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Simultaneous Confidence Intervals for All Differences of Coefficients of Variation of Two-Parameter Exponential Distributions

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Abstract

In this paper, a construction of simultaneous confidence intervals for differences of coefficients of variation of k independent two-parameter exponential distributions is considered. Parametric bootstrap approach, generalized confidence interval (GCI) approach, and method of variance estimates recovery (MOVER) approach are proposed to construct the simultaneous confidence intervals. Simulation results indicated that the GCI approach has satisfactory performance for all cases whereas the MOVER approach is preferable for large sample sizes.

Keywords: Coverage probability, simulation, parametric bootstrap approach, generalized confidence interval approach, method of variance estimates recovery approach.

1. Introduction

Two-parameter exponential distribution is a commonly used distribution in reliability analysis and lifetime data. This distribution is used to model the behavior of units. It is used to describe the lengths of the inter-arrival times. In addition, confidence interval provides more information on parameter than point estimate. Thus, many researchers are interested in constructing confidence interval of the two-parameter exponential distribution in several applications, namely Chiou (1997), Roy and Mathew (2005), Li and Zhang (2010), Kharrati-Kopaei et al. (2013), Singh and Singh (2013), Li et al. (2015), Sangnawakij and Niwitpong (2017), and Thangjai and Niwitpong (2017).

Coefficient of variation is a measure of relative variability. It is used rather than standard deviation because the coefficient of variation is free from the unit of measurement. The coefficient of variation is defined as the ratio of the standard deviation to the mean. It is used as a measure of precision in many situations. For instance, the coefficient of variation is used to

analyze the rainfall data in climatology. Furthermore, the coefficient of variation is particularly useful for comparing two distributions or two tests on different measures. Some applications are the coefficient of variation is used as a yardstick of precision of two measurement methods in chemical experiments. In finance, the coefficient of variation is used to test the equality of two stocks. In medical study, the coefficient of variation is used to measure precision within and between laboratories. Moreover, the test of coefficient of variation is also available for k populations. For example, the coefficient of variation is used to compare rainfall variability in k different regions. Confidence interval about the coefficient of variation has been considered by Vangel (1996), Wong and Wu (2002), Tian (2005), Mahmoudvand and Hassani (2009), Curto and Pinto (2009), Banik and Kibria (2011), Niwitpong (2013), Sangnawakij et al. (2015), Sangnawakij et al. (2016), Sangnawakij and Niwitpong (2017), and Thangjai and Niwitpong (2017).

Several researchers have looked at the problems of testing homogeneity and estimating simultaneous confidence interval of parameter of two-parameter exponential distributions. For example, Chen (1982) provided a test procedure for testing the null hypothesis and proposed the simultaneous confidence intervals of location parameters in two-parameter exponential distributions. Dhawan and Gill (1997) proposed simultaneous one-sided confidence intervals for the ordered pairwise difference of exponential location parameters. Gill and Dhawan (1999) presented a one-sided test for testing homogeneity of scale parameters of two-parameter exponential distributions. Singh and Gill (2004) provided the simultaneous confidence intervals for the ratios of scale parameters of two-parameter exponential distributions. Singh (2006) constructed simultaneous confidence intervals for the successive ratios of scale parameters of two-parameter exponential distributions. Kharrati-Kopaei et al. (2013) constructed the simultaneous confidence intervals for the successive differences of exponential location parameters based on simultaneous fiducial generalized confidence interval (SFGCI) approach. Li et al. (2015) proposed the simultaneous confidence intervals for differences of means from several two-parameter exponential distributions using parametric bootstrap approach. Kharrati-Kopaei (2015) provided the simultaneous confidence intervals for the differences of location parameters of successive exponential distributions. Thangjai et al. (2018) proposed the simultaneous confidence intervals for all differences of means of two-parameter exponential distributions. There is no previous work on constructing simultaneous confidence intervals for all differences of coefficients of variation of two-parameter exponential distributions. The main purpose of this paper is to construct the simultaneous confidence intervals for all differences of coefficients of variation of two-parameter exponential distributions using the GCI approach. Moreover, the GCI approach is compared with the parametric bootstrap approach and the method of variance estimates recovery (MOVER) approach.

This paper is organized as follows. In Section 2, we propose three approaches for simultaneous confidence intervals for differences of coefficients of variation from several two-parameter exponential distributions. Our simulation results are provided in Section 3. The proposed approaches are illustrated in Section 4 with real data example. Finally, some concluding remarks are presented in Section 5.

2. Simultaneous Confidence Intervals

Let $X_i = (X_{i1}, X_{i2}, \ldots, X_{in_i})$ be a random variable from the two-parameter exponential

distribution with a scale parameter λ_i and a location parameter β_i, where $i = 1, 2, \ldots, k$. Let $\overline{X}_i$ and $X_{(1)i} = \min(X_{i1}, X_{i2}, ..., X_{in_i})$ be a sample mean and a smallest sample, respectively. The maximum likelihood estimators of λ_i and β_i are $\hat{\lambda}_i = \overline{X}_i - X_{(1)i}$ and $\hat{\beta}_i = X_{(1)i}$, respectively.

The coefficient of variation of X_i can be written as

$$\theta_i = \frac{\lambda_i}{\lambda_i + \beta_i}. \tag{1}$$

Thus, an estimator of θ_i can be obtained by

$$\hat{\theta}_i = \frac{\overline{X}_i - X_{(1)i}}{\overline{X}_i} = 1 - \frac{X_{(1)i}}{\overline{X}_i}. \tag{2}$$

As pointed by Sangnawakij and Niwitpong (2017), variance of $\hat{\theta}_i$ can be written as

$$Var(\hat{\theta}_i) = \frac{\lambda_i^2 (n_i - 1)(\lambda_i^2 + n_i \beta_i^2 + 2\lambda_i \beta_i)}{n_i^3 (\lambda_i + \beta_i)^4}. \tag{3}$$

