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A New Generalization of Power Function Distribution: Properties and Estimation Based on Censored Samples

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Abstract

A new four-parameter power function distribution, named as exponentiated generalized power function (EGPF) is proposed. Some of its statistical properties are obtained including moments, probability weighted moments, incomplete moments and Rényi entropy measure. The estimation of the model parameters is performed based on type II censored samples. The maximum likelihood estimators are developed for estimating the model parameters. Asymptotic confidence interval estimators of the model parameters are developed. Simulation procedure and real data are performed to illustrate the theoretical purposes.

Keywords: Exponentiated generalized power function distribution, moments, order statistics, maximum likelihood estimation.

1. Introduction

In statistical literature, many of probability distributions are used for modeling business failure data and widely applied in a variety of contexts studies. Some particular distributions like, Weibull, gamma and lognormal occupy a central role because of their demonstrated advantages in variety of situations. Power function (PF) is a flexible life time distribution that arises in several scientific fields. The PF distribution is a special model from the uniform distribution. The PF is the inverse of Pareto distribution (see Dallas 1976). Estimation of the PF parameters has been done by various authors, for instance; Saran and Pandey (2004), Zaka and Akhter (2013), Zarrin et al. (2013), Naveed-Shahzad et al. (2015) and Hanif et al. (2015).

The probability density function (pdf) and cumulative distribution function (cdf) of two-parameter PF distribution with scale parameter λ and shape parameter θ are, respectively, given by

$$g(x;\lambda,\theta)=\frac{\theta x^{\theta-1}}{\lambda^{\theta}},\quad 0<x<\lambda,\ \theta>0,$$

and

$$G(x;\lambda,\theta)=\left(\frac{x}{\lambda}\right)^{\theta}. \tag{1}$$

Extensions of the PF distribution have been extensively studied by some authors. For instance; Meniconi and Barry (1996) proposed the two-parameter PF distribution as a simple alternative to the exponential distribution when it comes for modelling failure data related to electrical components. Cordeiro and Brito (2012) proposed the beta PF distribution. The Weibull PF distribution was suggested by Tahir et al. (2016). Oguntunde et al. (2015) suggested the Kumaraswamy PF distribution. Haq et al. (2016) introduced the transmuted PF distribution. The exponentiated Kumaraswamy PF distribution was presented by Bursa and Kadilar (2017). Okorie et al. (2017) proposed the modified PF distribution. Hassan and Assar (2017) proposed exponentiated Weibull PF distribution. Hassan et al. (2019) introduced the odd generalized exponential PF distribution.

Many classical distributions have been extensively used for modeling real data in many areas. However, in many situations; there is a clear need for extended forms of these distributions in order to improve the flexibility and goodness of fit of these distributions. For that reason, generated families of continuous distributions are development by introducing one or more additional shape parameter(s) to the baseline distribution. We present a list of some generated families as follows; the beta-G by Eugene et al. (2002) and Jones (2004), gamma-G (Type I) by Zografos and Balakrishanan (2009), Kumaraswamy-G by Cordeiro and Castro (2011), McDonald-G by Alexander et al. (2012), gamma-G (Type II) by Ristić and Balakrishnan (2012), transformed-transformer by Alzaatreh et al. (2013), exponentiated generalized (EG) by Cordeiro et al. (2013), Weibull-G by Bourguignon et al. (2014), Kumaraswamy Weibull-G by Hassan and Elgarhy (2016a), exponentiated Weibull-G by Hassan and Elgarhy (2016b), additive Weibull-G by Hassan and Hemeda (2016) and type II half logistic-G by Hassan et al. (2017), inverse Weibull-G by Hassan and Nassr (2018) and power Lindley-G by Hassan and Nassr (2019), among others.

The cdf of the EG class of distributions (see Cordeiro et al. 2013), is defined as

$$F(x;\alpha,\beta,\varsigma)=\left\{1-\left(1-G(x;\varsigma)\right)^{\alpha}\right\}^{\beta},\quad x>0\ ,\ \alpha,\beta>0, \tag{2}$$

where α and β are the two shape parameters and $G(x;\varsigma)$ is the cdf of any distribution. The associated pdf is given by

$$f(x;\alpha,\beta,\varsigma)=\alpha\beta g(x;\varsigma)\left[1-G(x;\varsigma)\right]^{\alpha-1}\left\{1-\left(1-G(x;\varsigma)\right)^{\alpha}\right\}^{\beta-1},x>0\ ,\ \alpha,\beta>0. \tag{3}$$

Special sub-models can be obtained from (3) as follows; For $\alpha=1,$ the pdf (3) gives Lehmann type I class. For $\beta=1,$ the pdf (3) gives the Lehmann type II class. For $\alpha=1$ and $\beta=1,$ the pdf (3) gives the baseline distribution $G(x;\varsigma).$ According to Cordeiro et al. (2013) the class of the EG distributions shares an attractive physical interpretation whenever α and β are positive integers. Consider a device made of independent components in a parallel system with each component is made of independent subcomponents identically distributed according to $G(x;\varsigma)$ in a series system. The device fails if all components fail and each component fails if any subcomponent fails. Let $X_{k1},\ldots,X_{ka}$ denote the lifetimes of the subcomponents within the k^{th}

component, $k=1,\ldots,\beta$ with common cdf $G(x;\varsigma)$. Let X_k denote the lifetime of the k^{th} component and let X denote the lifetime of the device. Therefore, the cdf of X is as follows

$$P(X\le x)=P(X_1\le x,...,X_\beta\le x)=P(X_1\le x)^\beta=\left[1-P(X_1>x)\right]^\beta$$
$$=[1-P(X_{11}>x,...,X_{1\alpha}>x)]^\beta=[1-P(X_{11}>x)^\alpha]^\beta$$
$$=[1-(1-P(X_{11}<x)^\alpha]^\beta,$$

and the lifetime of the device obeys the EG class. Further, the EG family allows for greater flexibility of its tails and can be widely applied to many areas of engineering and biology. So, the present work provides new generalizations of the PF distribution based on the EG family with more flexibility than the baseline (2). The generalization of classical distributions sometimes provides reasonable parametric fits to particular applications as in lifetimes and reliability studies. The new distribution has more sub-models when compared with baseline distribution and hence it allows us to study more comprehensive structural properties. Further, we consider the parameter estimation for EGPF distribution when the available data are of type II censoring (TIIC). Further, approximate confidence intervals (CIs) of the unknown parameters based on *s*-normal approximation are constructed. The rest of the paper contains the following sections. The formation of the pdf and cdf of the EGPF distribution is performed in Section 2. Some statistical properties of EGPF distribution are provided in Section 3. Estimation of parameters, simulation study and application to real data are presented in Section 4. The article ends with concluding remarks.

2. The EGPF Distribution

In this section, we introduce a four-parameter EGPF distribution. The cdf of the EGPF distribution is obtained by inserting cdf (1) in cdf (2) as follows

$$F(x;\Psi)=\left\{1-\left(1-\left(\frac{x}{\lambda}\right)^\theta\right)^\alpha\right\}^\beta,\quad \alpha,\beta,\lambda,\theta>0,\quad 0<x<\lambda,$$

where $\Psi\equiv(\alpha,\beta,\lambda,\theta)$. The corresponding pdf of the EGPF distribution is written as follows

$$f(x;\Psi)=\frac{\theta\alpha\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}\left[1-\left(\frac{x}{\lambda}\right)^\theta\right]^{\alpha-1}\left[1-\left(1-\left(\frac{x}{\lambda}\right)^\theta\right)^\alpha\right]^{\beta-1},\quad 0<x<\lambda;\ \alpha,\theta,\beta>0. \tag{4}$$

$X\sim\text{EGPF}(\alpha,\beta,\lambda,\theta)$ denotes a random variable with pdf (4). Some sub-models can be obtained from (4) as follows

- For $\beta=1,$ the pdf of the EGPF distribution reduces to exponentiated PF distribution.
- For $\beta=1$ and $\lambda=1,$ the pdf of the EGPF distribution reduces to exponentiated standard PF distribution.
- For $\alpha=1,$ the pdf of the EGPF distribution reduces to generalized PF distribution.

