Thailand Statistician
April 2020; 18(2): 150-164
http//:statassoc.or.th
Contributed paper

Bayesian Computation and Analysis of C-PAR(1) Time Series Model with Structural Break

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Received: 20 May 2018
Revised: 13 January 2019
Accepted: 6 April 2019

Abstract

Panel data consists of repeated observations over time on the same set of cross-sectional units and impact of covariate may influence the estimation and testing procedures. Present paper proposes a covariate panel autoregressive (C-PAR(1)) unit root tests and estimation considering structural break under Bayesian approach. We use Monte Carlo simulation method to estimate the parameters using conditional posterior distribution and compared with ordinary least square estimator. For testing the unit root hypothesis, posterior odds ratio is derived and recorded satisfactory results. We have also studied empirical analysis on import of fertilizers to get more applicability of the model.

Keywords: Bayesian inference, covariate, Gibbs sampler, panel data model, unit root test.

1. Introduction

Panel data contain repeated measures of the same variable, taken from same set of units over time. The growing interest in panel data modelling, especially over the 20th century, reflects increasing availability of such data and advances in computing technology. It also reflects the growing interest in estimating models of individual behaviour over time without having to use aggregate time series data. Panel data is more appropriate in comparison to univariate model because of it's applicability in controlling individuals heterogeneity which are not controlled alone through univariate time series model. It is also able to study the dynamics of adjustment which is capable to identify and measures the effects that are simply not editable in pure cross-section data.

In literature of classical approach, unit root test with covariate, Hensen (1995) developed unit root test with some stationary covariates for autoregressive parameter. He proposed covariate augmented Dickey-Fuller (CADF) unit root test and obtain the asymptotic local power function of CADF statistic. This CADF test was further extended to a point-optimal covariate (POC) unit root test by Elliott and Jansson (2003). Moreover, Chang et al. (2017) developed bootstrap unit root tests with covariate method to the CADF test to deal with the nuisance parameter dependency and provided a valid basis for inference based on the CADF test. Use of Bayesian approach for testing the unit root hypothesis was pointed out Sims (1988) and Sims and Uhling (1991). Chaturvedi et al. (2017) proposed univariate model for testing unit root hypothesis for an autoregressive model in the presence of stationary covariate under Bayesian framework.

Costantini and Lupi (2013) developed panel data model with stationary covariate which was the extension of model discussed by Hensen (1995). Recently, Kumar et al. (2018) described Bayesian approach for testing unit root hypothesis and estimation for Covariate Panel Data Autoregressive model without structural break using both simulated as well as real data. There are discussions in literature that sometimes, structure of the series changes due to various causes such as outliers, structural break, irregular dynamic pattern etc. Structural break is a long term shifting in the original structure of a series and impact on the coefficients of series and receiving attraction among the researchers in both single as well as multiple time points. Chib (1998) discussed the multiple change point under Bayesian approach and provided a comprehensive study for estimation and model selection for autoregressive (AR) model. Although, Wang and Zivot (2000), Newbold et al. (2001), Chaturvedi and Kumar (2007), Chen et al. (2011), Meligkotsiduo et al. (2017) etc. considered structural break in regression and time series model for detecting, estimating and/or testing the presence of break point as well as unit root hypothesis. Fossati (2013) extended covariate unit root tests in the presence of structural break in trend function and contributed inference of unit root hypothesis that improve the power of correlated stationary covariates. Under Bayesian approach, there are several advantages, especially with regards to use of computational methods, particularly Markov Chain Monte Carlo (MCMC). The use of MCMC techniques has allowed practitioners to draw samples from previously intractable probability distributions by running a constructed Markov chain for a long period of time. Whilst several ways exist to construct these chains, they are all, including the Gibbs Sampler (Geman 1984), a subset of the general MCMC technique of Metropolis et al. (1953) and Hastings (1970). Akaike's information criterion (AIC) and Bayesian information criterion (BIC) are used for comparing nested models of differing dimensions (Priestley 1981).

In this paper, we have proposed covariate panel autoregressive (C-PAR(1)) model considering structural break in mean and variance of model as well as in coefficient of covariate at known single time point under Bayesian approach, posterior odds ratio has been derived to test the unit root hypothesis and estimation methodology is proposed under assumed prior assumption. The Bayes estimates obtained by present methods are compared with the ordinary least square estimates for both simulated and empirical time series.

2. Model with Structural Break

The proposed autoregressive panel data model with covariate is the extension of univariate C-AR(1) model (Kumar et al. 2017). In which they considered the univariate model. Let $\{y_{it}; i=1,2,...,n; t=1,2,...,T\}$ be a time series with *n* panels considering structural break in mean at a single known time point T_B where mean of the series changes from μ_{i1} to μ_{i2}.

$$y_{it} = \begin{cases} \mu_{i1} + u_{it}; \, t = 1,2,...,T_B \\ \mu_{i2} + u_{it}; \, t = T_B+1, T_B+2,...,T, \end{cases} \tag{1}$$

where u_{it} is serially correlated panel AR(1) containing stationary correlated covariate $\{w_{it}\}$ which is given as

$$u_{it} = \begin{cases} \rho u_{it-1} + \sum_{j=-r_1+1}^{p_1} \lambda_{ij} w_{it-j} + \sigma_1 \varepsilon_{it}; \, t \le T_B \\ \rho u_{it-1} + \sum_{j=-r_2+1}^{p_2} \lambda_{ij} w_{it-j} + \sigma_2 \varepsilon_{it}; \, t > T_B. \end{cases} \tag{2}$$

Error process is shifted in both error variance and covariate's coefficients at same point T_B. Here, $\{\varepsilon_{it}; t = 1, 2, 3, ..., T\}$ are i.i.d. random variables each following normal distribution with mean zero and unknown variance. We can write the model (1) utilizing (2) as

$$y_{it} = \begin{cases} \rho y_{it-1} + (1-\rho)\mu_{i1} + \sum_{j=-r_1+1}^{p_1} \lambda_{ij} w_{it-j} + \sigma_1 \varepsilon_{it}; & t \le T_B \\ \rho y_{it-1} + (1-\rho)\mu_{i2} + \sum_{j=-r_2+1}^{p_2} \lambda_{ij} w_{it-j} + \sigma_2 \varepsilon_{it}; & t > T_B. \end{cases} \tag{3}$$

The stationary covariate $\{w_{it}\}$ which helps in increasing the power of the test and at time t, the model may involve some future values $w_{i(t+1)}, w_{i(t+2)}, ..., w_{i(t+r+1)}$. If future values are not available, the predicted values of w_{it} 's can be used in place of these values. We are also interested to test the unit root hypothesis for the above model, $H_0 : \rho = 1$ against the alternative $H_1 : \rho \in s$ with $s = \{a < \rho < 1; a > -1\}$. Under the null hypothesis of unit root, the model reduces to

$$\Delta y_{it} = \begin{cases} \sum_{j=-r_1+1}^{p_1} \lambda_{ij} w_{it-j} + \sigma_1 \varepsilon_{it}; t \le T_B \\ \sum_{j=-r_2+1}^{p_2} \lambda_j w_{it-j} + \sigma_2 \varepsilon_{it}; t > T_B. \end{cases} \tag{4}$$