This paper is interested in constructing simultaneous confidence intervals for $\theta_i - \theta_l$, where $i, l = 1, 2, \ldots, k$ and $i \neq l$. Note that

$$Var(\hat{\theta}_i - \hat{\theta}_l) = \frac{\lambda_i^2 (n_i - 1)(\lambda_i^2 + n_i \beta_i^2 + 2\lambda_i \beta_i)}{n_i^3 (\lambda_i + \beta_i)^4} + \frac{\lambda_l^2 (n_l - 1)(\lambda_l^2 + n_l \beta_l^2 + 2\lambda_l \beta_l)}{n_l^3 (\lambda_l + \beta_l)^4}. \tag{4}$$

An estimator of $Var(\hat{\theta}_i - \hat{\theta}_l)$ is

$$\begin{aligned} V_{il} = & \frac{(\overline{X}_i - X_{(1)i})^2 (n_i - 1)((\overline{X}_i - X_{(1)i})^2 + n_i X_{(1)i}^2 + 2X_{(1)i}(\overline{X}_i - X_{(1)i}))}{n_i^3 \overline{X}_i^4} \\ & + \frac{(\overline{X}_l - X_{(1)l})^2 (n_l - 1)((\overline{X}_l - X_{(1)l})^2 + n_l X_{(1)l}^2 + 2X_{(1)l}(\overline{X}_l - X_{(1)l}))}{n_l^3 \overline{X}_l^4}. \end{aligned} \tag{5}$$

2.1. Parametric bootstrap simultaneous confidence intervals

For $i = 1, 2, \ldots, k$, let $S_i = \overline{X}_i - X_{(1)i}$. For parametric bootstrap approach, let

$$X_{(1)i}^{PB} \sim Exp\left(x_{(1)i}, \frac{\overline{x}_i - x_{(1)i}}{n_i} \right), \tag{6}$$

and

$$\overline{X}_i^{PB} - X_{(1)i}^{PB} = S_i^{PB} \sim \frac{(\overline{x}_i - x_{(1)i}) \chi_{2n_i - 2}^2}{2n_i}, \tag{7}$$

where $\chi_{2n_i - 2}^2$ denotes a chi-squared distribution with $2n_i - 2$ degrees of freedom.

The parametric bootstrap pivotal variable is

$$T^{PB} = \max_{i \neq l} \left| \frac{\left(\frac{\overline{X}_i^{PB} - X_{(1)i}^{PB}}{\overline{X}_i^{PB}} - \frac{\overline{X}_l^{PB} - X_{(1)l}^{PB}}{\overline{X}_l^{PB}} \right) - \left(\frac{\overline{x}_i^{PB} - x_{(1)i}^{PB}}{\overline{x}_i^{PB}} - \frac{\overline{x}_l^{PB} - x_{(1)l}^{PB}}{\overline{x}_l^{PB}} \right)}{\sqrt{V_{il}^{PB}}} \right|, \tag{8}$$

where

$$V_{il}^{PB} = \frac{(\bar{X}_i^{PB} - X_{(1)i}^{PB})^2 (n_i - 1)((\bar{X}_i^{PB} - X_{(1)i}^{PB})^2 + n_i (X_{(1)i}^{PB})^2 + 2X_{(1)i}^{PB}(\bar{X}_i^{PB} - X_{(1)i}^{PB}))}{n_i^3 (\bar{X}_i^{PB})^4} + \frac{(\bar{X}_l^{PB} - X_{(1)l}^{PB})^2 (n_l - 1)((\bar{X}_l^{PB} - X_{(1)l}^{PB})^2 + n_l (X_{(1)l}^{PB})^2 + 2X_{(1)l}^{PB}(\bar{X}_l^{PB} - X_{(1)l}^{PB}))}{n_l^3 (\bar{X}_l^{PB})^4}. \quad (9)$$

Therefore, the $100(1-\alpha)\%$ two-sided simultaneous confidence intervals for $\theta_i - \theta_l$ based on parametric bootstrap approach are

$$SCI_{(il)PB} = [\hat{\theta}_i - \hat{\theta}_l - q_\alpha^{n.PB}\sqrt{V_{il}}, \hat{\theta}_i - \hat{\theta}_l + q_\alpha^{n.PB}\sqrt{V_{il}}], \quad (10)$$

where $q_\alpha^{n.PB}$ denotes the $(1-\alpha)^{\text{th}}$ quantile of the distribution of T^{PB}.

Theorem 1 *Suppose that* $X_i = (X_{i1}, X_{i2}, \ldots, X_{in_i}) \sim Exp(\lambda_i, \beta_i)$, *where* $i = 1, 2, \ldots, k$. *For* $i, l = 1, 2, \ldots, k$ *and* $i \neq l$, *let* θ_i *and* θ_l *be the coefficients of variation based on the* i^{th} *sample and* l^{th} *sample, respectively. And let* $\hat{\theta}_i$ *and* $\hat{\theta}_l$ *be the estimators of* θ_i *and* θ_l, *respectively. Let* $\theta_{il} = \theta_i - \theta_l$ *be a difference between* θ_i *and* θ_l, *therefore,*

$$P\left[\theta_i - \theta_l \in (\hat{\theta}_i - \hat{\theta}_l \pm q_\alpha^{n.PB}\sqrt{V_{il}}), \forall i \neq l\right] \to 1 - \alpha. \quad (11)$$

Proof: Let $Var(\hat{\theta}_{il}) = Var(\hat{\theta}_i - \hat{\theta}_l)$ be a variance of difference between θ_i and θ_l. And let V_{il} be an estimator of $Var(\hat{\theta}_{il})$. It is well known that

$$P\left[\theta_i - \theta_l \in (\hat{\theta}_i - \hat{\theta}_l \pm q_\alpha^{n.PB}\sqrt{V_{il}}), \forall i \neq l\right] = P\left[\max_{i \neq l}\left|\frac{(\hat{\theta}_i - \hat{\theta}_l) - (\theta_i - \theta_l)}{\sqrt{V_{il}}}\right| \leq q_\alpha^{n.PB}\right]$$
$$= P\left(Q_n \leq q_\alpha^{n.PB}\right).$$

For $i = 1, 2, \ldots, k$, let $N = n_1 + n_2 + \ldots + n_k$ and $\frac{n_i}{N} \to r_i \in (0,1)$ as $N \to \infty$. Thus, $N(\hat{\theta}_i - \theta_i)$ converges in distribution to $Z = (Z_1, \ldots, Z_k)$. Denote

$$N(\hat{\theta}_i - \theta_i) \to Z,$$

where Z_i denotes a normal distribution with mean zero and variance σ_i^2 / r_i.