- For $\alpha = 1$ and $\lambda = 1,$ the pdf of the EGPF distribution reduces to generalized standard PF distribution.
- For $\alpha = 1, \beta = 1,$ the pdf of the EGPF distribution provides PF distribution.

Figure 1 displays a variety of possible shapes of the pdf of the EGPF distribution for some selected values of parameters.

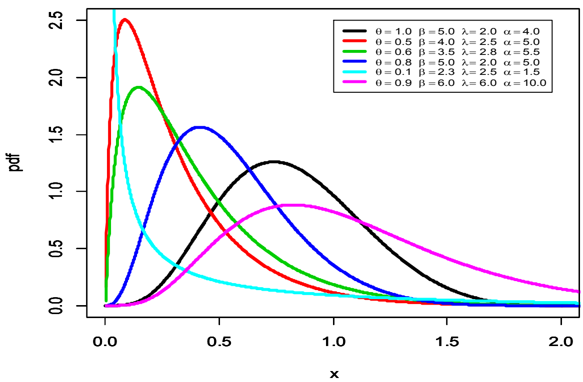

Figure 1 Plots of the EGPF density function for different values of α, β, λ and θ

Clearly, the EGPF densities take various shapes such as, uni-model, left skewed, reversed J shaped, U shaped and right skewed.

The survival and hazard rate function (hrf) of the EGPF distribution are respectively given by

$$\bar{F}(x;\Psi) = 1 - \left\{1 - \left(1 - \left(\frac{x}{\lambda}\right)^{\theta}\right)^{\alpha}\right\}^{\beta},$$

and

$$\varepsilon(x;\Psi) = \frac{\theta\alpha\beta x^{\theta-1}\left[1-\left(\frac{x}{\lambda}\right)^{\theta}\right]^{\alpha-1}\left\{1-\left(1-\left(\frac{x}{\lambda}\right)^{\theta}\right)^{\alpha}\right\}^{\beta-1}}{\lambda^{\theta}\left\{1-\left[1-\left(1-\left(\frac{x}{\lambda}\right)^{\theta}\right)^{\alpha}\right]^{\beta}\right\}}.$$

Figure 2 displays a variety of possible shapes of the hrf of the EGPF distribution for some selected values of parameters.

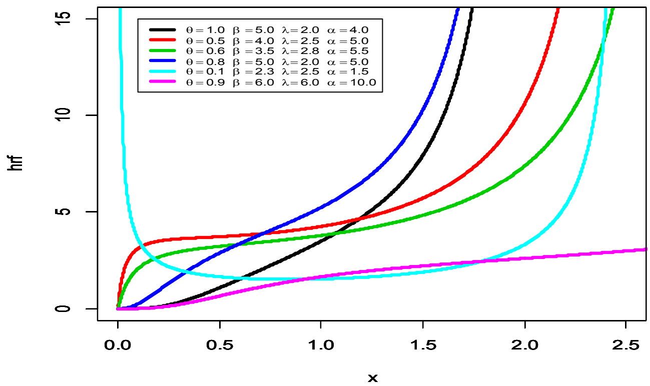

Figure 2 Plots of the EGPF hrf for different values of α, β, λ and θ

As seen from Figure 2 the hrfs of the EGPF distribution can have increasing, decreasing, and U-shaped. This fact implies that the EGPF can be very useful for modeling different types of data.

2.1. Shapes and asymptotics

To characterize the shape of the EGPF distribution, we start by obtaining the first derivative of its pdf

$$\frac{d\log f(x;\Psi)}{dx}=\frac{\theta-1}{x}-\frac{(\alpha-1)\theta x^{\theta-1}}{\lambda^{\theta}-x^{\theta}}+\frac{\alpha(\beta-1)\theta x^{\theta-1}(\lambda^{\theta}-x^{\theta})^{\alpha-1}}{\lambda^{\theta\alpha}-(\lambda^{\theta}-x^{\theta})^{\alpha}}.$$

The critical points of the density of $f(x)$ are the roots of the previous equation. It is often difficult to obtain an analytical solution for the critical points of this function. Therefore, it is common to obtain numerical solutions with high accuracy through optimization routines in most mathematical and statistical platforms.

The first derivative of the hrf of X is given by

$$\frac{d\log \varepsilon(x;\Psi)}{dx}=\frac{\theta-1}{x}-\frac{(\alpha-1)\theta x^{\theta-1}}{\lambda^{\theta}-x^{\theta}}+\frac{\alpha(\beta-1)\theta x^{\theta-1}(\lambda^{\theta}-x^{\theta})^{\alpha-1}}{\lambda^{\theta\alpha}-(\lambda^{\theta}-x^{\theta})^{\alpha}}+\frac{\alpha\beta\theta x^{\theta-1}(\lambda^{\theta}-x^{\theta})^{\alpha-1}\left[\lambda^{\theta\alpha}-(\lambda^{\theta}-x^{\theta})^{\alpha}\right]^{\beta-1}}{\lambda^{\theta\alpha\beta}-[\lambda^{\theta\alpha}-(\lambda^{\theta}-x^{\theta})^{\alpha}]^{\beta}}.$$

The critical values of $\varepsilon(x;\psi)$ are the roots of the previous equation.

2.2. Limiting behavior of EGPF density and hazard functions

Lemma 1. *The limit of the EGPF density function as* $x\to\infty$ *is* ∞ *and the limit as* $x\to\lambda$ *are*

$$\lim_{x\to\lambda} f(x;\Psi)=\begin{cases}0 & \text{for } 0<\alpha<1,\\ \dfrac{\theta\beta}{\lambda} & \text{for } \alpha=1,\\ \infty & \text{for } \alpha>1.\end{cases}$$

Proof: It is easy to demonstrate the result from the density function (4).

Lemma 2. *The limit of the EGPF hazard function as* $x \to \infty$ *is* ∞ *and the limit as* $x \to \lambda$ *are*

$$\lim_{x\to\lambda} \varepsilon(x;\Psi) = \begin{cases} 0 & \text{for } 0<\alpha<1, \\ \dfrac{\theta\beta}{\lambda} & \text{for } \alpha=1, \\ \infty & \text{for } \alpha>1. \end{cases}$$

Proof: It is straightforward to prove this result.

3. Statistical Properties

In this section, some important properties of the EGPF distribution are provided, specifically the r^{th} moment, the moment-generating function, probability weighted moments, incomplete moments and Rényi entropy.

3.1. Moments

Moments are very necessary and significant in any statistical analysis, especially for application studies. So, we concern here with the r^{th} moment and the moment generating function (MGF) of the EGPF distribution.