The model is given in Equations (3) and (4) may be written in matrix notation

$$\begin{aligned} Y_{T_B} &= \rho Y_{-1}^{T_B} + (1-\rho)\mathrm{M}_1(l_{T_B} \otimes I_n) + W_1\Lambda_1 + \sigma_1\xi_1; & t \le T_B \\ Y_{T-T_B} &= \rho Y_{-1}^{T-T_B} + (1-\rho)M_2(l_{T-T_B} \otimes I_n) + W_2\Lambda_2 + \sigma_2\xi_2;\ t > T_B \end{aligned} \tag{5}$$

and

$$\begin{aligned} \Delta Y_{T_B} &= W_1\Lambda_1 + \sigma_1\xi_1; & t \le T_B \\ \Delta Y_{T-T_B} &= W_1\Lambda_2 + \sigma_2\xi_2; & t > T_B, \end{aligned} \tag{6}$$

where

$$Y_{i,1} = \begin{pmatrix} y_{i,1} & y_{i,2} & \cdots & y_{i,T_B} \end{pmatrix}', \qquad Y_{T_B} = \begin{pmatrix} y_1 & y_2 & \cdots & y_{T_B} \end{pmatrix}',$$

$$Y_{i,-1} = \begin{pmatrix} y_{i,0} & y_{i,1} & \cdots & y_{i,T_B-1} \end{pmatrix}', \qquad Y_{T-T_B}^{T_B} = \begin{pmatrix} y_0 & y_1 & \cdots & y_{T_B-1} \end{pmatrix}',$$

$$Y_{T-T_B} = \begin{pmatrix} y_{i,T_B+1} & y_{i,T_B+2} & \cdots & y_{i,T} \end{pmatrix}', \quad Y_{-1}^{T-T_B} = \begin{pmatrix} y_{i,T_B} & y_{i,T_B+1} & \cdots & y_{i,T-1} \end{pmatrix}',$$

$$l_{T_B} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}', \qquad l_{T-T_B} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}',$$

$$M_1 = \left(\mu_{1,1},\ \mu_{1,2},\ ..., \mu_{1,n}\right)', \qquad M_2 = \left(\mu_{2,1},\ \mu_{2,2},\ ..., \mu_{2,n}\right)',$$

$$\Lambda_{i,1} = \begin{pmatrix} \lambda_{i,-r_1+1}^{(1)} & \lambda_{i,-r_1+2}^{(1)} & \cdots & \lambda_{i,p_1}^{(1)} \end{pmatrix}', \quad \Lambda_1 = \begin{pmatrix} \lambda_{1,1} & \lambda_{2,2} & \cdots & \lambda_{n,1} \end{pmatrix}',$$

$$\Lambda_{i,2} = \begin{pmatrix} \lambda_{i,-r_2+1}^{(2)} & \lambda_{i,-r_2+2}^{(2)} & \cdots & \lambda_{i,p_2}^{(2)} \end{pmatrix}', \quad \Lambda_2 = \begin{pmatrix} \lambda_{1,2} & \lambda_{2,2} & \cdots & \lambda_{n,2} \end{pmatrix}',$$

$$W_1 = \begin{pmatrix} W'_{1,T_B} & 0 & \cdots & 0 \\ 0 & W'_{2,T_B} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W'_{n,T_B} \end{pmatrix}, \qquad W_2 = \begin{pmatrix} W'_{1,T-T_B} & 0 & \cdots & 0 \\ 0 & W'_{2,T-T_B} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W'_{n,T-T_B} \end{pmatrix},$$

$$W_{i,T_B} = \begin{pmatrix} w_{i,r_1} & w_{i,r_1-1} & \cdots & w_{i,1-p_1} \\ w_{i,r_1-1} & w_{i,r_1} & \cdots & w_{i,2-p_1} \\ \vdots & \vdots & \ddots & \vdots \\ w_{i,T_B+r_1-1} & w_{i,T_B+r_1-2} & \cdots & w_{i,T_B-p_1} \end{pmatrix}, \quad W_{i,T-T_B} = \begin{pmatrix} w_{i,T_B+r_2} & w_{i,T_B+r_2-1} & \cdots & w_{i,T_B+1-p_2} \\ w_{i,T_B+r_2+1} & w_{i,T_B+r_2} & \cdots & w_{i,T_B+2-p_2} \\ \vdots & \vdots & \ddots & \vdots \\ w_{i,T_B+r_2-1} & w_{i,T_B+r_2-2} & \cdots & w_{i,T_B-p_2} \end{pmatrix},$$

$$\xi_1 = \begin{pmatrix} \varepsilon_{i,1} & \varepsilon_{i,2} & \ldots & \varepsilon_{i,T_B} \end{pmatrix}', \qquad \xi_2 = \begin{pmatrix} \varepsilon_{i,T_B+1} & \varepsilon_{i,T_B+2} & \ldots & \varepsilon_{i,T} \end{pmatrix}'.$$

3. Bayesian Analysis

Let us assume prior for mean term (μ_k) and error variance (σ_k^2) as conjugate normal prior $MN(\mu'_k, \sigma_k^2)$ and inverse gamma prior $IG(a_k, b_k)$, respectively and uniform prior for autoregressive coefficient (ρ) and covariate coefficient (λ_k). Then the joint prior distribution of model parameters is

$$P(\Theta) = \frac{b_1^{a_1} b_2^{a_2} (\sigma_1^2)^{-a_1-3/2} (\sigma_2^2)^{-a_2-3/2}}{(2\pi)\Gamma(a_1)\Gamma(a_2)(1-a)} \exp\left[-\frac{1}{2\sigma_1^2}\left((M_1-\mu'_1)^2+2b_1\right) - \frac{1}{2\sigma_2^2}\left((M_2-\mu'_2)^2+2b_2\right)\right], \tag{7}$$

where $k = 1, 2$.

In AR modelling, testing of stationary is needed before the parameter estimation. So, unit root hypothesis is here important before modelling and drawing any inferences. The unit root hypothesis basically test that as series is difference stationary or trend stationary by considering hypothesis $H_0 : \rho = 1$ against alternative $H_1 : \rho \in S;\ S = \{a < \rho < 1; a > -1\}$. We have tested this using posterior odds ratio (POR) which is the product of prior odds ratio and Bayes factor as given below

$$\beta_{01} = \frac{P(H_0)}{P(H_1)} \times \frac{P(Y \mid H_0)}{P(Y \mid H_1)} = \frac{p_0}{1-p_0} \times \frac{P(Y \mid H_0)}{P(Y \mid H_1)},$$

where prior odds ratio $p_0/(1-p_0)$ is the ratio of prior probability $P(H_0) = p_0$ and $P(H_1) = 1 - p_0$, in Bayesian framework and Bayes factor is ratio of posterior probability under null and alternative hypothesis with the known prior of parameters for more details please refer Chib (1998), Chaturvedi and Kumar (2007), Berger (2013) and Chen et al. (2015).