Using Slutsky's theorem, then Q_n converges in distribution to Q. Denote

$$Q_n \to Q,$$

where

$$Q = \max_{i \neq l}\left|\frac{Z_i - Z_l}{\sqrt{(\sigma_i^2 / r_i) + (\sigma_l^2 / r_l)}}\right|, \quad i, l = 1, 2, \ldots, k \text{ and } i \neq l.$$

Using Skorohod's theorem, let Y_n be the random variable on a common probability space with distribution Q_n and let Y be the random variable on a common probability space with distribution Q. Suppose that $Y_n \to Y$ and $Q_n \to Q$. Let Z_i and Z_i^* be independent and identically distributed

random variables. Then,

$$T(X, X^*, \mu, \sigma^2) \to Q^*,$$

where

$$Q^* = \max_{i \neq l} \left| \frac{Z_i^* - Z_l^*}{\sqrt{(\sigma_i^2 / r_i) + (\sigma_l^2 / r_l)}} \right|.$$

Suppose that $T(X, X^*, \mu, \sigma^2)$ is continuous distribution and converges in distribution to Q^*, and suppose that $q_\alpha^{n.PB} \to q_{1-\alpha}$, where $q_{1-\alpha}$ denotes the $(1-\alpha)^{\text{th}}$ quantile of Q^*. Therefore,

$$P(Q_n \leq q_\alpha^{n.PB}) \to P(Q \leq q_{1-\alpha}) = P(Q^* \leq q_{1-\alpha}) = 1-\alpha, \text{ as } N \to \infty.$$

Therefore, the $100(1-\alpha)\%$ two-sided simultaneous confidence intervals are

$$P\left[\theta_i - \theta_l \in (\hat{\theta}_i - \hat{\theta}_l \pm q_\alpha^{n.PB} \sqrt{V_{il}}), \forall i \neq l\right] \to 1-\alpha$$

implies that

$$P(\hat{\theta}_i - \hat{\theta}_l - q_\alpha^{n.PB} \sqrt{V_{il}} \leq \theta_{il} \leq \hat{\theta}_i - \hat{\theta}_l + q_\alpha^{n.PB} \sqrt{V_{il}}, \forall i \neq l) \to 1-\alpha.$$

Therefore, Theorem 1 is proved.

2.2. Simultaneous generalized confidence intervals

Definition Let $X = (X_1, X_2, ..., X_n)$ be a random sample from a distribution with the parameters $\xi = (\theta, \delta)$, where θ is the parameter of interest and δ is a vector of nuisance parameters. Let $x = (x_1, x_2, ..., x_n)$ be an observed value of $X = (X_1, X_2, \ldots, X_n)$. The random quantity $R(X; x, \theta, \delta)$ is called a generalized pivotal quantity (GPQ) if $R(X; x, \theta, \delta)$ has the following two properties; see Weerahandi (1993):

(i) For a given x, the distribution of $R(X; x, \theta, \delta)$ is free of all unknown parameters.

(ii) The observed value of $R(X; x, \theta, \delta)$, $X = x$, is the parameter of interest.

Let $R(\alpha)$ be the $(\alpha)^{\text{th}}$ quantile of $R(X; x, \theta, \delta)$. Hence, $[R(\alpha/2), R(1-\alpha/2)]$ becomes $100(1-\alpha)\%$ two-sided generalized confidence interval for parameter of interest.

From Sangnawakij and Niwitpong (2017), the GPQ for the coefficient of variation of two-parameter exponential distribution based on the i^{th} sample is

$$R_{\theta_i} = \left[1 + \frac{1}{2n_i}\left(\frac{x_{(1)i} V_i}{\bar{x}_i - x_{(1)i}} - U_i\right)\right]^{-1}. \tag{12}$$

Hence, the GPQ for the difference of two coefficients of variation can be written as

$$R_{\theta_{il}} = R_{\theta_i} - R_{\theta_l} = \left[1 + \frac{1}{2n_i}\left(\frac{x_{(1)i} V_i}{\bar{x}_i - x_{(1)i}} - U_i\right)\right]^{-1} - \left[1 + \frac{1}{2n_l}\left(\frac{x_{(1)l} V_l}{\bar{x}_l - x_{(1)l}} - U_l\right)\right]^{-1}. \tag{13}$$

Therefore, the $100(1-\alpha)\%$ two-sided simultaneous confidence intervals for $\theta_i - \theta_l$ based on GCI approach are

$$SCI_{il(GCI)} = [L_{il(GCI)}, U_{il(GCI)}] = [R_{\theta_{il}}(\alpha/2), R_{\theta_{il}}(1-\alpha/2)], \tag{14}$$

where $R_{\theta_{il}}(\alpha/2)$ and $R_{\theta_{il}}(1-\alpha/2)$ denote the $(\alpha/2)^{\text{th}}$ and $(1-\alpha/2)^{\text{th}}$ quantiles of $R_{\theta_{il}}$, respectively.