To obtain the r^{th} moment of the EGPF, we firstly obtain a simplified form of the pdf (4), since the generalized binomial theorem, for $b>0$ is real non integer and $|z|<1$,

$$(1-z)^{b-1} = \sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} z^i. \tag{5}$$

Then, by applying the binomial theorem (5) in (4), the pdf of the EGPF distribution where β is real non integer becomes

$$f(x;\Psi) = \sum_{j=0}^{\infty}(-1)^j \binom{\beta-1}{j} \frac{\theta\alpha\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}\left[1-\left(\frac{x}{\lambda}\right)^{\theta}\right]^{\alpha(j+1)-1}.$$

Employing the generalized binomial (5) another time, then the pdf of the EGPF distribution takes the following form

$$\left.\begin{aligned} f(x;\Psi) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} W_{i,j}\, g_{\theta(i+1)}(x), \\ W_{i,j} &= (-1)^{j+i}\binom{\beta-1}{j}\binom{\alpha(j+1)-1}{i}\frac{\alpha\beta}{(i+1)}, \end{aligned}\right\} \tag{6}$$

where $g_{\theta(i+1)}(x)$ denotes the pdf of the PF distribution with parameters $\theta(i+1)$ and λ. Hence the r^{th} moment of EGPF distribution is given by

$$\mu_r' = E(X^r) = \int_0^{\lambda} x^r \sum_{i,j=0}^{\infty} W_{i,j}\, g_{\theta(i+1)}(x)dx = \sum_{i,j=0}^{\infty} W_{i,j}\, \frac{\lambda^r\left(\theta(i+1)\right)}{r+\theta+\theta i}, r=1,2,... \tag{7}$$

Setting $r=1,2,3,4$ in (7), we can obtain the first four moments about zero. Generally, the MGF of the EGPF distribution is obtained as follows

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r)=\sum_{i,j=0}^{\infty}W_{i,j}\,\frac{t^r}{r!}\frac{\lambda^r\left(\theta(i+1)\right)}{r+\theta+\theta i}.$$

The important feature and characterizations of the EGPF distribution can thus be studied using (7). The mean, variance, skewness and kurtosis measures may be calculated from ordinary moments using well-known relationships. Table 1 lists numerical values of mean and variance of the EGPF distribution for some selected values of the parameters. Table 2 contains the skewness (α_3) and kurtosis (α_4) of the EGPF for some selected values of the parameters.

Table 1 Mean and variance of the EGPF distribution for various values of α,β,λ and θ

β	α	λ	$\theta=0.25$		$\theta=1.5$		$\theta=3$		$\theta=5$	
			μ	σ^2	μ	σ^2	μ	σ^2	μ	σ^2
0.25	0.5	0.5	0.066	0.018	0.180	0.029	0.251	0.027	0.308	0.021
		1.5	0.198	0.162	0.539	0.263	0.754	0.240	0.923	0.189
		3.0	0.395	0.647	1.078	1.053	1.507	0.961	1.846	0.758
	2.5	0.5	5.800*	0.690*	0.087	0.011	0.167	0.015	0.238	0.015
		1.5	0.017	6.211*	0.260	0.095	0.502	0.138	0.714	0.137
		3.0	0.035	0.025	0.520	0.378	1.003	0.553	1.427	0.548
	5.0	0.5	1.049*	0.048*	0.058	5.303*	0.136	0.011	0.209	0.012
		1.5	3.146*	0.432*	0.175	0.048	0.407	0.096	0.628	0.110
		3.0	6.293*	1.729*	0.350	0.191	0.814	0.386	1.256	0.440
2	0.5	0.5	0.306	0.028	0.439	5.914*	0.466	2.143*	0.479	0.915*
		1.5	0.919	0.253	1.318	0.053	1.399	0.019	1.437	8.230*
		3.0	1.838	1.013	2.635	0.213	2.798	0.077	2.873	0.033
	2.5	0.5	0.039	3.942*	0.265	9.171*	0.357	4.924*	0.406	2.496*
		1.5	0.116	0.035	0.794	0.083	1.071	0.044	1.219	0.022
		3.0	0.232	0.142	1.589	0.330	2.142	0.177	2.439	0.090
	5.0	0.5	7.437*	0.328*	0.186	6.075*	0.298	4.407*	0.364	2.537*
		1.5	0.022	2.947*	0.559	0.055	0.894	0.040	1.092	0.023
		3.0	0.045	0.012	1.118	0.219	1.787	0.159	2.185	0.091
5	0.5	0.5	0.425	9.610*	0.484	7.153*	0.492	0.205*	0.495	0.078*
		1.5	1.274	0.086	1.451	6.438*	1.475	1.847*	1.485	0.705*
		3.0	2.548	0.346	2.902	0.026	2.949	7.388*	2.969	2.820*
	2.5	0.5	0.077	6.461*	0.335	5.103*	0.407	2.051*	0.441	0.903*
		1.5	0.231	0.058	1.005	0.046	1.220	0.018	1.323	8.127*
		3.0	0.462	0.233	2.010	0.184	2.440	0.074	2.646	0.033
	5.0	0.5	0.016	0.665*	0.244	4.296*	0.346	2.292*	0.400	1.135*
		1.5	0.048	5.988*	0.731	0.039	1.037	0.021	1.199	0.010
		3.0	0.096	0.024	1.462	0.155	2.075	0.083	2.399	0.041

Note: * Indicate that the value multiply 10^{-3}

Table 2 Skewness and kurtosis of the EGPF distribution for various values of α, β, λ and θ

β	α	$\theta=0.5$		$\theta=3$		$\theta=5$		$\theta=7.5$	
		α_3	α_4	α_3	α_4	α_3	α_4	α_3	α_4
0.5	0.5	0.647	1.884	−0.678	2.330	−1.042	3.299	−1.835	7.061
	2.5	2.729	11.275	−5.794*	2.096	−0.449	2.504	−0.747	3.126
	5.0	4.061	24.989	0.155	2.249	−0.332	2.474	−0.633	3.009
	7.5	4.907	37.664	0.219	2.335	−0.274	2.477	−0.591	2.976
2.0	0.5	−0.979	2.830	−2.291	9.388	−2.558	11.601	−2.720	13.122
	2.5	1.184	3.922	−0.537	2.941	−0.782	3.562	−0.923	4.028
	5.0	2.130	8.849	−0.228	2.736	−0.494	3.143	−0.644	3.490
	7.5	2.687	13.425	−0.112	2.739	−0.390	3.059	−0.545	3.359
5.0	0.5	−2.371	9.477	−3.520	21.191	−3.682	23.478	−3.770	24.810
	2.5	0.591	2.725	−0.601	3.203	−0.740	3.573	−0.816	3.837
	5.0	1.454	5.604	−0.209	2.846	−0.372	3.056	−0.457	3.205
	7.5	1.928	8.391	−0.068	2.843	−0.243	2.983	−0.334	3.099
8.0	0.5	−3.212	16.899	−4.145	29.879	−4.251	30.073	−4.295	18.596
	2.5	0.377	2.528	−0.591	3.222	−0.695	3.497	−0.750	3.672
	5.0	1.220	4.775	−0.171	2.846	−0.299	2.984	−0.366	3.079
	7.5	1.669	7.048	−0.021	2.857	−0.162	2.934	−0.234	3.001

Note: * Indicate that the value multiply 10^{-3}

3.2. The probability weighted moments

The probability weighted moments (PWM) of a random variable X, say $\tau_{s,r}$, is formally defined by

$$\tau_{r,s} = E[X^r F(x)^s] = \int_{-\infty}^{\infty} x^r f(x)(F(x))^s \, dx. \tag{8}$$

