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New Extended Burr-III Distribution: Its Properties and Applications

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Abstract

In this article, we consider a three parameter extended Burr-III distribution and study some distributional, reliability properties and parameter estimation. Performance of estimation technique used for model parameters estimation is numerically investigated employing Monte Carlo simulation with different sample sizes and parameter values. Efficacy of this distribution in modeling from two real life data is evaluated in comparison to some existing extensions of Burr-III distribution employing well known goodness of fit tests and model selection criteria. Our findings show the proposed distribution as the best among the all the other extensions of Burr-III distribution considered in this study.

Keywords: Log-logistic-X, stress-strength reliability, stochastic ordering, Akaike information criterion.

1. Introduction

Burr (1942) defined the cumulative distribution function (cdf) and probability density function (pdf) (for $x > 0$) of the Burr-III (BIII) distribution respectively by

$$G^{\mathrm{BIII}}(x;\beta,\delta) = (1+x^{-\delta})^{-\beta}, \tag{1}$$

$$f^{\mathrm{BIII}}(x;\beta,\delta) = \beta\delta x^{-\delta-1}(1+x^{-\delta})^{-\beta-1}, \tag{2}$$

where $\beta > 0$ and $\delta > 0$ are both shape parameters.

In 2006, Gleaton and Lynch developed a new family of distribution named as generalized log-logistic family of distribution. Later on, this family was called as odd log-logistic family of distribution. The cdf of the odd log-logistic (OLL-X) family of distribution was given as

$$F(x;\alpha,\xi) = \frac{G(x,\xi)^{\alpha}}{[G(x,\xi)^{\alpha} + (1-G(x,\xi))^{\alpha}]}, \tag{3}$$

where $\alpha > 0$ is an additional shape parameter. $G(x,\xi)$ is the cdf of the parent distribution and ξ denotes the parameters of the parent distribution. The corresponding pdf of the OLL-X family is given as

$$f(x;\alpha,\xi)=\frac{\alpha g(x,\xi)G(x,\xi)^{\alpha-1}(1-G(x,\xi))^{\alpha-1}}{[G(x,\xi)^{\alpha}+(1-G(x,\xi))^{\alpha}]^{2}}. \tag{4}$$

If the parent distribution has closed form cdf then the newly developed model also possess a closed form cdf and $\frac{\log[F(x;\alpha,\xi)/\bar{F}(x;\alpha,\xi)]}{\log[G(x;\alpha,\xi)/\bar{G}(x;\alpha,\xi)]}=\alpha.$ Thus, α is the quotient of the log-odds ratio of the newly developed and parent distributions.

A number of extensions of the Burr-III distribution are proposed to offer better modeling. Some notable among them are the three-parameter Burr-III distribution (Shao et al. 2008), transmuted modified Burr-III (Ali and Ahmed 2015), Marshall Olkin modified Burr-III (Ali et al. 2015), Kumaraswamy Burr-III (Behairy et al. 2016), Marshall-Olkin extended Burr-III distribution (Al-Saiari et al. 2016), extended Burr-III distribution (Cordeiro et al. 2017) and generalized Marshall-Olkin Burr-III (Chakraborty et al. 2020) among others.

We define new extended Burr-III ($\text{NEBIII}(\alpha,\beta,\delta)$) model by using the odd log-logistic method for parameter induction. By inserting (1) in (3) we get the cdf of the $\text{NEBIII}(\alpha,\beta,\delta)$ model as

$$F^{\text{NEBIII}}(x;\alpha,\beta,\delta)=\frac{(1+x^{-\delta})^{-\alpha\beta}}{(1+x^{-\delta})^{-\alpha\beta}+\{1-(1+x^{-\delta})^{-\beta}\}^{\alpha}},\ \text{for}\ x>0, \tag{5}$$

where $\alpha>0,\beta>0$ and $\delta>0$ are the shape parameters. The corresponding pdf is given as

$$f^{\text{NEBIII}}(x;\alpha,\beta,\delta)=\frac{\alpha\beta\delta x^{-\delta-1}(1+x^{-\delta})^{-\alpha\beta-1}\{1-(1+x^{-\delta})^{-\beta}\}^{\alpha-1}}{[(1+x^{-\delta})^{-\alpha\beta}+\{1-(1+x^{-\delta})^{-\beta}\}^{\alpha}]^{2}},\ \text{for}\ x>0. \tag{6}$$