Theorem 2 *Let* $X_i = (X_{i1}, X_{i2}, \ldots, X_{in_i})$ *be random variable from two-parameter exponential distribution with scale parameter* λ_i *and location parameter* β_i, *where* $i = 1,2,\ldots,k$. *For* $i, l = 1,2,\ldots,k$ *and* $i \neq l$, *the coefficients of variation of* X_i *and* X_l *are* $\theta_i = \lambda_i/(\lambda_i + \beta_i)$ *and* $\theta_l = \lambda_l/(\lambda_l + \beta_l)$. *Let* $\hat{\theta}_i$ *and* $\hat{\theta}_l$ *be the estimators of* θ_i *and* θ_l, *respectively. Let* V_{il} *be an estimator of a variance of difference between* θ_i *and* θ_l. *Let* n_i *be the sample size based on the* i^{th} *sample and let* $N = n_1 + n_2 + \ldots + n_k$. *Suppose that* $\frac{n_i}{N} \to r_i \in (0,1)$ *as* $N \to \infty$. *Therefore,*

$$P\left[R_{\theta_{il}}(\alpha/2) \leq R_{\theta_{il}} \leq R_{\theta_{il}}(1-\alpha/2), \forall i \neq l\right] \to 1-\alpha. \tag{15}$$

Proof: Since

$$\begin{aligned} P\left[R_{\theta_{il}}(\alpha/2) \leq R_{\theta_{il}} \leq R_{\theta_{il}}(1-\alpha/2), \forall i \neq l\right] &= P\left[\theta_i - \theta_l \in (\hat{\theta}_i - \hat{\theta}_l \pm d_{1-\alpha}\sqrt{V_{il}}), \forall i \neq l\right] \\ &= P\left[\max_{i \neq l} \left|\frac{\hat{\theta}_i - \hat{\theta}_l - (\theta_i - \theta_l)}{\sqrt{V_{il}}}\right| \leq d_{1-\alpha}\right] \\ &= P\left(Q_n' \leq d_{1-\alpha}\right). \end{aligned}$$

Therefore,

$$P\left[\theta_i - \theta_l \in (\hat{\theta}_i - \hat{\theta}_l \pm d_{1-\alpha}\sqrt{V_{il}}), \forall i \neq l\right] \to 1-\alpha$$

implies that

$$P\left[R_{\theta_{il}}(\alpha/2) \leq R_{\theta_{il}} \leq R_{\theta_{il}}(1-\alpha/2), \forall i \neq l\right] \to 1-\alpha.$$

Therefore, Theorem 2 is proved.

2.3. Method of variance estimates recovery simultaneous confidence intervals

Donner and Zou (2010) proposed the MOVER approach to construct confidence interval of the difference of two parameters, i.e., $\theta_1 - \theta_2$ where θ_1 and θ_2 denote the parameters of interest. The concept of Donner and Zou (2010) is used to construct the simultaneous confidence intervals for $\theta_i - \theta_l$, where $i, l = 1,2,\ldots,k$ and $i \neq l$. Hence, the lower limit (L_{il}) and upper limit (U_{il}) of confidence interval for $\theta_i - \theta_l$ are

$$L_{il} = \hat{\theta}_i - \hat{\theta}_l - \sqrt{(\hat{\theta}_i - l_i)^2 + (u_l - \hat{\theta}_l)^2} \tag{16}$$

and

$$U_{il} = \hat{\theta}_i - \hat{\theta}_l + \sqrt{(u_i - \hat{\theta}_i)^2 + (\hat{\theta}_l - l_l)^2}, \tag{17}$$

where $\hat{\theta}_i = 1 - (X_{(1)i}/\bar{X}_i)$ and $\hat{\theta}_l = 1 - (X_{(1)l}/\bar{X}_l)$.

For $i, l = 1,2,\ldots,k$ and $i \neq l$, the simultaneous confidence intervals for $\theta_i - \theta_l$ are

$$SCI_{il(MOVER)} = \left[\hat{\theta}_i - \hat{\theta}_l - \sqrt{(\hat{\theta}_i - l_i)^2 + (u_l - \hat{\theta}_l)^2}, \hat{\theta}_i - \hat{\theta}_l + \sqrt{(u_i - \hat{\theta}_i)^2 + (\hat{\theta}_l - l_l)^2} \right]. \quad (18)$$

From Thangjai and Niwitpong (2017), we have two confidence intervals for the coefficient of variation of two-parameter exponential distribution. First, the lower limit and the upper limit of the confidence interval based on the i^{th} sample are

$$l_{1i} = \frac{\hat{\delta}_{1i}\hat{\delta}_{2i} - \sqrt{(\hat{\delta}_{1i}\hat{\delta}_{2i})^2 - l_{01i}u_{02i}(2\hat{\delta}_{1i} - l_{01i})(2\hat{\delta}_{2i} - u_{02i})}}{u_{02i}(2\hat{\delta}_{2i} - u_{02i})} \quad (19)$$

and

$$u_{1i} = \frac{\hat{\delta}_{1i}\hat{\delta}_{2i} + \sqrt{(\hat{\delta}_{1i}\hat{\delta}_{2i})^2 - u_{01i}l_{02i}(2\hat{\delta}_{1i} - u_{01i})(2\hat{\delta}_{2i} - l_{02i})}}{l_{02i}(2\hat{\delta}_{2i} - l_{02i})}, \quad (20)$$

where $\hat{\delta}_{1i} = \bar{x}_i - x_{(1)i}$, $\hat{\delta}_{2i} = \bar{x}_i$, $l_{01i} = \dfrac{n_i(\bar{x}_i - x_{(1)i})}{\sqrt{n_i - 1}(z_{\alpha/2} + \sqrt{n_i - 1})}$,

$$u_{01i} = \frac{n_i(\bar{x}_i - x_{(1)i})}{\sqrt{n_i - 1}(-z_{\alpha/2} + \sqrt{n_i - 1})},$$

$$l_{02i} = \bar{x}_i - \sqrt{\left[(\bar{x}_i - x_{(1)i}) - \frac{n_i(\bar{x}_i - x_{(1)i})}{z_{\alpha/2}\sqrt{n_i - 1} + (n_i - 1)}\right]^2 + \left[\frac{(\bar{x}_i - x_{(1)i})}{n_i}\ln(\alpha/2)\right]^2},$$

and

$$u_{02i} = \bar{x}_i + \sqrt{\left[\frac{n_i(\bar{x}_i - x_{(1)i})}{-z_{\alpha/2}\sqrt{n_i - 1} + (n_i - 1)} - (\bar{x}_i - x_{(1)i})\right]^2 + \left[\frac{(\bar{x}_i - x_{(1)i})}{n_i}\ln(1-\alpha/2)\right]^2}.$$

Second, the lower limit and the upper limit of the confidence interval for two-parameter exponential coefficient of variation based on the i^{th} sample are

$$l_{2i} = \frac{n_i}{n_i - 1}\left(1 - \frac{x_{(1)i}}{\bar{x}_i}\right) - z_{\alpha/2}\hat{c}_i \quad (21)$$

and

$$u_{2i} = \frac{n_i}{n_i - 1}\left(1 - \frac{x_{(1)i}}{\bar{x}_i}\right) + z_{\alpha/2}\hat{c}_i, \quad (22)$$

where $\hat{c}_i = \dfrac{(\bar{x}_i - x_{(1)i})}{(\bar{x}_i)^2}\sqrt{\dfrac{(\bar{x}_i - x_{(1)i})^2 + n_i(x_{(1)i})^2 + 2(\bar{x}_i - x_{(1)i})x_{(1)i}}{(n_i - 1)n_i}}$.