To obtain the PWM of the EGPF, we firstly obtain a simplified form of $\left[F(x;\Psi)\right]^s$ for s is a positive integer as follows

$$\left[F(x;\Psi)\right]^s = \sum_{k,m=0}^{\infty} (-1)^{k+m} \binom{\beta s}{k} \binom{\alpha k}{m} G_{\theta m}(x), \tag{9}$$

where, $G_{\theta m}(x)$ denotes the cdf of the PF distribution with parameters θm, and λ. Substituting pdf (6) and cdf (9), in (8) then the PWM of the EGPF is given by

$$\tau_{r,s} = \sum_{i,j,k,m=0}^{\infty} (-1)^{k+m} \binom{\beta s}{k} \binom{\alpha k}{m} W_{i,j} \int_0^{\lambda} \frac{\theta(i+1)}{\lambda} x^r \left(\frac{x}{\lambda}\right)^{\theta(i+m+1)-1} dx.$$

Hence, the PWM of the EGPF distribution takes the following form

$$\tau_{r,s} = \sum_{i,j,k,m=0}^{\infty} (-1)^{k+m} \binom{\beta s}{k} \binom{\alpha k}{m} \frac{W_{i,j} \ \lambda^r \theta(i+1)}{r+\theta(i+m+1)}.$$

3.3. Incomplete and conditional moments

The s^{th} incomplete moment of the EGPF distribution, say $K_s(t)$, is given by

$$K_s(t) = \int_0^t x^s f(x)\, dx.$$

Using (6), then $K_s(t)$ can be written as

$$K_s(t) = \sum_{i,j=0}^{\infty} W_{i,j} \frac{\theta(i+1)t^{\theta(i+1)+s}}{\lambda^{\theta(i+1)}\left(\theta(i+1)+s\right)}. \tag{10}$$

The first incomplete moments of the EGPF distribution is obtained by setting $s=1$ in (10). The mean deviations provide useful information about the characteristics of a population and it can be calculated from the first incomplete moments. Indeed, the amount of dispersion in a population may be measured to some extent by the totality of the deviations from the mean and median. The mean deviations of X about the mean μ and about the median m can be calculated from the following relations

$$\delta_1(X) = 2\mu F(\mu) - 2T(\mu)$$

and

$$\delta_2(X) = \mu - 2T(m),$$

where $T(p)$ is the first incomplete moments. An important application of the first incomplete moments is related to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The Lorenz and Bonferroni curves are obtained, respectively, as follows

$$L_F(x) = \frac{K_1(x)}{E(X)} = \frac{\sum_{i,j=0}^{\infty} W_{i,j} \frac{x^{\theta(i+1)+1}}{\lambda^{\theta(i+1)}} \frac{\theta(i+1)}{\theta(i+1)+1}}{\sum_{i,j=0}^{\infty} W_{i,j} \frac{\lambda(\theta(i+1))}{\theta+\theta i+1}},$$

and

$$B_F(x) = \frac{L_F(x)}{F(x)} = \frac{\sum_{i,j=0}^{\infty} W_{i,j} \frac{x^{\theta(i+1)+1}}{\lambda^{\theta(i+1)}} \frac{\theta(i+1)}{\theta(i+1)+1}}{\left[1-\left(1-\left(\frac{x}{\lambda}\right)^{\theta}\right)^{\alpha}\right]^{\beta} \sum_{i,j=0}^{\infty} W_{i,j} \frac{\lambda(\theta(i+1))}{\theta+\theta i+1}}.$$

Further, the conditional moments, say $\nu_s(t)$, is defined by

$$\nu_s(t) = \int_t^{\infty} x^s f(x)\, dx.$$

Hence, by using pdf (6), leads to

$$\nu_s(t) = \sum_{i,j=0}^{\infty} W_{i,j} \frac{\theta(i+1)}{\lambda^{\theta(i+1)}} \left[\frac{\lambda^{s+\theta(i+1)} - t^{s+\theta(i+1)}}{\left(s+\theta(i+1)\right)}\right].$$

3.4. Rényi entropy

The entropy of random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. It has been used in many fields such as physics, engineering and economics. The Rényi entropy of random variable X, is defined by

$$I_\rho(X)=\frac{1}{1-\rho}\log\int_{-\infty}^{\infty} f(x)^\rho\,dx,\quad \rho>0 \text{ and } \rho\neq 1.$$

The pdf, $f(x;\Psi)^\rho$, of the EGPF distribution after simplification can be expressed as follows

$$\left(f(x;\Psi)\right)^\rho=\sum_{j,k=0}^{\infty}\xi_{j,k}\left(\frac{x}{\lambda}\right)^{\rho(\theta-1)+\theta k},$$

where

$$\xi_{j,k}=(-1)^{j+k}\binom{\rho(\beta-1)}{j}\binom{\rho(\alpha-1)+\alpha j}{k}\left(\frac{\theta\alpha\beta}{\lambda}\right)^\rho.$$

Therefore, the Rényi entropy of the EGPF distribution is given by

$$I_\rho(X)=\frac{1}{1-\rho}\log\left[\int_0^\lambda\sum_{j,k=0}^{\infty}\xi_{j,k}\left(\frac{x}{\lambda}\right)^{\rho(\theta-1)+\theta k}dx\right]=\frac{1}{1-\rho}\log\left[\sum_{j,k=0}^{\infty}\xi_{j,k}\frac{\lambda}{\rho(\theta-1)+\theta k+1}\right].$$

3.5. Order statistics

Suppose that $X_{1:n}<X_{2:n}<\ldots<X_{n:n}$ be the order statistics (OS) of a random sample of size n from the EGPF distribution. The pdf of the h^{th} OS is defined by

$$f_{h:n}(x)=\frac{f(x)}{B(h,n-h+1)}\sum_{v=0}^{n-h}(-1)^v\binom{n-h}{v}F(x)^{h+v-1}, \tag{11}$$

where $B(.,.)$ is the beta function. Subsituting (6) and (9) in (11) by replacing s with $v+h-1$, leads to

$$\left.\begin{aligned} f_{h:n}(x)&=\frac{1}{B(h,n-h+1)}\sum_{v=0}^{n-h}\sum_{i,j,k,m=0}^{\infty}M_{i,j,k,m}\frac{\theta(i+1)}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta(m+i+1)-1},\\ M_{i,j,k,m}&=(-1)^{v+k+m}W_{i,j}\binom{n-h}{v}\binom{\beta(v+h-1)}{k}\binom{\alpha k}{m}.\end{aligned}\right\} \tag{12}$$

In particuler, the the pdf of the smallest and largest OSs are obtained by subsituting $h=1$ and $h=n$ in (12), respectively, as follows

$$f_{1:n}(x)=n\sum_{v=0}^{n-1}\sum_{i,j,k,,m=0}^{\infty}\eta_{v,i,j,k,m}\frac{\theta(i+1)}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta(m+i+1)-1},$$

$$\eta_{v,i,j,k,m}=(-1)^{v+k+m}W_{i,j}\binom{n-1}{v}\binom{\beta v}{k}\binom{\alpha k}{m},$$

and

$$f_{n:n}(x)=n\sum_{i,j,k,,m=0}^{\infty}E_{v,i,j,k,m}\frac{\theta(i+1)}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta(m+i+1)-1},$$

$$E_{\nu,i,j,k,m} = (-1)^{\nu+k+m} W_{i,j} \binom{\beta(\nu+n-1)}{k} \binom{\alpha k}{m}.$$

4. Estimation Based on TIIC Samples

In many life tests and reliability studies, the experimenter may not always obtain complete information on failure times for all experimental units due to the cost and time considerations. Data obtained from such experiments are called censored data. Life tests terminated after a specified number of failures are known as TIIC. With possibility of censoring, their advantages arise, first, reduction of the cost of the test; second, giving us a good chance to reach a decision in a shorter time or with fewer observations and the third advantage gives us analysis in complete data if some observations are missed.