Let us define the following notations:

$$\Sigma_1 = I - W_1'\left(W_1'W_1\right)^{-1} W_1$$

$$\Sigma_2 = I - W_2'\left(W_2'W_2\right)^{-1} W_2$$

$$A(\rho) = \left(Y_{T_B} - \rho Y_{-1}^{T_B}\right)' \Sigma_1 \left(Y_{T_B} - \rho Y_{-1}^{T_B}\right) - \left[\left(l_{T_B} \otimes I_n\right)(1-\rho)\Sigma_1\left(Y_{T_B} - \rho Y_{-1}^{T_B}\right) + \mu'_1\right]'$$
$$\left[\left(l_{T_B} \otimes I_n\right)(1-\rho)^2 \Sigma_1 \left(l_{T_B} \otimes I_n\right) + 1\right]^{-1} \left[\left(l_{T_B} \otimes I_n\right)(1-\rho)\Sigma_1\left(Y_{T_B} - \rho Y_{-1}^{T_B}\right) + \mu'_1\right] + (\mu'_1)^2 + 2b_1$$

$$
\begin{aligned}
B(\rho) = & \left(Y_{T-T_B} - \rho Y_{-1}^{T-T_B}\right)' \Sigma_2 \left(Y_{T-T_B} - \rho Y_{-1}^{T-T_B}\right) \\
& - \left[\left(l_{T-T_{\mathbf{B}}} \otimes I_n\right)(1-\rho)\Sigma_2 \left(Y_{T-T_B} - \rho Y_{-1}^{T-T_B}\right) + \mu_2'\right]' \\
& \left[\left(l_{T-T_{\mathbf{B}}} \otimes I_n\right)(1-\rho)^2 \Sigma_2 \left(l_{T-T_{\mathbf{B}}} \otimes I_n\right) + 1\right]^{-1} \left[\left(l_{T-T_{\mathbf{B}}} \otimes I_n\right)(1-\rho)\Sigma_2 \left(Y_{T-T_B} - \rho Y_{-1}^{T-T_B}\right) + \mu_2'\right] \\
& + (\mu_2')^2 + 2b_2.
\end{aligned}
$$

Theorem 1 *For testing the unit root hypothesis* $H_0 : \rho = 1$ *against the alternative hypothesis* $H_1 : \rho \in S; S = \{a < \rho < 1; a > -1\}$ *with prior odds ratio* $p_0/(1-p_0)$, *POR for the covariate model is derived as*

$$
\beta_{01} = \frac{p_0}{1-p_0} \frac{(1-a)}{\left(\Delta Y_{T_B}' \Sigma_1 \Delta Y_{T_B} + 2b_1\right)^{nT_B/2-(p_1+r_1)/2+a_1} \left(\Delta Y_{T-T_B}' \Sigma_2 \Delta Y_{T-T_B} + 2b_2\right)^{n(T-T_B)/2-(p_2+r_2)/2+a_2}} \times
$$
$$
\left[\int_a^1 \frac{1}{A(\rho)^{nT_B/2-(p_1+r_1)/2+a_1} B(\rho)^{n(T-T_B)/2-(p_2+r_2)/2+a_2} \left|\left(l_{T_{\mathbf{B}}} \otimes I_n\right)(1-\rho)^2 \Sigma_1 \left(l_{T_{\mathbf{B}}} \otimes I_n\right)+1\right|^{1/2} \left|\left(l_{T-T_{\mathbf{B}}} \otimes I_n\right)(1-\rho)^2 \Sigma_2 \left(l_{T-T_{\mathbf{B}}} \otimes I_n\right)+1\right|^{1/2}} d\rho\right]^{-1}. \tag{8}
$$

Proof: The detail poof of this theorem is given in Appendix.

In Bayesian study, estimates of model parameters are obtained by using posterior probabilities. To obtain posterior probability, integrating the joint posterior distribution under assumed prior. Using mathematical manipulation, we obtained conditional posterior distribution of $M_1, M_2, \Lambda_1, \Lambda_2, \sigma_1^2, \sigma_2^2$ and ρ as follows:

$$
\begin{aligned}
& M_1 \sim MN\left(V_1 Q_1^{-1}, \sigma_1^2 Q_1^{-1}\right); \\
& M_2 \sim MN\left(V_2 Q_2^{-1}, \sigma_2^2 Q_2^{-1}\right); \\
& \Lambda_1 \sim MN\left(V_3 Q_3^{-1}, \sigma_1^2 Q_3^{-1}\right); \\
& \Lambda_2 \sim MN\left(V_4 Q_4^{-1}, \sigma_2^2 Q_4^{-1}\right); \\
& \sigma_1^2 \sim IG\left(A_1, B_1\right); \\
& \sigma_2^2 \sim IG\left(A_2, B_2\right); \\
& \rho \sim TN\left(\left(\frac{V_5}{\sigma_1^2} + \frac{V_6}{\sigma_2^2}\right)\left(\frac{Q_5}{\sigma_1^2} + \frac{Q_6}{\sigma_2^2}\right)^{-1}, \left(\frac{Q_5}{\sigma_1^2} + \frac{Q_6}{\sigma_2^2}\right)^{-1}, a, 1\right),
\end{aligned} \tag{9}
$$

where

$$
A_1 = \frac{nT_B + 1}{2} + a_1,
$$

$$
A_2 = \frac{n(T - T_B) + 1}{2} + a_2,
$$

$$
\begin{aligned}
B_1 = \frac{1}{2}\Big\{ & \left(y_{T_B} - \rho y_{-1}^{T_B} - (1-\rho) M_1 (l_{T_B} \otimes I_n) - W_1 \Lambda_1\right)' \left(y_{T_B} - \rho y_{-1}^{T_B} - (1-\rho) M_1 (l_{T_B} \otimes I_n) - W_1 \Lambda_1\right) \\
& + \left(M_1 - \mu_1'\right)^2 + 2b_1 \Big\},
\end{aligned}
$$

$$B_2 = \frac{1}{2}\Big\{\left(y_{T-T_B} - \rho y_{-1}^{T-T_B} - (1-\rho)M_2(l_{T_B} \otimes I_n) - W_2\,\Lambda_2\right)'\left(y_{T-T_B} - \rho y_{-1}^{T-T_B} - (1-\rho)M_2(l_{T_B} \otimes I_n) - W_2\Lambda_2\right)$$
$$+\left(M_2 - \mu_2'\right)^2 + 2b_2\Big\},$$