Therefore, the $100(1-\alpha)\%$ two-sided simultaneous confidence intervals for $\theta_i - \theta_l$ based on MOVER approach are

$$SCI_{il(MOVER1)} = \left[\hat{\theta}_i - \hat{\theta}_l - \sqrt{(\hat{\theta}_i - l_{1i})^2 + (u_{1l} - \hat{\theta}_l)^2}, \hat{\theta}_i - \hat{\theta}_l + \sqrt{(u_{1i} - \hat{\theta}_i)^2 + (\hat{\theta}_l - l_{1l})^2} \right] \quad (23)$$

and

$$SCI_{il(MOVER2)} = \left[\hat{\theta}_i - \hat{\theta}_l - \sqrt{(\hat{\theta}_i - l_{2i})^2 + (u_{2l} - \hat{\theta}_l)^2}, \hat{\theta}_i - \hat{\theta}_l + \sqrt{(u_{2i} - \hat{\theta}_i)^2 + (\hat{\theta}_l - l_{2l})^2} \right]. \quad (24)$$

Theorem 3 *Let* $X_i = (X_{i1}, X_{i2}, \ldots, X_{in_i}) \sim Exp(\lambda_i, \beta_i)$, *where* $i = 1,2,\ldots,k$. *For* $i, l = 1,2,\ldots,k$ *and* $i \neq l$, *let* $\theta_i = \lambda_i/(\lambda_i + \beta_i)$ *and* $\theta_l = \lambda_l/(\lambda_l + \beta_l)$ *be the coefficients of variation of two-parameter exponential distributions based on the* i^{th} *sample and the* l^{th} *sample, respectively. Let* $\hat{\theta}_i$ *and* $\hat{\theta}_l$ *be the estimators of* θ_i *and* θ_l, *respectively. Let* $L_{il} = \hat{\theta}_i - \hat{\theta}_l - \sqrt{(\hat{\theta}_i - l_i)^2 + (u_l - \hat{\theta}_l)^2}$ *and* $U_{il} = \hat{\theta}_i - \hat{\theta}_l + \sqrt{(u_i - \hat{\theta}_i)^2 + (\hat{\theta}_l - l_l)^2}$ *be the lower limit and the upper limit of the confidence interval for* $\theta_{il} = \theta_i - \theta_l$. *Therefore,*

$$P\left[L_{il} \leq \theta_{il} \leq U_{il}, \forall i \neq l\right] \to 1 - \alpha. \tag{25}$$

Proof: For $i, l = 1,2,\ldots,k$ and $i \neq l$, the lower limit L_{il} of the confidence interval for $\theta_{il} = \theta_i - \theta_l$ is

$$L_{il} = \hat{\theta}_i - \hat{\theta}_l - \sqrt{(\hat{\theta}_i - l_i)^2 + (u_l - \hat{\theta}_l)^2} = \hat{\theta}_{il} - \sqrt{(\hat{\theta}_i - l_i)^2 + (u_l - \hat{\theta}_l)^2}.$$

The upper limit U_{il} of the confidence interval for $\theta_{il} = \theta_i - \theta_l$ is

$$U_{il} = \hat{\theta}_i - \hat{\theta}_l + \sqrt{(u_i - \hat{\theta}_i)^2 + (\hat{\theta}_l - l_l)^2} = \hat{\theta}_{il} + \sqrt{(u_i - \hat{\theta}_i)^2 + (\hat{\theta}_l - l_l)^2}.$$

For $i, l = 1,2,\ldots,k$ and $i \neq l$, the variance estimates for $\hat{\theta}_i$ at $\theta_i = l_i$ and $\hat{\theta}_l$ at $\theta_l = l_l$ are

$$\widehat{Var}(\hat{\theta}_i) = \frac{(\hat{\theta}_i - l_i)^2}{z_{\alpha/2}^2} \text{ and } \widehat{Var}(\hat{\theta}_l) = \frac{(\hat{\theta}_i - l_l)^2}{z_{\alpha/2}^2},$$

where $z_{\alpha/2}$ is the $(\alpha/2)^{\text{th}}$ quantile of the standard normal distribution.

Moreover, the variance estimates for $\hat{\theta}_i$ at $\theta_i = u_i$ and $\hat{\theta}_l$ at $\theta_l = u_l$ are

$$\widehat{Var}(\hat{\theta}_i) = \frac{(u_i - \hat{\theta}_i)^2}{z_{\alpha/2}^2} \text{ and } \widehat{Var}(\hat{\theta}_l) = \frac{(u_l - \hat{\theta}_l)^2}{z_{\alpha/2}^2}.$$

Then,

$$L_{il} = \hat{\theta}_{il} - z_{\alpha/2}\sqrt{\frac{(\hat{\theta}_i - l_i)^2}{z_{\alpha/2}^2} + \frac{(u_l - \hat{\theta}_l)^2}{z_{\alpha/2}^2}} = \hat{\theta}_{il} - z_{\alpha/2}\sqrt{Var(\hat{\theta}_i) + Var(\hat{\theta}_l)}$$

and

$$U_{il} = \hat{\theta}_{il} + z_{\alpha/2}\sqrt{\frac{(u_i - \hat{\theta}_i)^2}{z_{\alpha/2}^2} + \frac{(\hat{\theta}_l - l_l)^2}{z_{\alpha/2}^2}} = \hat{\theta}_{il} + z_{\alpha/2}\sqrt{Var(\hat{\theta}_i) + Var(\hat{\theta}_l)}.$$