The main motivation behind the proposed family is to obtain an extension of the Burr-III distribution with one additional parameters to bring in more flexibility with respect to skewness, kurtosis, tail weight and length; which encompasses number known distributions as special and related cases, also to ensure that it provides better alternative in the data modeling not only to its sub models including the BIII distribution, but to other recent extensions.

The rest of this article is organized in five more sections. In Section 2, linear representation of the cdf and pdf of the proposed family. In Section, 3 we discuss some mathematical and statistical properties of the proposed family. In Section 4, maximum likelihood methods of estimation of parameters is presented. The data fitting applications and simulation are presented in Section 5. The article ends with a conclusion in Section 6.

2. Linear Representations

The cdf and pdf of the $\text{NEBIII}(\alpha,\beta,\delta)$ can be expressed as infinite linear mixture of the corresponding functions of BIII distribution as follows. We know that

$$(1+z)^{-n}=\sum_{k=0}^{\infty}(-1)^{k}\binom{n+k-1}{k}z^{k}, \tag{7}$$

$$(1-z)^{t}=\sum_{i=0}^{\infty}(-1)^{i}\binom{t}{i}z^{i}\ \text{for}\ |z|>0. \tag{8}$$

By using (7) the cdf of $\text{NEBIII}(\alpha,\beta,\delta)$ can be written as

$$F^{\text{NEBIII}}(x;\alpha,\beta,\delta)=\sum_{k=0}^{\infty}\sum_{j=0}^{\alpha k}\vartheta_{jk}[G^{\text{BIII}}(x;\beta,\delta)]^{j+\alpha k}, \tag{9}$$

where $\vartheta_{jk}=(-1)^{j+k}\binom{\alpha k}{j}$. The corresponding pdf is given by

$$f^{\mathrm{NEBIII}}(x;\alpha,\beta,\delta)=\sum_{k=0}^{\infty}\sum_{j=0}^{\alpha k}\vartheta_{jk}(j+\alpha k)[G^{\mathrm{BIII}}(x;\beta,\delta)]^{j+\alpha\,k-1}g^{\mathrm{BIII}}(x;\beta,\delta)$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\alpha k}\vartheta'_{jk}g^{\mathrm{BIII}}(x;\beta,\delta)[G^{\mathrm{BIII}}(x;\beta,\delta)]^{j+\alpha k-1} \quad (10)$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\alpha k}\vartheta_{jk}\frac{d}{dt}[G^{\mathrm{BIII}}(x;\beta,\delta)]^{j+\alpha k}, \quad (11)$$

where ϑ_{jk} is defined above and $\vartheta'_{jk}=\vartheta_{jk}(j+\alpha k)$.

Alternatively, density function of the proposed model can be expressed as by using (7)

$$f^{\mathrm{NEBIII}}(x;\alpha,\beta,\delta)=\alpha\beta\delta x^{-\delta-1}\sum_{i=0}^{\infty}(-1)^{i}(i+1)\left\{1-(1+x^{-\delta})^{-\beta}\right\}^{\alpha(i+1)-1}(1+x^{-\delta})^{\alpha\beta(i+1)-1}.$$

By using (8)

$$f^{\mathrm{NEBIII}}(x;\alpha,\beta,\delta)=\alpha\beta\delta x^{-\delta-1}\sum_{i,j=0}^{\infty}(-1)^{i+j}(i+1)\binom{\alpha(i+1)-1}{j}(1+x^{-\delta})^{\alpha\beta(i+1)-\beta j-1}$$

$$=\alpha\beta\delta\sum_{i,j=0}^{\infty}\pi_{i,j}x^{-\delta-1}(1+x^{-\delta})^{\alpha\beta(i+1)-\beta j-1}, \quad (12)$$

where $\pi_{i,j}=(-1)^{i+j}(i+1)\binom{\alpha(i+1)-1}{j}$. Shapes of the pdf and hazard rate function (hrf) of $\mathrm{NEBIII}(\alpha,\beta,\delta)$:

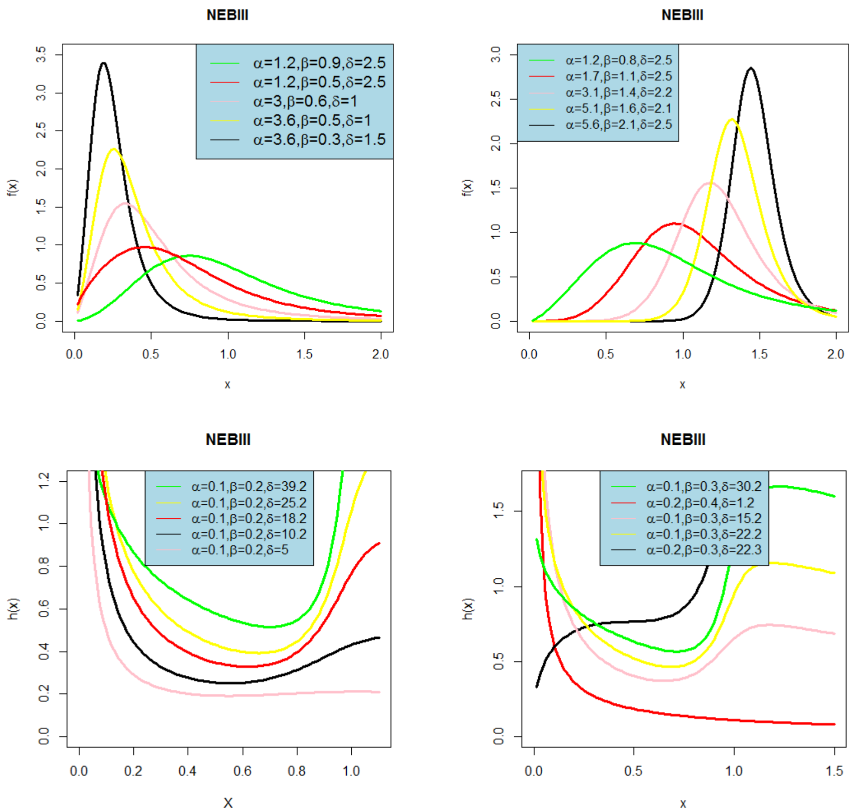


Figure 1 Plot of the pdf and hrf of $\mathrm{NEBIII}(\alpha,\beta,\delta)$

3. Properties

This section explores unambiguous expressions for some major properties of newly developed distribution.

3.1. Quantile function and random sample generation

From (5) the p^{th} quantile t_p for NEBIII(α,β,δ) distribution can be respectively obtained as

$$t_p=\left[\left\{\left(\frac{1}{p}-1\right)^{1/\alpha}+1\right\}^{1/\beta}-1\right]^{-1/\delta}. \tag{13}$$

As a result a random number T from NEBIII(α,β,δ) distribution can be easily generated starting with a uniform random number U~Uniform(0,1). By using inversion method as

$$T=\left[\left\{\left(\frac{1}{p}-1\right)^{1/\alpha}+1\right\}^{1/\beta}-1\right]^{-1/\delta}.$$

We can also compute the first quartile, median and third quartile of the observed distribution by inserting $u=0.25$, 0.5 and 0.75, respectively in (13).