$$Q_1 = (l_{T_B} \otimes I_n)'(l_{T_B} \otimes I_n)(1-\rho)^2 + I_n,$$
$$Q_2 = (l_{T-T_B} \otimes I_n)'(l_{T-T_B} \otimes I_n)(1-\rho)^2 + I_n,$$
$$Q_3 = W_1'W_1,$$
$$Q_4 = W_2'W_2,$$
$$Q_5 = \left(y_{-1}^{T_B} - M_1(l_{T_B} \otimes I_n)\right)'\left(y_{-1}^{T_B} - M_1(l_{T-T_B} \otimes I_n)\right),$$
$$Q_6 = \left(y_{-1}^{T-T_B} - M_2(l_{T-T_B} \otimes I_n)\right)'\left(y_{-1}^{T-T_B} - M_2(l_{T-T_B} \otimes I_n)\right),$$
$$V_1 = (1-\rho)(l_{T_B} \otimes I_n)'\left(y_{T_B} - \rho y_{-1}^{T_B} - W_1\Lambda_1\right) + \mu_1' I_n,$$
$$V_2 = (1-\rho)(l_{T-T_B} \otimes I_n)'\left(y_{T-T_B} - \rho y_{-1}^{T-T_B} - W_2\Lambda_2\right) + \mu_2' I_n,$$
$$V_3 = W_1'\left(y_{T_B} - \rho y_{-1}^{T_B} - (1-\rho)M_1(l_{T_B} \otimes I_n)\right),$$
$$V_4 = W_2'\left(y_{T-T_B} - \rho y_{-1}^{T-T_B} - (1-\rho)M_2(l_{T-T_B} \otimes I_n)\right),$$
$$V_5 = \left(y_{-1}^{T_B} - M_1(l_{T_B} \otimes I_n)\right)'\left(y_{T_B} - M_1(l_{T_B} \otimes I_n) - W_1\Lambda_1\right),$$
$$V_6 = \left(y_{-1}^{T-T_B} - M_2(l_{T-T_B} \otimes I_n)\right)'\left(y_{T-T_B} - M_2(l_{T-T_B} \otimes I_n) - W_2\Lambda_2\right).$$

4. Simulation Study

Simulation is a technique, tools, and strategies which can be applied in designing the structure of the process in different areas. It has been widely useful in the fields of econometrics and time series. In time series, simulation offers good scope for estimation, testing and prediction. The realistic scenarios of this technique is well developed in various mathematical software's like R, STATA, MATLAB etc. which provides relevant results to verify the specific problem through simulated samples. In present study, we have performed simulation study for proposed C-PAR(1) time series model. For series generation from C-PAR(1) model (5), different size of series $T =$ (80, 100) with one know structural break at three scenarios of $T_B = (T/4, T/2, 3T/4)$ and covariate series having similar break point T_B with error term ζ_{it} distributed normally which is as follows

$$w_{it} = \begin{cases} 0.05 + 0.07 w_{i,t-1} + \zeta_{it}, & t \le T_B \\ 0.7 + 0.08 w_{i,t-1} + \zeta_{it}, & t > T_B. \end{cases} \tag{10}$$

We have considered panel of size three with initial value of y_{i0} = (0.20, 0.40, 0.60) and w_{i0} = (0.3, 0.5, 0.7) and ρ = (0.90, 0.92, 0.94, 0.97, 0.98, 0.99) with other model parameters, M_1 = (0.3, 0.5, 0.7), M_2 = (0.4, 0.6, 0.8), Λ_1 = (0.5, 1, 1.5), Λ_2 = (2, 2.5, 3), σ_1 = 0.5 and σ_2 = 1. The simulation study targets here to get more generalized view on the model so we have generated the time series for small values of ρ = (0.90, 0.92, 0.94) and values close to 1, ρ = (0.97, 0.98, 0.99) and reported the respective values in Tables 1 and 2, respectively. Tables 1 and 2 provides POR and estimates of autoregressive coefficient in respect to T and T_B. The POR strongly favours the rejection of unit root hypothesis in all cases and correctly concluding the stationarity.

Table 1 Bayesian unit root testing with lower value of ρ

T_B	ρ	T =80		T =100	
		$\hat{\rho}$	POR	$\hat{\rho}$	POR
$T/4$	0.90	0.8991	8.01E-04	0.9020	6.29E-04
	0.92	0.9196	6.29E-04	0.9187	5.70E-04
	0.94	0.9363	6.37E-04	0.9402	5.06E-04
$T/2$	0.90	0.8977	5.95E-04	0.8979	5.52E-04
	0.92	0.9165	5.43E-04	0.9159	5.18E-04
	0.94	0.9400	4.52E-04	0.9384	4.66E-04
$3T/4$	0.90	0.9015	5.10E-04	0.9006	3.70E-04
	0.92	0.9204	4.47E-04	0.9200	4.27E-04
	0.94	0.9375	4.00E-04	0.9362	3.46E-04

Table 2 Bayesian unit root testing with higher value of ρ

T_B	ρ	T =80		T =100	
		$\hat{\rho}$	POR	$\hat{\rho}$	POR
$T/4$	0.97	0.9732	4.60E-04	0.9704	4.07E-04
	0.98	0.9813	4.68E-04	0.9795	4.65E-04
	0.99	0.9880	1.90E-01	0.9873	4.12E-03
$T/2$	0.97	0.9679	4.06E-04	0.9665	4.25E-04
	0.98	0.9778	4.41E-04	0.9788	4.98E-04
	0.99	0.9870	7.16E-04	0.9904	9.37E-04
$3T/4$	0.97	0.9684	3.13E-04	0.9683	2.04E-04
	0.98	0.9764	3.20E-04	0.9794	2.24E-04
	0.99	0.9909	5.23E-04	0.9859	2.90E-04

In Bayesian approach, loss function was introduces to minimize the error in posterior density. Various researchers used different types of loss function for their convenience. In this paper, we have used both symmetric as well as asymmetric loss function for better interpretation of the proposed model (Ali et al. 2013). The following loss function is considered (1) Squared error loss function (SELF) (2) Absolute loss function (ALF) and (3) Precautionary loss function (PLF). For these loss functions, conditional posterior distribution of the parameters is obtained and getting the estimated value using Gibbs sampling technique. Due to limitation of the manuscript the estimated result are reported here for ρ = (0.90, 0.95, 0.99). Tables 3 to 5 reported the mean squared error (MSE) and absolute bias (ABS) with credible interval (CI) for different value of ρ. Based on the results, observed that OLS performance is not much performing than Bayesian methods because additional information are given to obtained the estimates of estimators in Bayesian in terms of MSE and ABS. Under the loss function, SELF and ALF having approximately same magnitude but PLF shows a better estimator due to minimum MSE and ABS. The confidence interval truly captures the true value of model parameters.

5. Empirical Study

A rapid growth in agricultural sector is due to advance improvement in fertilizers, seeds, new technology etc., especially in developing country like India. India is the well-developed in agriculture field to produce better quality of agriculture commodities using advance techniques. But fertilizers are also useful to improve the production of yields. In India, productions of fertilizers are very less which do not sufficient to fulfil the demand. Therefore, import of fertilizers is necessary to complete the requirement. In present study, we have been taken data series of fertilizers of NPK (Nitrogen, Phosphate and Potassium) from Department of Fertilizers and Department of Agriculture, Cooperation & Farmers Welfare (DAC&FW), India during 1981-2016. This data series recorded the consumptions, productions and imports of fertilizers in annual basis. Due to non availability of production of potassium fertilizers, considered consumptions and imports for analysis purpose only. In these data sets, different types of fertilizers are considered as a panel and assuming import as a covariate.