This section considers the maximum likelihood (ML) estimators of the unknown parameters for the EGPF distribution under TIIC samples. Approximate CIs are obtained. Simulation study and application to real data are also provided.

4.1. ML estimators

The length of the life tests of items cannot be observed failure times exactly. Generally there are constraints on the length of life tests or other reliability studies. During the analysis of highly reliable items, the testing has to be stopped before all of the items have failed as there is limited availability of test time. Life tests terminated after a specified number of failures are known as TIIC or failure censoring. Let $X = (X_{(1)} < X_{(2)} < ... < X_{(r)})$ is a TIIC of size r from a life test on n items whose lifetimes have the EGPF distribution with set of parameters $\Psi = (\alpha, \theta, \beta, \lambda)$. The likelihood function of r failures and $(n-r)$ censored values is given by

$$L = \frac{n!}{(n-r)!} \prod_{i=1}^{r} \frac{\theta\alpha\beta}{\lambda}\left(\frac{x_i}{\lambda}\right)^{\theta-1}\left[1-\left(\frac{x_i}{\lambda}\right)^{\theta}\right]^{\alpha-1}\left[1-\left(1-\left(\frac{x_i}{\lambda}\right)^{\theta}\right)^{\alpha}\right]^{\beta-1}\left\{1-\left[1-\left(1-\left(\frac{x_i}{\lambda}\right)^{\theta}\right)^{\alpha}\right]^{\beta}\right\}^{n-r},$$

for simplicity, we write x_i, instead of $x_{(i)}$. The log-likelihood function for the vector of parameters $\Psi = (\alpha, \theta, \beta, \lambda)$ is

$$\ln L(\Psi) = \ln\left(\frac{n!}{(n-r)!}\right) + r\ln\alpha + r\ln\beta + r\ln\theta - r\theta\ln\lambda + (\theta-1)\sum_{i=1}^{r}\ln(x_i) + (\alpha-1)\sum_{i=1}^{r}\ln(y_i)$$
$$+(\beta-1)\sum_{i=1}^{r}\ln(1-y_i^{\alpha}) + (n-r)\ln\left[1-(1-y_r^{\alpha})^{\beta}\right],$$

where $y_i = 1-\left(\frac{x_i}{\lambda}\right)^{\theta}$ and $y_r = 1-\left(\frac{x_r}{\lambda}\right)^{\theta}$.

It is known that, the estimate of λ is the sample maxima. The elements of the score function $U(\Psi) = (U_\alpha, U_\beta, U_\theta)$ are given by

$$U_\alpha = \frac{r}{\alpha} + \sum_{i=1}^{r}\ln(y_i) - (\beta-1)\sum_{i=1}^{r}\frac{\ln(y_i)}{(y_i^{-\alpha}-1)} + \beta(n-r)\ln(y_r)y_r^{\alpha}(1-y_r^{\alpha})^{\beta-1}\left[1-(1-y_r^{\alpha})^{\beta}\right]^{-1},$$

$$U_{\beta}=\frac{r}{\beta}+\sum_{i=1}^{r}\ln(1-y_i^{\alpha})-(n-r)\ln(1-y_r^{\alpha})(1-y_r^{\alpha})^{\beta}\left[1-(1-y_r^{\alpha})^{\beta}\right]^{-1},$$

and

$$\begin{aligned}U_{\theta}=&\frac{r}{\theta}-r\ln\lambda+\sum_{i=1}^{r}\ln(x_i)-(\alpha-1)\sum_{i=1}^{r}\ln\left(\frac{x_i}{\lambda}\right)\left(\frac{x_i}{\lambda}\right)^{\theta}y_i^{-1}\\&+\alpha(\beta-1)\sum_{i=1}^{r}\frac{y_i^{\alpha-1}}{(1-y_i^{\alpha})}\ln\left(\frac{x_i}{\lambda}\right)\left(\frac{x_i}{\lambda}\right)^{\theta}\\&-\alpha\beta(n-r)y_r^{\alpha-1}(1-y_r^{\alpha})^{\beta-1}\ln\left(\frac{x_r}{\lambda}\right)\left(\frac{x_r}{\lambda}\right)^{\theta}\left[1-(1-y_r^{\alpha})^{\beta}\right]^{-1}.\end{aligned}$$

However, it is not easy to obtain a closed form solution for the above equations; therefore an iterative technique is applied to obtain the ML estimates.

The most common method to set confidence bounds for the parameters is to use the asymptotic normal distribution of the ML estimators (see Vander Wiel and Meeker 1990). The asymptotic variances and covariance matrix of the ML estimators of the parameters can be approximated by numerically inverting the asymptotic Fisher-information matrix F. It is composed of the negative second and mixed derivatives of the natural logarithm of the likelihood function evaluated at the ML estimators. So, the elements of the Fisher information are given by

$$\begin{aligned}U_{\alpha\alpha}=&-\frac{r}{\alpha^2}-(\beta-1)\sum_{i=1}^{r}\frac{y_i^{-\alpha}(\ln y_i)^2}{(y_i^{-\alpha}-1)^2}+\beta(n-r)\frac{y_r^{\alpha}(1-y_r^{\alpha})^{\beta-1}(\ln y_r)^2}{\left[1-(1-y_r^{\alpha})^{\beta}\right]}\\&-\beta(\beta-1)(n-r)\frac{y_r^{2\alpha}(1-y_r^{\alpha})^{\beta-2}(\ln y_r)^2}{\left[1-(1-y_r^{\alpha})^{\beta}\right]}-\beta^2(n-r)\frac{y_r^{2\alpha}(1-y_r^{\alpha})^{2\beta-2}(\ln y_r)^2}{\left[1-(1-y_r^{\alpha})^{\beta}\right]^2},\end{aligned}$$

$$U_{\beta\beta}=-\frac{r}{\beta^2}-(n-r)\frac{\left[\ln(1-y_r^{\alpha})\right]^2(1-y_r^{\alpha})^{\beta}}{\left[1-(1-y_r^{\alpha})^{\beta}\right]}-(n-r)\frac{\left[\ln(1-y_r^{\alpha})\right]^2(1-y_r^{\alpha})^{2\beta}}{\left[1-(1-y_r^{\alpha})^{\beta}\right]^2},$$