3.2. Moments and related measures

By definition of moments using (7)

$$\mu_r'=E(X^r)=\int_0^\infty x^r f(x;\alpha,\beta,\delta)dx=\int_0^\infty \alpha\beta\delta\sum_{i=j=0}^{\infty}\pi_{i,j}x^{r-\delta-1}(1+x^{-\delta})^{\alpha\,\beta(i+1)-\beta j-1}dx.$$

Let $x^{-\delta}=y$ and $x=y^{-\frac{1}{\delta}}$ and $dx=-\frac{1}{\delta}y^{-\frac{1}{\delta}-1}dy,$

$$\mu_r'=\int_0^\infty \alpha\beta\sum_{i,j=0}^{\infty}\pi_{i,j}y^{-\frac{r}{\delta}}(1+y)^{\alpha\,\beta(i+1)-\beta j-1}dy.$$

Put $y=\frac{w}{1-w},\ w=\frac{y}{1+y}$ and $dy=\frac{1}{(1-w)^2}dw$

$$\mu_r'=\int_0^1 \alpha\beta\sum_{i,j=0}^{\infty}\pi_{i,j}\frac{w^{-\frac{r}{\delta}}}{(1-w)^{-\frac{r}{\delta}}}\left(\frac{1}{1-w}\right)^{\alpha\beta(i+1)-\beta j-1}\frac{1}{(1-w)^2}dw$$

$$=\int_0^1 \alpha\beta\sum_{i,j=0}^{\infty}\pi_{i,j}w^{1-\frac{r}{\delta}-1}(1-w)^{\beta j-\alpha\beta(i+1)+\frac{r}{\delta}-1}dw.$$

Hence $\mu_r'=\alpha\beta\sum_{i,j=0}^{\infty}\pi_{i,j}B\left(1-\frac{r}{\delta},\beta j-\alpha\beta(i+1)+\frac{r}{\delta}\right)$, for $\delta>r$, where $B(m,n)$, $m>0$, $n>0$ is the beta function. The moment generating function (mgf) is derived using the above relation as

$$M_x(t)=E(e^{tx})=\int_0^\infty e^{tx}f(x;\alpha,\beta,\delta)dx=\int_0^\infty\sum_{r=0}^{\infty}\frac{t^r}{r!}x^r f(x;\alpha,\beta,\delta)dx$$

$$=\alpha\beta\delta\sum_{i,j=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}\pi_{i,j}B\left(1-\frac{r}{\delta},\beta j-\alpha\beta(i+1)+\frac{r}{\delta}\right)\text{ for }\delta>r.$$

The s^{th} cumulants (k_s) and central moments (μ_s) of the NEBIII(α,β,δ) distribution are obtained from above expression as

$$k_s = \mu_s' - \sum_{k=1}^{s-1}\binom{s-1}{k-1} k_k \mu_{s-k}', \quad \mu_s = -\sum_{k=0}^{s}\binom{s}{k}(-1)^k \mu_1'^{\,k} \mu_{s-k}',$$

where $\mu_1' = k.$ Thus,

$$k_2 = \mu_2' - (\mu_1')^2, \;\; k_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3, \;\; k_4 = \mu_4' - 4\mu_3'\mu_1' - 3(\mu_2')^2 + 12\mu_2'(\mu_1')^2 - 6(\mu_1')^4.$$

The skewness and kurtosis of the NEBIII(α,β,δ) distribution can be obtained from third and fourth standardized cumulants by using formulae $\gamma_1 = k_3 / k_2^{3/2}$ and $\gamma_2 = k_4 / k_2^2$, respectively.

3.3. Rényi entropy

If a random variable X follows NEBIII(α,β,δ) distribution, then its Rényi entropy is defined as

$$I_R(\vartheta) = \frac{1}{1-\vartheta}\log\{I(\vartheta)\} = \frac{1}{1-\vartheta}\log\int_{-\infty}^{\infty} f^{\vartheta}(x)\,dx \quad \text{for } \vartheta > 0 \text{ and } \vartheta \neq 1, \tag{14}$$

where $I(\vartheta) = \int_{-\infty}^{\infty} f^{\vartheta}(x)\,dx.$ Now, $I(\vartheta) = \int_0^{\infty}\left[\frac{\alpha\beta\delta x^{-\delta-1}(1+x^{-\delta})^{-\alpha\beta-1}\{1-(1+x^{-\delta})^{-\beta}\}^{\alpha-1}}{[(1+x^{-\delta})^{-\alpha\beta} + \{1-(1+x^{-\delta})^{-\beta}\}^{\alpha}]^2}\right]^{\vartheta} dx.$