Present paper deals the structural break in C-PAR(1) with single known break point. So, first we have identified the break point using the R-command "strucchange" developed by Kleiber et al. (2002) in R-software and observed that break occur at time point T_B = 23. After identifying the break point, C-PAR(1) model is fitted under various loss functions and reported in Table 6. For testing the unit root, POR is obtained for different estimated values of ρ estimated by OLS, SELF, ALF and PLF and recorded that series are trend stationary. Therefore, no unit root present in the series of fertilizers. Information criteria also calculated for model comparison and observed that minimum AIC and BIC is obtained through PLF methods with higher ρ = 0.954. For proposed model, estimation is carried out using different methods which are shown in Table 7 and observed that Bayesian approach provides better estimates with respect to OLS and then calculated confidence interval of estimators.

6. Conclusions

Present paper has developed a Bayesian approach for C-PAR(1) model considering structural break in mean, error variance and covariate series. A posterior odds ratio is derived to test the unit root hypothesis and then computed the conditional posterior distribution for estimation under different loss functions. A Monte Carlo simulation is carried out to know the performance of model and results shows that Bayesian estimators improved the efficiency of the parameters. We have applied the model on agricultural data series to find out the appropriate inference of the model. The result shows that data series is stationary and Bayesian estimates of parameters gives more efficient result then OLS method. The model may be further extended for multiple structural breaks.

Table 3 Parameter estimation and credible interval different loss function at $\rho = 0.90$

	SELF			ALF			PLF			OLS		
P	MSE	ABS	CI	MSE	ABS	CI	MSE	ABS	CI	MSE	ABS	CI
ρ	0.6832	0.5545	(0.76, 0.83)	0.6844	0.5541	(0.81, 0.83)	0.6831	0.5546	(0.81, 0.83)	0.6975	0.5522	(0.81, 0.83)
μ_{11}	1.3225	0.8331	(−0.60, 1.15)	1.3243	0.8335	(−0.28, 0.91)	0.7029	0.5670	(−0.27, 0.93)	1.4161	0.8804	(0.65, 1.09)
μ_{12}	1.1293	0.7383	(−0.45, 1.31)	1.1294	0.7379	(−0.21, 0.96)	0.7054	0.5886	(−0.20, 0.97)	1.2442	0.8070	(0.65, 1.17)
μ_{13}	0.8149	0.6231	(−0.11, 1.59)	0.8135	0.6222	(0.10, 1.27)	0.6471	0.6126	(0.12, 1.28)	0.8907	0.6826	(0.67, 1.41)
μ_{21}	1.4315	0.8717	(−1.20, 1.92)	1.4332	0.8735	(−0.51, 1.30)	0.6847	0.6398	(−0.63, 1.22)	1.8513	1.0396	(0.77, 1.44)
μ_{22}	1.1800	0.7683	(−0.91, 2.01)	1.1825	0.7703	(−0.28, 1.35)	0.6992	0.6707	(−0.29, 1.33)	1.6008	0.9588	(0.81, 1.53)
μ_{23}	0.9698	0.7309	(−0.69, 2.40)	0.9697	0.7312	(−0.13, 1.61)	0.7366	0.7133	(−0.13, 1.61)	1.3771	0.9285	(0.80, 1.75)
λ_{11}	0.9740	0.6347	(0.32, 0.68)	0.9740	0.6347	(0.33, 0.68)	0.8658	0.5873	(0.32, 0.68)	0.9753	0.6350	(0.45, 0.74)
λ_{12}	0.6568	0.6105	(0.82, 1.19)	0.6570	0.6103	(0.83, 1.19)	0.6499	0.6279	(0.81, 1.18)	0.6574	0.6107	(0.86, 1.21)
λ_{13}	0.8124	0.8324	(1.31, 1.66)	0.8126	0.8326	(1.30, 1.66)	0.8426	0.8509	(1.30, 1.66)	0.8127	0.8327	(1.35, 1.70)
λ_{21}	1.5364	1.1470	(1.69, 2.33)	1.5370	1.1474	(1.69, 2.34)	1.6189	1.1762	(1.71, 2.35)	1.5383	1.1476	(1.75, 2.38)
λ_{22}	2.7104	1.5075	(2.22, 2.83)	2.7095	1.5072	(2.22, 2.83)	2.8099	1.5350	(2.23, 2.84)	2.7025	1.5050	(2.25, 2.85)
λ_{23}	4.4288	1.9529	(2.68, 3.34)	4.4272	1.9520	(2.68, 3.35)	4.5408	1.9793	(2.68, 3.34)	4.4208	1.9509	(2.71, 3.38)
σ_1	0.8229	0.5725	(0.40, 0.61)	0.8382	0.5767	(0.51, 0.73)	0.8146	0.5704	(0.50, 0.73)	0.9440	0.6158	(0.52, 0.74)
σ_2	0.6515	0.6789	(0.77, 1.30)	0.6455	0.6654	(0.92, 1.47)	0.6562	0.6877	(0.88, 1.42)	0.6473	0.6036	(0.93, 1.50)

Table 4 Parameter estimation and credible interval different loss function at $\rho = 0.95$