$$\begin{aligned}U_{\theta\theta}=&-\frac{r}{\theta^2}-(\alpha-1)\sum_{i=1}^{r}\left[\ln\left(\frac{x_i}{\lambda}\right)\right]^2\left\{\left(\frac{x_i}{\lambda}\right)^{\theta}y_i^{-1}+\left(\frac{x_i}{\lambda}\right)^{2\theta}y_i^{-2}\right\}\\&-\alpha(\alpha-1)(\beta-1)\sum_{i=1}^{r}\frac{y_i^{\alpha-2}}{(1-y_i^{\alpha})}\left[\ln\left(\frac{x_i}{\lambda}\right)\right]^2\left(\frac{x_i}{\lambda}\right)^{2\theta}+\alpha(\beta-1)\sum_{i=1}^{r}\frac{y_i^{\alpha-1}}{(1-y_i^{\alpha})}\left[\ln\left(\frac{x_i}{\lambda}\right)\right]^2\left(\frac{x_i}{\lambda}\right)^{\theta}\\&-\alpha^2(\beta-1)\sum_{i=1}^{r}\frac{y_i^{2\alpha-2}}{(1-y_i^{\alpha})^2}\left[\ln\left(\frac{x_i}{\lambda}\right)\right]^2\left(\frac{x_i}{\lambda}\right)^{2\theta}\\&-\alpha\beta(n-r)\frac{y_r^{\alpha-1}(1-y_r^{\alpha})^{\beta-1}}{\left[1-(1-y_r^{\alpha})^{\beta}\right]}\left[\ln\left(\frac{x_r}{\lambda}\right)\right]^2\left(\frac{x_r}{\lambda}\right)^{\theta}\\&+\alpha\beta(\alpha-1)(n-r)\frac{y_r^{\alpha-2}(1-y_r^{\alpha})^{\beta-1}}{\left[1-(1-y_r^{\alpha})^{\beta}\right]}\left[\ln\left(\frac{x_r}{\lambda}\right)\right]^2\left(\frac{x_r}{\lambda}\right)^{2\theta}\\&-\alpha^2\beta(\beta-1)(n-r)\frac{y_r^{2\alpha-2}(1-y_r^{\alpha})^{\beta-2}}{\left[1-(1-y_r^{\alpha})^{\beta}\right]}\left[\ln\left(\frac{x_r}{\lambda}\right)\right]^2\left(\frac{x_r}{\lambda}\right)^{2\theta}\\&-\alpha^2\beta^2(n-r)\frac{y_r^{2\alpha-2}(1-y_r^{\alpha})^{2\beta-2}}{\left[1-(1-y_r^{\alpha})^{\beta}\right]^2}\left[\ln\left(\frac{x_r}{\lambda}\right)\right]^2\left(\frac{x_r}{\lambda}\right)^{2\theta},\end{aligned}$$

$$U_{\alpha\theta}=-\sum_{i=1}^{r}\ln\left(\frac{x_i}{\lambda}\right)\left(\frac{x_i}{\lambda}\right)^{\theta}y_i^{-1}+\alpha(\beta-1)\sum_{i=1}^{r}\frac{y_i^{-\alpha-1}\ln(y_i)}{(y_i^{-\alpha}-1)^2}\ln\left(\frac{x_i}{\lambda}\right)\left(\frac{x_i}{\lambda}\right)^{\theta}$$
$$+(\beta-1)\sum_{i=1}^{r}(y_i^{-\alpha+1}-y_i)^{-1}\ln\left(\frac{x_i}{\lambda}\right)\left(\frac{x_i}{\lambda}\right)^{\theta}-\beta(n-r)\frac{(1-y_r^{\alpha})^{\beta-1}y_r^{\alpha-1}}{\left[1-(1-y_r^{\alpha})^{\beta}\right]}\ln\left(\frac{x_r}{\lambda}\right)\left(\frac{x_r}{\lambda}\right)^{\theta}$$
$$+\alpha\beta^2(n-r)\frac{(1-y_r^{\alpha})^{2\beta-2}y_r^{2\alpha-1}\ln(y_r)}{\left[1-(1-y_r^{\alpha})^{\beta}\right]^2}\ln\left(\frac{x_r}{\lambda}\right)\left(\frac{x_r}{\lambda}\right)^{\theta}+\alpha\beta(\beta-1)(n-r)\frac{(1-y_r^{\alpha})^{\beta-2}y_r^{2\alpha-1}\ln(y_r)}{\left[1-(1-y_r^{\alpha})^{\beta}\right]}$$
$$-\alpha\beta(n-r)\frac{(1-y_r^{\alpha})^{\beta-1}y_r^{\alpha-1}\ln(y_r)}{\left[1-(1-y_r^{\alpha})^{\beta}\right]}\ln\left(\frac{x_r}{\lambda}\right)\left(\frac{x_r}{\lambda}\right)^{\theta},$$

$$U_{\alpha\beta}=-\sum_{i=1}^{r}\frac{\ln(y_i)}{(y_i^{-\alpha}-1)}+(n-r)\sum_{i=1}^{r}\frac{y_i^{\alpha}(1-y_i^{\alpha})^{\beta-1}\ln(y_i)}{\left[1-(1-y_i^{\alpha})^{\beta}\right]}$$
$$+\beta(n-r)\frac{y_r^{\alpha}(1-y_r^{\alpha})^{\beta-1}\ln(y_r)\ln(1-y_r^{\alpha})}{\left[1-(1-y_r^{\alpha})^{\beta}\right]}-\beta(n-r)\frac{y_r^{\alpha}(1-y_r^{\alpha})^{2\beta-1}\ln(y_r)\ln(1-y_r^{\alpha})}{\left[1-(1-y_r^{\alpha})^{\beta}\right]^2},$$

and

$$U_{\beta\theta}=\alpha\sum_{i=1}^{r}\frac{y_i^{\alpha-1}}{(1-y_i^{\alpha})}\ln\left(\frac{x_i}{\lambda}\right)\left(\frac{x_i}{\lambda}\right)^{\theta}-\alpha(n-r)\frac{y_r^{\alpha-1}(1-y_r^{\alpha})^{\beta-1}}{\left[1-(1-y_r^{\alpha})^{\beta}\right]}\left(\frac{x_r}{\lambda}\right)^{\theta}\ln\left(\frac{x_r}{\lambda}\right)$$
$$-\alpha\beta(n-r)\frac{y_r^{\alpha-1}(1-y_r^{\alpha})^{\beta-1}\ln(1-y_r^{\alpha})}{\left[1-(1-y_r^{\alpha})^{\beta}\right]}\left(\frac{x_r}{\lambda}\right)^{\theta}\ln\left(\frac{x_r}{\lambda}\right)$$
$$-\alpha\beta(n-r)\frac{y_r^{\alpha-1}(1-y_r^{\alpha})^{2\beta-1}\ln\left(1-y_r^{\alpha}\right)}{\left[1-(1-y_r^{\alpha})^{\beta}\right]^2}\left(\frac{x_r}{\lambda}\right)^{\theta}\ln\left(\frac{x_r}{\lambda}\right).$$

In relation to the asymptotic variance-covariance matrix of the ML estimators of the parameters, it can be approximated by numerically inverting the above Fisher's information matrix F. Thus, the approximate $100(1-\gamma)\%$ two-sided CIs for α,θ and β can be, respectively, easily obtained by

$$\hat{\alpha}\pm Z_{\gamma/2}\,\sigma_{\hat{\alpha}},\ \ \hat{\theta}\pm Z_{\gamma/2}\,\sigma_{\hat{\theta}}\ \text{ and }\ \hat{\beta}\pm Z_{\gamma/2}\,\sigma_{\hat{\beta}},$$

where Z is the $\left[100(1-\gamma)/2\right]^{\text{th}}$ standard normal percentile and $\sigma(.)$ is the standard deviation for the ML estimates.

4.2. Simulation study

A simulation study is carried out to compare the performance of the estimates based on different sampling schemes. The performance of the resulting estimates of the unknown parameters (α,θ,β) has been considered in terms of their mean squared errors (MSEs), absolute relative bias (ARB), standard errors (SEs), and average lengths (AL) of CIs. The simulation procedure is described as follows:

- 1000 random samples of sizes 30, 50, 100 and 300 are generated from the EGPF distribution is very simple, if U has a uniform $(0,1)$ random number, then $Y=\lambda\left[1-\left(1-U^{1/\beta}\right)^{1/\alpha}\right]^{1/\theta}$ follows an EGPF distribution.