Therefore,

$$\begin{aligned}
&P(L_{il} \leq \theta_{il} \leq U_{il}) \\
&= P\left(\hat{\theta}_{il} - z_{\alpha/2}\sqrt{Var(\hat{\theta}_i) + Var(\hat{\theta}_l)} \leq \theta_{il} \leq \hat{\theta}_{il} + z_{\alpha/2}\sqrt{Var(\hat{\theta}_i) + Var(\hat{\theta}_l)}\right) \\
&= P\left(\theta_{il} \in \left(\hat{\theta}_{il} \pm z_{\alpha/2}\sqrt{Var(\hat{\theta}_i) + Var(\hat{\theta}_l)}\right), \forall i \neq l\right) \\
&= P\left(\max_{i \neq l}\left|\frac{\hat{\theta}_{il} - \theta_{il}}{\sqrt{Var(\hat{\theta}_i) + Var(\hat{\theta}_l)}}\right| \leq z_{\alpha/2}\right)
\end{aligned}$$

$$= \quad P(Q_n'' \le z_{\alpha/2}).$$

For $i = 1,2,\ldots,k$, let $N = n_1 + n_2 + \ldots + n_k$ and $\frac{n_i}{N} \to r_i \in (0,1)$ as $N \to \infty$. Then $N(\hat{\theta}_i - \theta_i)$ converges in distribution to $Z = (Z_1,\ldots,Z_k)$, where Z_i is a normal distribution with mean zero and variance σ_i^2 / r_i. Using Slutsky's theorem, Q_n'' converges in distribution to Q''. Denote

$$Q_n'' \to Q'',$$

where

$$Q'' = \max_{i \ne l} \left| \frac{Z_i - Z_l}{\sqrt{(\sigma_i^2 / r_i) + (\sigma_l^2 / r_l)}} \right|; \ i,l = 1,2,\ldots,k \text{ and } i \ne l.$$

Using Skorohod's theorem, let Y_n and Y be the random variables on a common probability space are distributed as Q_n'' and Q'', respectively. Then, Y_n converges to Y, denoted $Y_n \to Y$, and Q_n'' converges to Q'', denoted $Q_n'' \to Q''$. Suppose that random variables Z_i and Z_i^* are independent and identically distributed random variables. Therefore, $T(X, X^*, \mu, \sigma^2)$ converges to Q''^*. Denote

$$T(X, X^*, \mu, \sigma^2) \to Q''^*,$$

where

$$Q''^* = \max_{i \ne l} \left| \frac{Z_i^* - Z_l^*}{\sqrt{(\sigma_i^2 / r_i) + (\sigma_l^2 / r_l)}} \right|; \ i,l = 1,2,\ldots,k \text{ and } i \ne l.$$

Since the limiting distribution of $T(X, X^*, \mu, \sigma^2)$ is continuous and $z_{\alpha/2}(X)$ converges to $q_{\alpha/2}$, where $q_{\alpha/2}$ is the $(\alpha/2)^{\text{th}}$ quantile of the distribution of Q''^*. Therefore,

$$P(Q_n'' \le z_{\alpha/2}) \to P(Q'' \le q_{\alpha/2}) = P(Q''^* \le q_{\alpha/2}) = 1 - \alpha, \text{ as } N \to \infty.$$

Therefore,

$$P\left(\theta_{il} \in \left(\hat{\theta}_{il} \pm z_{\alpha/2}\sqrt{Var(\hat{\theta}_i) + Var(\hat{\theta}_l)}\right), \forall i \ne l\right) \to 1 - \alpha.$$

That is

$$P\left[\hat{\theta}_{il} - \sqrt{(\hat{\theta}_i - l_i)^2 + (u_l - \hat{\theta}_l)^2} \le \theta_{il} \le \hat{\theta}_{il} + \sqrt{(u_i - \hat{\theta}_i)^2 + (\hat{\theta}_l - l_l)^2}, \forall i \ne l\right] \to 1 - \alpha$$

implies that

$$P\left(L_{il} \le \theta_{il} \le U_{il}, \forall i \ne l\right) \to 1 - \alpha.$$

Therefore, Theorem 3 is proved.

3. Simulation Studies

In this section, the results of our simulation studies are presented. The performances of the MOVER approach (SCI_{MOVER1} and SCI_{MOVER2}) are compared with the parametric bootstrap approach (SCI_{PB}) and the GCI approach (SCI_{GCI}). For parametric bootstrap approach and the

GCI approach, the critical value is evaluated using 1,000 Monte Carlo samples. Our simulations were based on 5,000 runs.

The following algorithm was used to estimate the coverage probabilities of four simultaneous confidence intervals.

Algorithm 1

Step 1. Generate X_i, a random sample of sample size n_i from two-parameter exponential population with parameters λ_i and β_i, where $i = 1, 2, \ldots k$. Calculate $\bar{x}_i$ and $x_{(1)i}$ (the observed values of $\bar{X}_i$ and $X_{(1)i}$).

Step 2. For each approach, construct the simultaneous confidence intervals and record whether or not all the values of $\theta_i - \theta_l \ \ (i \neq l)$ fall in their corresponding confidence intervals.

Step 3. Repeat steps 1 and 2 with $M =$ 5,000 times. Therefore, for each approach, the fraction of times that all $\theta_i - \theta_l \ \ (i \neq l)$ are in their corresponding the simultaneous confidence intervals provides the coverage probability.

In our simulation, the configuration factors are considered: 1) sample cases: $k = 3$ and $k =$ 5; 2) scale parameters: $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 1$; 3) location parameters: $\beta_i = \lambda_i(1-\theta_i)/\theta_i$; 4) population coefficients of variation: $\theta_1, \theta_2, \ldots, \theta_k$; and 5) sample sizes: $n_1, n_2, \ldots, n_k$. Specific combinations of configurations are presented in the following tables.