	SELF			ALF			PLF			OLS		
P	MSE	ABS	CI	MSE	ABS	CI	MSE	ABS	CI	MSE	ABS	CI
ρ	0.6730	0.5563	(0.78, 0.85)	0.6740	0.5558	(0.83, 0.85)	0.6730	0.5563	(0.83, 0.85)	0.6846	0.5527	(0.83, 0.85)
μ_{11}	1.3220	0.8361	(−0.65, 1.36)	1.3220	0.8359	(−0.30, 0.95)	0.6882	0.5661	(−0.31, 0.93)	1.4708	0.9059	(0.66, 1.10)
μ_{12}	1.1097	0.7323	(−0.44, 1.42)	1.1120	0.7338	(−0.11, 1.10)	0.6941	0.5958	(−0.10, 1.10)	1.2387	0.8091	(0.67, 1.23)
μ_{13}	0.8897	0.6430	(−0.38, 1.62)	0.8887	0.6429	(0.05, 1.22)	0.6637	0.6212	(0.07, 1.24)	1.0246	0.7441	(0.70, 1.37)
μ_{21}	1.4061	0.8640	(−1.56, 2.05)	1.4063	0.8647	(−0.53, 1.27)	0.6478	0.6209	(−0.56, 1.26)	1.9517	1.0712	(0.83, 1.45)
μ_{22}	1.0899	0.7572	(−0.95, 2.27)	1.0932	0.7575	(−0.31, 1.40)	0.6681	0.6636	(−0.30, 1.42)	1.5586	0.9828	(0.85, 1.60)
μ_{23}	0.9663	0.7107	(−0.82, 2.82)	0.9628	0.7097	(−0.08, 1.70)	0.7261	0.7135	(−0.08, 1.65)	1.5696	0.9692	(0.85, 1.79)
λ_{11}	0.9416	0.6235	(0.33, 0.69)	0.9407	0.6229	(0.34, 0.70)	0.8428	0.5824	(0.35, 0.70)	0.9412	0.6227	(0.46, 0.76)
λ_{12}	0.6348	0.5998	(0.81, 1.18)	0.6356	0.6002	(0.82, 1.19)	0.6283	0.6180	(0.80, 1.17)	0.6350	0.5996	(0.88, 1.23)
λ_{13}	0.8225	0.8359	(1.31, 1.67)	0.8214	0.8353	(1.34, 1.71)	0.8523	0.8547	(1.31, 1.68)	0.8215	0.8355	(1.34, 1.70)
λ_{21}	1.4984	1.1276	(1.69, 2.31)	1.4981	1.1273	(1.69, 2.31)	1.5795	1.1563	(1.70, 2.31)	1.4992	1.1281	(1.73, 2.34)
λ_{22}	2.6848	1.4996	(2.20, 2.81)	2.6858	1.5000	(2.22, 2.84)	2.7846	1.5281	(2.21, 2.83)	2.6897	1.5012	(2.24, 2.85)
λ_{23}	4.352	1.9328	(2.70, 3.33)	4.3515	1.9326	(2.66, 3.30)	4.4635	1.9599	(2.67, 3.32)	4.3468	1.9313	(2.70, 3.32)
σ_1	0.8241	0.5776	(0.39, 0.59)	0.8403	0.5822	(0.49, 0.72)	0.8155	0.5753	(0.48, 0.70)	0.9465	0.6221	(0.51, 0.74)
σ_2	0.6490	0.6794	(0.74, 1.22)	0.6438	0.6659	(0.92, 1.42)	0.6536	0.688	(0.88, 1.37)	0.6469	0.6042	(0.94, 1.44)

Table 5 Parameter estimation and credible interval different loss function at $\rho = 0.99$

	SELF			ALF			PLF			OLS		
P	MSE	ABS	CI	MSE	ABS	CI	MSE	ABS	CI	MSE	ABS	CI
ρ	0.6655	0.5593	(0.80, 0.86)	0.6664	0.5589	(0.85, 0.87)	0.6654	0.5593	(0.84, 0.87)	0.6786	0.5567	(0.85, 0.87)
μ_{11}	1.3273	0.8590	(−0.96, 1.35)	1.3265	0.8586	(−0.39, 0.88)	0.668	0.5606	(−0.36, 0.92)	1.5418	0.9536	(0.69, 1.10)
μ_{12}	1.0811	0.7298	(−0.50, 1.77)	1.0806	0.7303	(−0.19, 1.11)	0.6894	0.6111	(−0.24, 1.05)	1.2687	0.8506	(0.68, 1.25)
μ_{13}	0.8874	0.6570	(−0.19, 1.96)	0.8855	0.6563	(0.15, 1.38)	0.6693	0.6312	(0.15, 1.37)	1.0708	0.7751	(0.72, 1.46)
μ_{21}	1.3473	0.8623	(−1.50, 2.40)	1.3446	0.8619	(−0.55, 1.23)	0.6657	0.6412	(−0.61, 1.16)	2.0619	1.1235	(0.85, 1.50)
μ_{22}	1.1751	0.7901	(−1.26, 2.83)	1.1726	0.7898	(−0.37, 1.36)	0.7191	0.6972	(−0.42, 1.32)	2.0282	1.1248	(0.85, 1.61)
μ_{23}	0.9507	0.7121	(−1.11, 2.74)	0.9519	0.7129	(−0.27, 1.56)	0.7409	0.7291	(−0.27, 1.55)	1.7421	1.0417	(0.86, 1.77)
λ_{11}	0.9550	0.6284	(0.34, 0.69)	0.9539	0.6279	(0.35, 0.70)	0.8513	0.5862	(0.35, 0.70)	0.9556	0.6288	(0.44, 0.74)
λ_{12}	0.6456	0.6073	(0.79, 1.15)	0.6462	0.6076	(0.82, 1.18)	0.6393	0.6244	(0.80, 1.16)	0.6459	0.6079	(0.87, 1.21)
λ_{13}	0.8131	0.8296	(1.33, 1.70)	0.8134	0.8297	(1.32, 1.69)	0.8431	0.8479	(1.32, 1.69)	0.8110	0.8284	(1.36, 1.72)
λ_{21}	1.5128	1.1381	(1.68, 2.29)	1.5130	1.1382	(1.67, 2.29)	1.5955	1.1668	(1.68, 2.29)	1.5127	1.1379	(1.73, 2.34)
λ_{22}	2.6668	1.4930	(2.18, 2.85)	2.6665	1.493	(2.14, 2.81)	2.7647	1.5198	(2.16, 2.82)	2.6646	1.4922	(2.20, 2.86)
λ_{23}	4.4090	1.9465	(2.65, 3.32)	4.4097	1.9466	(2.67, 3.34)	4.5226	1.9736	(2.66, 3.33)	4.4012	1.9436	(2.71, 3.38)
σ_1	0.8160	0.5717	(0.40, 0.62)	0.8318	0.5762	(0.52, 0.77)	0.8077	0.5695	(0.49, 0.73)	0.9363	0.6144	(0.53, 0.78)
σ_2	0.6540	0.6770	(0.71, 1.24)	0.6481	0.6635	(0.90, 1.45)	0.6586	0.6855	(0.88, 1.43)	0.6501	0.6043	(0.91, 1.47)

Table 6 Model Comparison using POR, AIC and BIC

Method	OLS	SELF	ALF	PLF
ρ	0.8716	0.9541	0.9449	0.9548
POR	0.1506	0.0552	0.0455	0.0545
AIC	1638.4458	663.1125	682.6000	646.5079
BIC	1700.5195	725.1862	744.6737	708.5816

Table 7 Estimates and confidence interval for parameters

P	OLS	SELF	ALF	PLF	CI
ρ	0.8716	0.9541	0.9449	0.9548	(0.88, 1.00)
μ_{11}	1.0393	0.8514	0.8467	0.9744	(0.00, 1.85)
μ_{12}	0.2192	0.2811	0.2712	0.5569	(−0.67, 1.17)
μ_{13}	−0.0034	0.1081	0.0981	0.4830	(−0.72, 1.11)
μ_{21}	1.7401	1.5581	1.5710	1.6708	(0.45, 2.78)
μ_{22}	0.5466	0.5892	0.5785	0.8300	(−0.51, 1.80)
μ_{23}	−0.1666	0.2438	0.2371	0.6613	(−0.92, 1.40)
λ_{11}	0.0421	0.2799	0.2623	0.7353	(−0.98, 1.70)
λ_{12}	0.3617	0.2829	0.2614	0.9246	(−1.38, 2.12)
λ_{13}	0.1664	0.0255	0.0288	0.6496	(−1.26, 1.24)
λ_{21}	0.0554	0.0881	0.0817	0.5464	(−1.08, 1.08)
λ_{22}	0.1419	0.0958	0.1139	0.6450	(−1.15, 1.33)
λ_{23}	0.2464	0.0250	0.0206	0.6174	(−1.18, 1.20)
σ_1	0.0013	0.0474	0.0447	0.0495	(0.03, 0.07)
σ_2	0.0026	0.1184	0.1082	0.1290	(0.04, 0.22)

Acknowledgements

The authors would like to thank the editor and two anonymous referees for their constructive suggestions that greatly improved the last version of this manuscript.