- In TIIC, the numbers of the failure unites r are selected as $r = 0.7n, 0.9n$ for different samples under TIIC data. Also, we take $r = n,$ i.e., complete sample.
- Select different values of the unknown parameters (α, θ, β) as
  Case I $\equiv (\alpha = 0.5, \theta = 0.25, \beta = 0.7),$ Case II $\equiv (\alpha = 0.75, \theta = 0.25, \beta = 0.7),$
  Case III $\equiv (\alpha = 0.5, \theta = 0.25, \beta = 0.5),$ Case IV $\equiv (\alpha = 0.5, \theta = 0.5, \beta = 0.7).$
- Iterative technique is used for solving the three nonlinear equations for α, θ and β to obtain ML estimates under complete and TIIC data.
- The MSEs, ARBs, SEs, and AL with confidence level $\gamma = 0.95$ for all sample sizes and for the all selected sets of parameters are listed in Tables 3 and 4.

We conclude the following based on Tables 3, 4 and Figures 3 to 8.

1. For all cases, it is clear that MSEs and SEs decrease as sample size increases (see Tables 3 to 4).
2. The MSEs for the numbers of the failure unites at $r = n$ for all parameters values, are the smallest among the other numbers of the failure unites $r = 0.7, 0.9$ (see, Tables 3, 4 and see for example Figures 3 and 4).
3. For all cases, it is clear that AL of CIs for the unknown parameters decreases as n increases (see Tables 3, 4 and see for example Figures 5 and 6).
4. For all cases of parameters, as the values of r and n increase, MSEs of all estimates decrease (see, Tables 3, 4 and see for example Figures 3 to 6).
5. The MSEs of α estimates are smaller than the corresponding MSEs for the other estimates for θ and β in almost all of the cases (see Tables 3 and 4).
6. The MSE and AL for all estimates in Case I have the smallest values corresponding to the other cases and hence it has good statistical properties (see Tables 3 and 4).

4.3. Applications to real data

In this subsection, two real data sets are provided to illustrate the importance of the EGPF distribution. To check the validity of the fitted model, Kolmogorov-Smirnov goodness of fit test is performed for each data set and the p-values in each case indicates that the model fits the data very well. The data I represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). The data are:

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1. 59, 1 .6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 51, 2.53, 2.54, 2.3, 2.31, 2.4, 2.45, 2. 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

The data II reported by Jorgensen (1982) will be considered. It consists of 40 observations of the active repair times (in hours) for airborne communication transceiver. The data are:

0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50.

The ML estimates of the parameters and their SEs for the real data based on TIIC are listed in Table 5.

Table 3 The ARBs, MSEs, SEs and AL of the estimates for Case I and Case II

n	r	Para-meters	$(\alpha=0.5,\theta=0.25,\beta=0.7)$ Properties			95% CIs	$(\alpha=0.75,\theta=0.25,\beta=0.7)$ Properties			95% CIs
			MSE	ARB	SE	AL	MSE	ARB	SE	AL
30	21	α	0.025	0.116	4.904*	0.331	0.114	0.314	8.091*	0.591
		θ	0.047	0.860	0.752*	0.088	0.032	0.710	0.960*	0.113
		β	0.442	0.945	2.109*	0.248	0.194	0.622	2.216*	0.261
	27	α	0.016	0.069	4.106*	0.271	0.052	0.141	6.695*	0.473
		θ	0.035	0.749	0.676*	0.079	0.037	0.764	0.853*	0.100
		β	0.538	1.044	2.104*	0.247	0.229	0.678	2.013	0.237
	30	α	0.014	0.060	3.808*	0.250	0.040	0.098	6.158*	0.429
		θ	0.035	0.744	0.647*	0.076	0.063	0.999	0.803*	0.094
		β	0.503	1.009	2.156*	0.254	0.200	0.634	1.790*	0.210
50	35	α	0.015	0.063	2.395*	0.263	0.053	0.142	4.068*	0.480
		θ	0.015	0.487	3.670*	0.072	0.021	0.573	0.541*	0.106
		β	0.437	0.941	1.186*	0.232	0.088	0.416	1.157*	0.227
	45	α	0.010	0.047	1.969*	0.212	0.027	0.046	3.208*	0.365
		θ	0.016	0.495	3.412*	0.067	0.027	0.650	0.449*	0.088
		β	0.435	0.938	1.135*	0.222	0.101	0.448	1.039*	0.204
	50	α	8.718*	0.040	1.824*	0.195	0.022	0.037	2.914*	0.327
		θ	0.017	0.513	3.394*	0.067	0.028	0.669	0.425*	0.083
		β	0.350	0.841	1.155*	0.226	0.130	0.511	0.996*	0.195
100	70	α	8.422*	1.266*	0.918*	0.196	0.024	0.019	1.540*	0.348
		θ	0.012	0.440	0.160*	0.063	0.015	0.482	2.422*	0.095
		β	0.233	0.687	0.442*	0.173	9.698*	0.125	4.546*	0.178
	90	α	5.407*	8.460*	0.734*	0.154	0.014	5.558*	1.190*	0.261
		θ	0.010	0.399	0.150*	0.059	0.012	0.439	0.208*	0.082
		β	0.197	0.631	0.457*	0.179	0.031	0.244	0.446*	0.175
	100	α	4.492*	5.287*	0.669*	0.140	0.012	5.649*	1.104*	0.240
		θ	0.010	0.397	0.157*	0.062	0.017	0.511	1.994*	0.078
		β	0.132	0.515	0.467*	0.183	0.031	0.244	4.177*	0.164
300	210	α	3.290*	0.026	0.186*	0.116	8.527*	0.018	0.304*	0.195
		θ	5.329*	0.287	0.096*	0.052	7.065*	0.326	0.068*	0.080
		β	0.043	0.292	0.106*	0.125	1.338*	0.015	0.117*	0.138
	270	α	1.921*	2.203*	0.146*	0.090	5.083*	0.013	0.236*	0.148
		θ	2.930*	0.209	0.048*	0.057	5.550*	0.290	0.057*	0.067
		β	0.026	0.225	0.119*	0.140	3.213*	0.065	0.113*	0.133
	300	α	1.574*	4.942*	0.132*	0.081	4.055*	9.854*	0.211*	0.132
		θ	1.816*	0.159	0.052*	0.061	5.510*	0.291	0.051*	0.060
		β	0.019	0.191	0.131*	0.154	5.724*	0.097	0.109*	0.128

