\sigma^2 - 0.00021\sigma^2 d^2 + 15.9504\sigma^2 d^4$
0.0	$0.00230\sigma^2 - 0.00028\sigma^2 d^2 + 19.57267\sigma^2 d^4$
0.1	$0.00278\sigma^2 - 0.00035\sigma^2 d^2 + 23.32455\sigma^2 d^4$
0.2	$0.00331\sigma^2 - 0.00041\sigma^2 d^2 + 26.80029\sigma^2 d^4$
0.3	$0.00388\sigma^2 - 0.00046\sigma^2 d^2 + 29.70527\sigma^2 d^4$
0.4	$0.00450\sigma^2 - 0.00048\sigma^2 d^2 + 31.70051\sigma^2 d^4$
0.5	$0.00516\sigma^2 - 0.00048\sigma^2 d^2 + 32.40271\sigma^2 d^4$
0.6	$0.00586\sigma^2 - 0.00046\sigma^2 d^2 + 31.38424\sigma^2 d^4$
0.7	$0.00662\sigma^2 - 0.00040\sigma^2 d^2 + 28.17317\sigma^2 d^4$
0.8	$0.00741\sigma^2 - 0.00031\sigma^2 d^2 + 22.25322\sigma^2 d^4$
0.9	$0.00825\sigma^2 - 0.00018\sigma^2 d^2 + 13.06380\sigma^2 d^4$

Table 2 Study of variance function of SORD under tri-diagonal correlated structure of errors using a pair of PBIBD for different values of ' ρ ' and d by taking $\sigma^2 = 1$

$$D_1 = (v = 6, b_1 = 4, r_1 = 2, k_1 = 3, \lambda_{11} = 1, \lambda_{12} = 0),$$

$$D_2 = (v = 6, b_2 = 3, r_2 = 1, k_2 = 2, \lambda_{21} = 0, \lambda_{22} = 1); N = 88$$

ρ	$d = 0.1$	$d = 0.2$	$d = 0.3$	$d = 0.4$	$d = 0.5$	$d = 0.6$	$d = 0.7$	$d = 0.8$	$d = 0.9$	$d = 1.0$
– 0.9	0.0000	0.0001	0.0001	0.0002	0.0003	0.0004	0.0007	0.0010	0.0015	0.0022
– 0.8	0.0001	0.0002	0.0003	0.0005	0.0009	0.0017	0.0029	0.0046	0.0071	0.0106
– 0.7	0.0003	0.0004	0.0006	0.0011	0.0022	0.0042	0.0073	0.0121	0.0190	0.0285
– 0.6	0.0005	0.0006	0.0011	0.0021	0.0043	0.0083	0.0147	0.0245	0.0387	0.0584
– 0.5	0.0008	0.0010	0.0017	0.0035	0.0073	0.0141	0.0253	0.0424	0.0671	0.1016
– 0.4	0.0011	0.0014	0.0025	0.0053	0.0112	0.0218	0.0392	0.0658	0.1045	0.1584
– 0.3	0.0015	0.0019	0.0034	0.0074	0.0159	0.0311	0.0561	0.0945	0.1502	0.2279
– 0.2	0.0020	0.0025	0.0045	0.0099	0.0213	0.0419	0.0758	0.1277	0.2031	0.3083
– 0.1	0.0025	0.0031	0.0057	0.0127	0.0273	0.0538	0.0973	0.1642	0.2613	0.3967
0.0	0.0031	0.0038	0.0070	0.0156	0.0335	0.0662	0.1199	0.2023	0.3221	0.4892
0.1	0.0037	0.0046	0.0084	0.0185	0.0398	0.0786	0.1424	0.2402	0.3824	0.5808
0.2	0.0044	0.0054	0.0097	0.0213	0.0457	0.0901	0.1631	0.2752	0.4381	0.6653
0.3	0.0052	0.0063	0.0110	0.0238	0.0508	0.0998	0.1805	0.3044	0.4845	0.7357
0.4	0.0060	0.0071	0.0122	0.0258	0.0545	0.1067	0.1927	0.3245	0.5163	0.7838
0.5	0.0068	0.0080	0.0132	0.0271	0.0563	0.1096	0.1972	0.3318	0.5274	0.8003
0.6	0.0078	0.0089	0.0139	0.0273	0.0556	0.1071	0.1918	0.3219	0.5110	0.7749
0.7	0.0088	0.0098	0.0142	0.0262	0.0516	0.0977	0.1737	0.2903	0.4598	0.6964
0.8	0.0098	0.0106	0.0141	0.0236	0.0436	0.0800	0.1399	0.2319	0.3656	0.5522
0.9	0.0109	0.0114	0.0134	0.0190	0.0307	0.0520	0.0872	0.1411	0.2195	0.3289

Table 2 (Continued) $D_1 = (v = 10, b_1 = 8, r_1 = 4, k_1 = 5, \lambda_{11} = 2, \lambda_{12} = 0)$, $D_2 = (v = 10, b_2 = 5, r_2 = 1, k_2 = 2, \lambda_{21} = 0, \lambda_{22} = 1); N = 296$