References

Ali S, Aslam M, Kazmi SM. A study of the effect of the loss function on Bayes Estimate, posterior risk and hazard function for Lindley distribution. Appl Math Model. 2013; 37(8): 6068-6078.

Berger JO. Statistical decision theory and Bayesian analysis. New York: Springer Science & Business Media; 2013.

Chang Y, Sickles R. C, Song W. Bootstrapping unit root tests with covariates. Economet Rev. 2017; 36(1-3): 136-155.

Chaturvedi A, Kumar J. Bayesian unit root test for time series models with structural breaks. Am J Math Manage Sci. 2007; 27(1-2): 243-268.

Chaturvedi A, Kumar J, Kumar A. Testing for unit root in the presence of stationary covariate: a Bayesian approach. Int J Intell Tech Appl Stat. 2017; 10(1): 1-8.

Chen CWS, Gerlach R, Liu FC. Detection of structural breaks in a time-varying heteroskedastic regression model, J Stat Plan Infer. 2011; 141, 3367-3381.

Chen CWS, Lee S, Chen SY. Local non-stationarity test in mean for Markov switching GARCH models: an approximate Bayesian approach, Comput Stat. 2015; 31, 1-24.

Chib S. Estimation and comparison of multiple change-point models, J Econom. 1998; 86: 221-241.

Costantini M, LupiC. A simple panel-CADF test for unit roots. Oxf Bull Econ Stat. 2013; 75(2): 276-296.

Elliott G, Jansson M. Testing for unit roots with stationary covariates. J Econom. 2003; 115(1): 75-89.

Fossati S. Unit root testing with stationary covariates and a structural break in the trend function. J Time Ser Anal. 2013; 34(3): 368-384.

Geman S. Gibbs distribution, and the Bayesian restoration of images. IEEE Proc. Pattern Analysis and Machine Intelligence. 1984; 774-778.

Hansen BE. Rethinking the univariate approach to unit root testing: Using covariates to increase power. Econom Theory. 1995; 11(5): 1148-1171.

Hastings WK. Monte Carlo sampling methods using Markov chains and their applications. Biometrika. 1970; 57(1): 97-109.

Kleiber C, Hornik K, Leisch F, Zeileis A. Strucchange: An R package for testing for structural change in linear regression models. J Stat Softw. 2002; 7(2): 1-38.

Kumar J, Kumar A, Agiwal V, Shangodoyin DK. Bayesian inference of C-AR(1) time series model with structural break. Afr Stat. 2017; 12(3): 1467-1479.

Kumar J, Kumar A, Agiwal V. Bayesian unit root test for panel AR(1) model with stationary covariate. Int J StatEcon. 2018; 19(2): 48-60.

Meligkotsidou L, Tzavalis E, Vrontos I. On Bayesian analysis and unit root testing for autoregressive models in the presence of multiple structural breaks. Econ Stat. 2017; 4: 70-90.

Metropolis N, Rosenbluth AW, Rosenbluth MN, Teller AH, Teller E. Equation of state calculations by fast computing machines. J Chem Phys. 1953; 21(6): 1087-1092.

Newbold P, Leybourne S, Wohar ME. Trend-stationarity, difference-stationarity, or neither: further diagnostic tests with an application to US Real GNP, 1875-1993. J Econ Bus. 2001; 53(1): 85-102.

Priestley MB. Spectral analysis and time series. New York: Academic Press; 1981.

Sims CA, Uhlig H. Understanding unit rooters: a helicopter tour. Econometrica. 1991; 59(6): 1591-1599.

Sims CA. Bayesian skepticism on unit root econometrics. J Econ Dyn Control. 1988; 12(2-3): 463-474.

Wang J, Zivot E. A Bayesian time series model of multiple structural changes in level, trend, and variance. J Bus Econ Stat. 2000; 18(3): 374-386.

Appendix

In this appendix, derived the posterior probability with the help of likelihood function and prior distribution which is as follows:

Likelihood function under unit root hypothesis

$$P(Y \mid \Lambda_1, \Lambda_2, \sigma_1, \sigma_2) = \left(\frac{1}{2\pi\sigma_1^2}\right)^{nT_B/2} \exp\left[-\frac{1}{2\sigma_1^2}\left\{\left(\Delta Y_{T_B} - W_1\Lambda_1\right)'\left(\Delta Y_{T_B} - W_1\Lambda_1\right)\right\}\right]\times \left(\frac{1}{2\pi\sigma_2^2}\right)^{n(T-T_B)/2} \exp\left[-\frac{1}{2\sigma_2^2}\left\{\left(\Delta Y_{T-T_B} - W_2\Lambda_2\right)'\left(\Delta Y_{T-T_B} - W_2\Lambda_2\right)\right\}\right]. \tag{A1}$$

Posterior distribution under H_0

$$P(Y \mid H_0)=\int_0^\infty\int_0^\infty\int_{R^{p_2+r_2}}\int_{R^{p_1+r_1}}\left(\frac{b_1^{a_1}b_2^{a_2}\left(\sigma_1^2\right)^{-a_1-1}\left(\sigma_2^2\right)^{-a_2-1}}{(2\pi)^{nT/2}\Gamma(a_1)\Gamma(a_2)}\right)\exp\left[-\frac{1}{2\sigma_1^2}\left\{\left(\Delta Y_{T_B}-W_1\Lambda_1\right)'\left(\Delta Y_{T_B}-W_1\Lambda_1\right)\right\}\right]\times$$
$$\exp\left[-\frac{1}{2\sigma_2^2}\left\{\left(\Delta Y_{T-T_B}-W_2\Lambda_2\right)'\left(\Delta Y_{T-T_B}-W_2\Lambda_2\right)\right\}\right]\exp\left(-\frac{b_1}{\sigma_1^2}\right)\exp\left(-\frac{b_2}{\sigma_1^2}\right)d\Lambda_1 d\Lambda_2 d\sigma_1 d\sigma_2. \tag{A2}$$