The simulation results for $k = 3$ and $k = 5$ are given in Tables 1 and 2, respectively. The performance of the proposed confidence intervals is compared. From Table 1, the coverage probabilities of SCI_{PB} are in the range from 0.96-0.98. Therefore, SCI_{PB} is conservative confidence interval. The coverage probabilities of SCI_{GCI} are close to the nominal confidence level 0.95 for all cases. The coverage probabilities of SCI_{MOVER1} are close to 1.00 when all the coefficients of variation are positive value, whereas the coverage probabilities of SCI_{MOVER1} are close to the nominal confidence level 0.95 when the sample sizes are large and some the coefficient of variation is negative value**.** Moreover, SCI_{MOVER2} have the coverage probabilities close to the nominal confidence level 0.95 when the sample sizes are large and all the coefficients of variation are positive value. The coverage probabilities of SCI_{MOVER2} provide less than the nominal confidence level 0.95 for all sample cases when some the coefficient of variation is negative value.

Next, Table 2 is considered. Again, SCI_{PB} and SCI_{MOVER1} are conservative confidence interval because the coverage probabilities are close to 1.00. The coverage probabilities of SCI_{GCI} are stable. The coverage probabilities of SCI_{MOVER2} are close to the nominal confidence level 0.95 when the sample sizes are large and all the coefficients of variation are positive value, whereas SCI_{MOVER2} have the coverage probabilities under nominal confidence level 0.95 for all sample cases when some the coefficient of variation is negative value.

Table 1 The coverage probabilities (CP), average lengths (AL) and standard errors (s.e.) of 95% of two-sided simultaneous confidence intervals for all differences of coefficients of variation of two-parameter exponential distributions: 3 sample cases

(n_1,n_2,n_3)	$(\theta_1,\theta_2,\theta_3)$	SCI_{PB}		SCI_{GCI}		SCI_{MOVER1}		SCI_{MOVER2}	
		CP	AL (s.e.)	CP	AL (s.e.)	CP	AL (s.e.)	CP	AL (s.e.)
(30,30,30)	(1.0,0.5,0.3)	0.9769	0.2735 (0.0153)	0.9495	0.2357 (0.0099)	1.0000	1.7276 (0.5473)	0.9489	0.2318 (0.0111)
	(−1.0,1.0,0.5)	0.9667	2.4144 (1.0077)	0.9683	3.1298 (1.4441)	0.9767	2.3256 (0.4876)	0.8910	1.2229 (0.4907)
(50,50,50)	(1.0,0.5,0.3)	0.9807	0.1981 (0.0155)	0.9534	0.1674 (0.0117)	1.0000	0.9470 (0.2595)	0.9528	0.1674 (0.0116)
	(−1.0,1.0,0.5)	0.9725	1.3334 (0.5444)	0.9542	1.3114 (0.5721)	0.9657	1.3230 (0.2128)	0.9105	0.8695 (0.3501)
(30,30,50)	(1.0,0.5,0.3)	0.9756	0.2666 (0.0209)	0.9501	0.2195 (0.0151)	0.9996	1.7101 (0.5575)	0.9225	0.2165 (0.0168)
	(−1.0,1.0,0.5)	0.9656	2.4068 (1.0324)	0.9663	3.2099 (1.5016)	0.9705	2.3393 (0.5349)	0.8761	1.2167 (0.5060)
(100,100,100)	(1.0,0.5,0.3)	0.9792	0.1316 (0.0133)	0.9487	0.1104 (0.0109)	0.9999	0.5540 (0.1396)	0.9520	0.1110 (0.0105)
	(−1.0,1.0,0.5)	0.9672	0.7713 (0.3145)	0.9465	0.6700 (0.2810)	0.9537	0.7567 (0.0909)	0.9259	0.5793 (0.2349)
(30,50,100)	(1.0,0.5,0.3)	0.9724	0.2298 (0.0156)	0.9499	0.1807 (0.0134)	0.9995	1.6304 (0.6163)	0.8821	0.1773 (0.0137)
	(−1.0,1.0,0.5)	0.9655	2.4044 (1.0909)	0.9689	2.9668 (1.4177)	0.9695	1.6552 (0.3471)	0.8825	1.1697 (0.5185)
(200,200,200)	(1.0,0.5,0.3)	0.9787	0.0896 (0.0103)	0.9499	0.0754 (0.0086)	1.0000	0.3605 (0.0874)	0.9520	0.0757 (0.0084)
	(−1.0,1.0,0.5)	0.9744	0.5001 (0.2051)	0.9471	0.4312 (0.1793)	0.9438	0.4994 (0.0540)	0.9393	0.4036 (0.1651)

4. An Empirical Verification

In this section, an example previously considered by Maurya et al. (2011) is used to illustrate the proposed approaches. In Table 3, the data show the survival days of patients for different types of lung cancer. The summary statistics are as follows: $n_1 = 9,\ \bar{x}_1 = 51.0000,\ x_{(1)1} = 8.0000,\ \hat{\theta}_1 = 0.8431,\ n_2 = 9,\ \bar{x}_2 = 22.1111,\ x_{(1)2} = 13.0000,\ \hat{\theta}_2 = 0.4121,\ n_3 = 9,\ \bar{x}_3 = 72.8889,\ x_{(1)3} = 3.0000,\ \hat{\theta}_3 = 0.9588,\ n_4 = 9,\ \bar{x}_4 = 197.8889,\ x_{(1)4} = 103.0000,$ and $\hat{\theta}_4 = 0.4795$. Table 4 shows the 95% simultaneous confidence intervals for the six differences of coefficients of variation of two- parameter exponential distributions, $\theta_{il} = \theta_i - \theta_l$, where $i, l = 1,2,3,4$ and $i \neq l$. The results show that the lengths of MOVER approach (CI_{MOVER2}) are shorter than the lengths of other approaches. Therefore, this result confirms our simulation study in the previous section for small samples sizes.