Note: * Indicate that the value multiply 10^{-3}

Table 4 The ARBs, MSEs, SEs and AL of the estimates for Case III and Case IV

n	r	Para-meters	$(\alpha=0.5,\theta=0.25,\beta=0.5)$ Properties			95% CIs	$(\alpha=0.5,\theta=0.5,\beta=0.7)$ Properties			95% CIs
			MSE	ARB	SE	AL	MSE	ARB	SE	AL
30	21	α	0.049	0.256	5.977*	0.414	0.030	0.210	4.601*	0.308
		θ	0.012	0.434	0.782*	0.092	0.050	0.444	0.971*	0.114
		β	0.243	0.978	1.950*	0.229	1.137	1.520	2.480*	0.292
	27	α	0.026	0.139	4.902*	0.331	0.017	0.097	3.965*	0.261
		θ	0.013	0.441	0.634*	0.075	0.079	0.558	0.974*	0.114
		β	0.315	1.117	1.829	0.215	1.016	1.436	2.300	0.270
	30	α	0.021	0.122	4.421*	0.294	0.013	0.061	3.673*	0.240
		θ	0.016	0.498	0.625*	0.073	0.108	0.655	0.980*	0.115
		β	0.258	1.009	1.783	0.210	0.797	1.272	2.191*	0.258
50	35	α	0.030	0.141	3.178*	0.361	0.014	0.084	2.225*	0.242
		θ	7.817*	0.344	0.402*	0.079	0.037	0.383	0.526*	0.103
		β	0.165	0.807	0.962*	0.189	0.893	1.347	1.224*	0.240
	45	α	0.015	0.051	2.437*	0.268	9.928*	0.060	1.902*	0.204
		θ	0.011	0.408	0.352*	0.069	0.052	0.454	0.523*	0.102
		β	0.160	0.794	0.888*	0.174	0.765	1.246	1.169*	0.229
	50	α	0.012	0.042	2.166*	0.235	8.182*	0.046	1.749*	0.186
		θ	0.012	0.425	0.346*	0.068	0.062	0.495	4.865*	0.095
		β	0.135	0.730	0.883*	0.173	0.636	1.137	1.181*	0.231
100	70	α	0.015	0.044	1.218*	0.268	6.805*	0.021	0.818*	0.173
		θ	3.496*	0.225	0.183*	0.072	0.029	0.335	0.219*	0.086
		β	0.086	0.581	0.384*	0.151	0.475	0.983	0.468*	0.184
	90	α	8.408*	0.016	0.913*	0.195	4.757*	2.346*	0.690*	0.144
		θ	0.049	0.256	5.977*	0.414	0.030	0.210	4.601*	0.308
		β	0.012	0.434	0.782*	0.092	0.050	0.444	0.971*	0.114
	100	α	0.243	0.978	1.950*	0.229	1.137	1.520	2.480*	0.292
		θ	0.026	0.139	4.902*	0.331	0.017	0.097	3.965*	0.261
		β	0.013	0.441	0.634*	0.075	0.079	0.558	0.974*	0.114
300	210	α	0.315	1.117	1.829	0.215	1.016	1.436	2.300	0.270
		θ	0.021	0.122	4.421*	0.294	0.013	0.061	3.673*	0.240
		β	0.016	0.498	0.625*	0.073	0.108	0.655	0.980*	0.115
	270	α	0.258	1.009	1.783	0.210	0.797	1.272	2.191*	0.258
		θ	0.030	0.141	3.178*	0.361	0.014	0.084	2.225*	0.242
		β	7.817*	0.344	0.402*	0.079	0.037	0.383	0.526*	0.103
	300	α	0.165	0.807	0.962*	0.189	0.893	1.347	1.224*	0.240
		θ	0.015	0.051	2.437*	0.268	9.928*	0.060	1.902*	0.204
		β	0.011	0.408	0.352*	0.069	0.052	0.454	0.523*	0.102

Note: * Indicate that the value multiply 10^{-3}

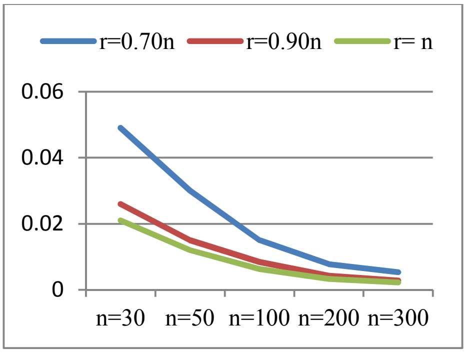

Figure 3 The MSE for α of Case III

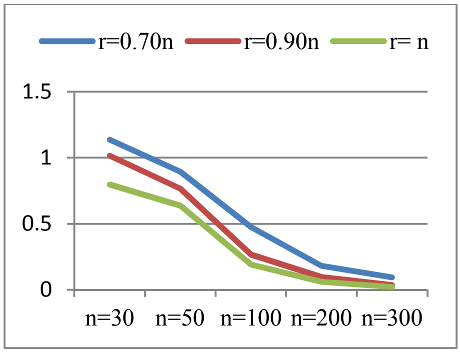

Figure 4 The MSE for β of Case IV

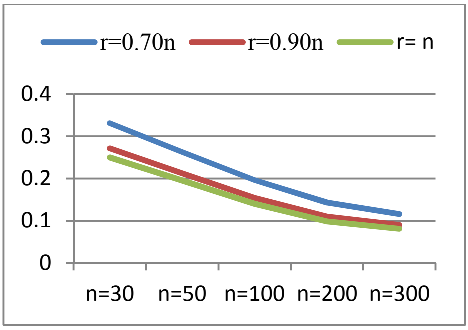

Figure 5 AL for α of Case I

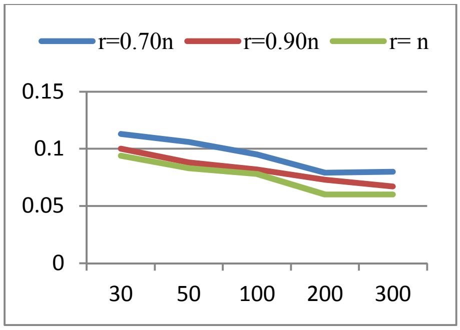

Figure 6 AL for β of Case II

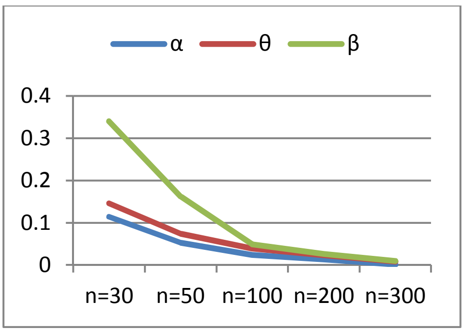

Figure 7 The MSE of (α,θ,β) for Case II at $r=0.7n$

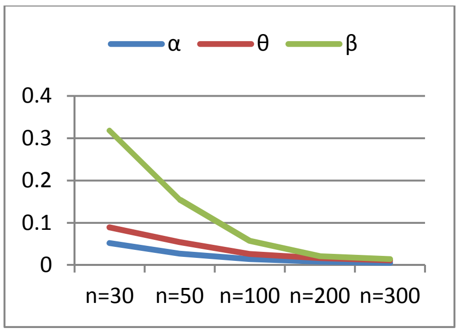

Figure 8 The MSE of (α,θ,β) for Case II at $r=0.9n$

In previous table, the numbers of the failure unites r are selected as $r=0.9n$ for both two real data sets under TIIC data. The estimate of λ is the maximum value, which are 5.55 for data I and 24.5 for data II. The estimates of α,β,θ and their SEs are as given in Table 5. We notice that the SEs of θ take the smallest values corresponding to the SEs of α and β.

Table 5 ML estimates and their SEs based on TIIC data

Real data	n	r	Estimator	Estimate	SEs
I	72	65	$\hat{\alpha}$	4.168	0.019
			$\hat{\lambda}$	5.550	-
			$\hat{\theta}$	0.860	0.006
			$\hat{\beta}$	4.592	0.086
II	40	28	$\hat{\alpha}$	3.793	0.023
			$\hat{\lambda}$	24.500	-
			$\hat{\theta}$	0.357	0.003
			$\hat{\beta}$	5.767	0.087

5. Concluding Remarks

We introduce a new generalization for the PF distribution called the exponentiated generalized power function. Some statistical properties are obtained. The estimation of the model parameters is established based on complete and TIIC samples. The maximum likelihood estimators and asymptotic confidence interval of the model parameters are obtained. Simulation study is conducted to compare the performance of estimates under TIIC with different censoring levels. Applications to real data are given.

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