ρ	$d = 0.1$	$d = 0.2$	$d = 0.3$	$d = 0.4$	$d = 0.5$	$d = 0.6$	$d = 0.7$	$d = 0.8$	$d = 0.9$	$d = 1.0$
– 0.9	0.0000	0.0001	0.0002	0.0005	0.0011	0.0023	0.0041	0.0070	0.0112	0.0170
– 0.8	0.0001	0.0003	0.0010	0.0029	0.0068	0.0140	0.0259	0.0440	0.0703	0.1071
– 0.7	0.0003	0.0008	0.0029	0.0085	0.0203	0.0417	0.0770	0.1311	0.2097	0.3195
– 0.6	0.0005	0.0016	0.0061	0.0181	0.0435	0.0897	0.1658	0.2824	0.4521	0.6887
– 0.5	0.0008	0.0027	0.0107	0.0323	0.0779	0.1607	0.2971	0.5063	0.8105	1.2349
– 0.4	0.0012	0.0042	0.0169	0.0513	0.1237	0.2554	0.4723	0.8049	1.2887	1.9636
– 0.3	0.0017	0.0060	0.0246	0.0747	0.1805	0.3727	0.6892	1.1748	1.8809	2.8660
– 0.2	0.0022	0.0081	0.0335	0.1021	0.2466	0.5095	0.9423	1.6062	2.5718	3.9188
– 0.1	0.0028	0.0104	0.0435	0.1324	0.3199	0.6609	1.2224	2.0838	3.3365	5.0842
0.0	0.0035	0.0129	0.0539	0.1643	0.3970	0.8202	1.5171	2.5861	4.1408	6.3097
0.1	0.0042	0.0155	0.0644	0.1960	0.4737	0.9786	1.8102	3.0857	4.9407	7.5287
0.2	0.0050	0.0179	0.0742	0.2256	0.5450	1.1258	2.0822	3.5494	5.6830	8.6598
0.3	0.0058	0.0202	0.0825	0.2506	0.6049	1.2492	2.3102	3.9378	6.3049	9.6072
0.4	0.0066	0.0220	0.0886	0.2680	0.6464	1.3345	2.4677	4.2060	6.7339	10.2608
0.5	0.0074	0.0232	0.0913	0.2748	0.6619	1.3657	2.5247	4.3027	6.8884	10.4958
0.6	0.0083	0.0235	0.0896	0.2674	0.6425	1.3246	2.4479	4.1710	6.6768	10.1729
0.7	0.0091	0.0228	0.0821	0.2419	0.5788	1.1914	2.2002	3.7478	5.9984	9.1384
0.8	0.0099	0.0207	0.0676	0.1939	0.4601	0.9442	1.7414	2.9644	4.7429	7.2242
0.9	0.0107	0.0170	0.0446	0.1187	0.2751	0.5594	1.0276	1.7458	2.7903	4.2476

Table 2 (Continued) $D_1 = (v = 12, b_1 = 8, r_1 = 4, k_1 = 6, \lambda_{11} = 2, \lambda_{12} = 0)$, $D_2 = (v = 12, b_2 = 6, r_2 = 1, k_2 = 2, \lambda_{21} = 0, \lambda_{22} = 1); N = 560$

ρ	$d = 0.1$	$d = 0.2$	$d = 0.3$	$d = 0.4$	$d = 0.5$	$d = 0.6$	$d = 0.7$	$d = 0.8$	$d = 0.9$	$d = 1.0$
– 0.9	0.0000	0.0001	0.0005	0.0015	0.0037	0.0076	0.0141	0.0239	0.0383	0.0584
– 0.8	0.0001	0.0007	0.0029	0.0091	0.0220	0.0454	0.0840	0.1432	0.2293	0.3494
– 0.7	0.0003	0.0019	0.0085	0.0264	0.0641	0.1326	0.2455	0.4186	0.6704	1.0217
– 0.6	0.0006	0.0039	0.0180	0.0562	0.1366	0.2828	0.5236	0.8929	1.4300	2.1794
– 0.5	0.0010	0.0068	0.0320	0.1000	0.2432	0.5037	0.9326	1.5906	2.5475	3.8825
– 0.4	0.0015	0.0107	0.0506	0.1582	0.3850	0.7973	1.4765	2.5182	4.0331	6.1467
– 0.3	0.0020	0.0154	0.0736	0.2300	0.5600	1.1600	2.1481	3.6638	5.8680	8.9432
– 0.2	0.0027	0.0210	0.1003	0.3137	0.7638	1.5822	2.9301	4.9975	8.0042	12.1990
– 0.1	0.0034	0.0271	0.1298	0.4062	0.9890	2.0488	3.7942	6.4714	10.3648	15.7967
0.0	0.0043	0.0336	0.1608	0.5033	1.2255	2.5388	4.7016	8.0191	12.8437	19.5747
0.1	0.0051	0.0401	0.1917	0.5998	1.4605	3.0255	5.6028	9.5563	15.3057	23.3270
0.2	0.0060	0.0462	0.2204	0.6893	1.6782	3.4765	6.4379	10.9804	17.5866	26.8032
0.3	0.0068	0.0514	0.2445	0.7643	1.8603	3.8535	7.1359	12.1709	19.4931	29.7087
0.4	0.0077	0.0552	0.2612	0.8160	1.9857	4.1127	7.6156	12.9887	20.8028	31.7045
0.5	0.0084	0.0570	0.2676	0.8346	2.0302	4.2044	7.7848	13.2770	21.2642	32.4074
0.6	0.0090	0.0561	0.2600	0.8092	1.9673	4.0731	7.5410	12.8606	20.5967	31.3896
0.7	0.0094	0.0517	0.2348	0.7278	1.7673	3.6577	6.7708	11.5461	18.4907	28.1794
0.8	0.0096	0.0430	0.1876	0.5770	1.3982	2.8913	5.3503	9.1221	14.6075	22.2603
0.9	0.0096	0.0291	0.1141	0.3427	0.8247	1.7013	3.1448	5.3591	8.5793	13.0719