Table 2 The coverage probabilities (CP), average lengths (AL) and standard errors (s.e.) of 95% of two-sided simultaneous confidence intervals for all differences of coefficients of variation of two-parameter exponential distributions: 5 sample cases

$(n_1,n_2,n_3,$ $n_4,n_5)$	$(\theta_1,\theta_2,\theta_3,$ $\theta_4,\theta_5)$	SCI_{PB}		SCI_{GCI}		SCI_{MOVER1}		SCI_{MOVER2}	
		CP	AL (s.e.)	CP	AL (s.e.)	CP	AL (s.e.)	CP	AL (s.e.)
(30,30,30, 30,30)	(1.0,0.5,0.5, 0.3,0.3)	0.9919	0.3278 (0.0097)	0.9514	0.2404 (0.0058)	1.0000	1.2854 (0.2716)	0.9533	0.2365 (0.0063)
	(−1.0,1.0,1.0, 0.5,0.5)	0.9831	1.7120 (0.5276)	0.9603	1.9719 (0.7060)	0.9851	2.3381 (0.3239)	0.9155	0.8341 (0.2423)
(50,50,50, 50,50)	(1.0,0.5,0.5, 0.3,0.3)	0.9928	0.2420 (0.0092)	0.9500	0.1752 (0.0061)	0.9999	0.7352 (0.1295)	0.9533	0.1746 (0.0060)
	(−1.0,1.0,1.0, 0.5,0.5)	0.9856	0.9667 (0.2875)	0.9505	0.8599 (0.2832)	0.9794	1.2337 (0.1206)	0.9287	0.5893 (0.1722)
(30,30,30, 50,50)	(1.0,0.5,0.5, 0.3,0.3)	0.9916	0.3131 (0.0142)	0.9513	0.2206 (0.0095)	0.9996	1.2555 (0.2799)	0.9349	0.2181 (0.0100)
	(−1.0,1.0,1.0, 0.5,0.5)	0.9829	1.6512 (0.5250)	0.9609	2.0073 (0.7344)	0.9815	7.1734 (2.3412)	0.9001	0.8104 (0.2470)
(100,100, 100,100, 100)	(1.0,0.5,0.5, 0.3,0.3)	0.9932	0.1639 (0.0077)	0.9514	0.1183 (0.0055)	0.9999	0.4403 (0.0698)	0.9540	0.1184 (0.0052)
	(−1.0,1.0,1.0, 0.5,0.5)	0.9866	0.5907 (0.1774)	0.9499	0.4483 (0.1400)	0.9745	0.7268 (0.0606)	0.9396	0.3930 (0.1169)
(30,30,50, 100,100)	(1.0,0.5,0.5, 0.3,0.3)	0.9891	0.2788 (0.0177)	0.9502	0.1892 (0.0122)	0.9989	1.1834 (0.2955)	0.9100	0.1870 (0.0124)
	(−1.0,1.0,1.0, 0.5,0.5)	0.9821	1.6043 (0.5417)	0.9587	1.8433 (0.6890)	0.9807	1.8204 (0.3044)	0.8813	0.7683 (0.2510)
(200,200, 200,200, 200)	(1.0,0.5,0.5, 0.3,0.3)	0.9933	0.1128 (0.0059)	0.9494	0.0817 (0.0043)	0.9998	0.2892 (0.0439)	0.9511	0.0818 (0.0042)
	(−1.0,1.0,1.0, 0.5,0.5)	0.9888	0.3893 (0.1182)	0.9491	0.2863 (0.0884)	0.9681	0.4750 (0.0369)	0.9453	0.2697 (0.0814)

Table 3 Survival days of patients based on types of lung cancer

Type	Survival days								
Squamous	72	10	81	110	100	42	8	25	11
Small	30	13	23	16	21	18	20	27	31
Adeno-type	8	92	35	117	132	12	162	3	95
Large	177	162	553	200	156	182	143	105	103

Table 4 The 95% simultaneous confidence intervals for all pairwise differences of coefficients of variation of two-parameter exponential distributions

Parameters	CI_{PB}	CI_{GCI}	CI_{MOVER1}	CI_{MOVER2}
$\theta_2 - \theta_1$	[−0.8188, −0.0433]	[−0.9470, −0.1837]	[−2.2797, 4.2758]	[−0.6907, −0.0859]
$\theta_3 - \theta_1$	[−0.3084, 0.5398]	[−0.3433, 0.6573]	[−2.8859, 2.8708]	[−0.2436, 0.4504]
$\theta_4 - \theta_1$	[−0.7628, 0.0355]	[−0.8442, −0.1240]	[−2.3262, 2.7369]	[−0.6423, −0.0169]
$\theta_3 - \theta_2$	[0.1486, 0.9449]	[0.3147, 1.0942]	[−4.3266, 2.4425]	[0.1790, 0.8045]
$\theta_4 - \theta_2$	[−0.3040, 0.4389]	[−0.2275, 0.3539]	[−4.2444, 2.4374]	[−0.2221, 0.3406]
$\theta_4 - \theta_3$	[−0.8886, −0.0701]	[−1.0480, −0.2485]	[−2.4864, 2.8686]	[−0.7563, −0.1101]

5. Discussion and Conclusions

In this paper, the parametric bootstrap approach, the GCI approach, and the MOVER approach are proposed to construct the simultaneous confidence intervals for differences of coefficients of variation of two-parameter exponential distributions. The parametric bootstrap approach provides a conservative confidence interval. Hence, parametric bootstrap approach is not recommended to construct the simultaneous confidence intervals for differences of coefficients of variation. The GCI approach is well in term of the coverage probability for all cases. Therefore, the GCI approach should be used to evaluate the simultaneous confidence intervals for the differences of coefficients of variation. Moreover, the MOVER approach can be considered to use for the confidence interval estimation when the sample sizes are large**.** The MOVER approach (SCI_{MOVER1}) can be used when some the coefficient of variation is negative value, whereas the MOVER approach (SCI_{MOVER2}) performs well when all the coefficients of variation are positive value.

As a final note, Sangnawakij and Niwitpong (2017) indicated that the GCI approach is recommended to construct the confidence intervals for single and difference of coefficients of variation of two-parameter exponential distributions. Moreover, Thangjai and Niwitpong (2017) proposed that the GCI approach should be chosen to construct the confidence interval for the weighted coefficients of variation of two-parameter exponential distributions. This paper indicated that the GCI approach can be used for constructing the simultaneous confidence intervals for differences of coefficients of variation of two-parameter exponential distributions.

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