Solving equation (A2) for $\Lambda_1,\Lambda_2,\sigma_1$ and σ_2 by supposing
$\hat{\Lambda}_1=W_1'\left(W_1'W_1\right)^{-1}\Delta Y_{T_B}$ and $\hat{\Lambda}_2=W_2'\left(W_2'W_2\right)^{-1}\Delta Y_{T-T_B}$. We will get required solution

$$P(Y \mid H_0)$$
$$=\frac{2^{a_1+a_2}b_1^{a_1}b_2^{a_2}\Gamma\left(nT_B/2-(p_1+r_1)/2+a_1\right)\Gamma\left(n(T-T_B)/2-(p_2+r_2)/2+a_2\right)|W_1'W_1|^{-1/2}|W_2'W_2|^{-1/2}}{\Gamma(a_1)\Gamma(a_2)\pi^{nT/2-(p_1+r_1)/2-(p_2+r_2)/2}\left[\Delta Y_{T_B}'\Sigma_1\Delta Y_{T_B}+2b_1\right]^{nT_B/2-(p_1+r_1)/2+a_1}\left[\Delta Y_{T-T_B}'\Sigma_2\Delta Y_{T-T_B}+2b_2\right]^{n(T-T_B)/2-(p_2+r_2)/2+a_2}}. \tag{A3}$$

Likelihood function under alternative hypothesis

$$P(Y \mid \rho,\sigma_1,\sigma_2,M_1,M_2,\Lambda_1,\Lambda_2,)$$
$$=\left(\frac{1}{2\pi\sigma_1^2}\right)^{nT_B/2}\exp\left[-\frac{1}{2\sigma_1^2}\left\{\begin{matrix}\left(Y_{T_B}-\rho Y_{-1}^{T_B}-(1-\rho)M_1\left(l_{T_\mathbf{B}}\otimes I_n\right)-W_1\Lambda_1\right)'\\ \left(Y_{T_B}-\rho Y_{-1}^{T_B}-(1-\rho)M_1\left(l_{T_\mathbf{B}}\otimes I_n\right)-W_1\Lambda_1\right)\end{matrix}\right\}\right]\times$$
$$\left(\frac{1}{2\pi\sigma_2^2}\right)^{n(T-T_B)/2}\exp\left[-\frac{1}{2\sigma_2^2}\left\{\begin{matrix}\left(Y_{T-T_B}-\rho Y_{-1}^{T-T_B}-(1-\rho)M_2\left(l_{T-T_\mathbf{B}}\otimes I_n\right)-W_2\Lambda_2\right)'\\ \left(Y_{T-T_B}-\rho Y_{-1}^{T-T_B}-(1-\rho)M_2\left(l_{T-T_\mathbf{B}}\otimes I_n\right)-W_2\Lambda_2\right)\end{matrix}\right\}\right]. \tag{A4}$$

Posterior distribution under H_1

$$P(Y \mid H_1)=\frac{b_1^{a_1}b_2^{a_2}}{(2\pi)^{T/2+1}\Gamma(a_1)\Gamma(a_2)}\int_0^1\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{R^{p_2+r_2}}\int_{R^{p_1+r_1}}\frac{\left(\sigma_1^2\right)^{-nT_B/2-a_1-1/2}\left(\sigma_2^2\right)^{-n(T-T_B)/2-a_2-1/2}}{1-a}\times$$
$$\exp\left[-\frac{1}{2\sigma_1^2}\left\{\left(Y_{T_B}-\rho Y_{-1}^{T_B}-(1-\rho)M_1\left(l_{T_B}\otimes I_n\right)-W_1\Lambda_1\right)'\left(Y_{T_B}-\rho Y_{-1}^{T_B}-(1-\rho)M_1\left(l_{T_B}\otimes I_n\right)-W_1\Lambda_1\right)\right\}\right]\times$$
$$\exp\left[-\frac{1}{2\sigma_2^2}\left\{\left(Y_{T-T_B}-\rho Y_{-1}^{T-T_B}-(1-\rho)M_2\left(l_{T-T_B}\otimes I_n\right)-W_2\Lambda_2\right)'\left(Y_{T-T_B}-\rho Y_{-1}^{T-T_B}-(1-\rho)M_2\left(l_{T-T_B}\otimes I_n\right)-W_2\Lambda_2\right)\right\}\right]$$
$$\exp\left(-\frac{b_1}{\sigma_1^2}\right)\exp\left(-\frac{b_2}{\sigma_1^2}\right)\exp\left(-\frac{1}{2\sigma_1^2}\left(M_1-\mu_1'\right)^2\right)\exp\left(-\frac{1}{2\sigma_2^2}\left(M_2-\mu_2'\right)^2\right)d\Lambda_1 d\Lambda_2 dM_1 dM_2 d\sigma_1 d\sigma_2 d\rho. \tag{A5}$$

Solving equation (A5) for $\Lambda_1,\Lambda_2,M_1,M_2,\sigma_1,\sigma_2$ and ρ by supposing

$$\tilde{\Lambda}_1=W_1'\left(W_1'W_1\right)^{-1}\left(Y_{T_B}-\rho Y_{-1}^{T_B}-(1-\rho)M_1\left(l_{T_B}\otimes I_n\right)\right),$$
$$\tilde{\Lambda}_2=W_2'\left(W_2'W_2\right)^{-1}\left(Y_{T-T_B}-\rho Y_{-1}^{T-T_B}-(1-\rho)M_2\left(l_{T_B}\otimes I_n\right)\right),$$

$$\tilde{M}_1 = \left[\left(l_{T_B} \otimes I_n\right)(1-\rho)\Sigma_1\left(Y_{T_B} - \rho Y_{-1}^{T_B}\right) + \mu_1'\right]\left[V(1-\rho)^2\Sigma_1\left(l_{T_B} \otimes I_n\right)+1\right]^{-1} \text{ and}$$

$$\tilde{M}_2 = \left[\left(l_{T_B} \otimes I_n\right)(1-\rho)\Sigma_2\left(Y_{T-T_B} - \rho Y_{-1}^{T-T_B}\right) + \mu_2'\right]\left[\left(l_{T-T_B} \otimes I_n\right)(1-\rho)^2\Sigma_2\left(l_{T-T_B} \otimes I_n\right)+1\right]^{-1}.$$

We will get required solution

$$P\left(Y \mid H_1\right) = \frac{2^{a_1+a_2} b_1^{a_1} b_2^{a_2} \Gamma\left(nT_B/2-(p_1+r_1)/2+a_1\right)\Gamma\left(n(T-T_B)/2-(p_2+r_2)/2+a_2\right)|W_1'W_1|^{-1/2}|W_2'W_2|^{-1/2}}{\Gamma(a_1)\Gamma(a_2)\pi^{T/2-(p_1+r_1)/2-(p_2+r_2)/2}} \times$$

$$\int_a^1 \frac{1}{(1-a)A(\rho)^{nT_B/2-(p_1+r_1)/2+a_1} B(\rho)^{n(T-T_B)/2-(p_2+r_2)/2+a_2} \left|\left(l_{T_B} \otimes I_n\right)(1-\rho)^2\Sigma_1\left(l_{T_B} \otimes I_n\right)+1\right|^{1/2}\left|\left(l_{T-T_B} \otimes I_n\right)(1-\rho)^2\Sigma_2\left(l_{T-T_B} \otimes I_n\right)+1\right|^{1/2}} d\rho.$$