On simplification by using the expansions of (7) and (8) we get

$$I(\vartheta) = (\alpha\beta\delta)^{\vartheta}\sum_{i,j=0}^{\infty}\varphi_{i,j}\int_0^{\infty} x^{-\vartheta(\delta+1)}(1+x^{-\delta})^{\alpha\beta(\vartheta+i)-\beta j-\vartheta}dx,$$

where $\varphi_{i,j} = \frac{\Gamma(2\vartheta+i)}{\Gamma(2\vartheta)\Gamma(i+1)}(-1)^{i+j}\binom{\alpha(\vartheta+i)-\vartheta}{j}.$

Let $x^{-\delta} = y$ and $x = y^{-\frac{1}{\delta}}$ and $dx = -\frac{1}{\delta}y^{-\frac{1}{\delta}-1}dy,$ then

$$I(\vartheta) = (\alpha\beta\delta)^{\vartheta}\sum_{i=j=0}^{\infty}\varphi_{i,j}\int_0^{\infty} y^{\frac{\vartheta}{\delta}-\frac{1}{\delta}}(1+y)^{\alpha\beta(\vartheta+i)-\beta j-\vartheta}dy.$$

Putting $y = \frac{w}{1-w}, w = \frac{y}{1+y}$ and $dy = \frac{1}{(1-w)^2}dw$ we get

$$I(\vartheta) = \alpha^{\vartheta}\beta^{\vartheta}\delta^{\vartheta-1}\sum_{i,j=0}^{\infty}\varphi_{i,j}\int_0^1 \frac{w^{\frac{\vartheta}{\delta}-\frac{1}{\delta}}}{(1-w)^{\frac{\vartheta}{\delta}-\frac{1}{\delta}}}\left(\frac{1}{1-w}\right)^{\alpha\beta(\vartheta+i)-\beta j-\vartheta}\frac{1}{(1-w)^2}dw$$

$$= \alpha^{\vartheta}\beta^{\vartheta}\delta^{\vartheta-1}\sum_{i,j=0}^{\infty}\varphi_{i,j}\int_0^1 w^{1+\frac{\vartheta}{\delta}-\frac{1}{\delta}-1}(1-w)^{-\alpha\beta(\vartheta+i)+\beta j+\vartheta-\frac{\vartheta}{\delta}+\frac{1}{\delta}-1-1}dw$$

$$= \alpha^{\vartheta}\beta^{\vartheta}\delta^{\vartheta-1}\sum_{i,j=0}^{\infty}\varphi_{i,j}B\left(1+\frac{\vartheta}{\delta}-\frac{1}{\delta}, \beta j+\vartheta-\alpha\beta(\vartheta+i)-\frac{\vartheta}{\delta}+\frac{1}{\delta}-1\right). \tag{15}$$

By inserting (15) in (14) we get the Rényi entropy of NEBIII(α,β,δ) distribution as

$$I_R(\delta) = \frac{\vartheta\log\alpha}{1-\vartheta} + \frac{\vartheta\log\beta}{1-\vartheta} - \log(\delta)$$

$$+\frac{1}{1-\vartheta}\log\left[\sum_{i,j=0}^{\infty}\varphi_{i,j}B\left(1+\frac{\vartheta}{\delta}-\frac{1}{\delta},\beta j+\vartheta-\alpha\beta(\vartheta+i)-\frac{\vartheta}{\delta}+\frac{1}{\delta}-1\right)\right]. \tag{16}$$

3.4. Moment generating function

Using the result in (11) of Section 2 we can express the mgf of $\text{NEBIII}(\alpha,\beta,\delta)$ as

$$M_X(s)=E[e^{sX}]=\int_0^{\infty}e^{sx}\sum_{k=0}^{\infty}\sum_{j=0}^{\alpha k}\vartheta_{jk}\frac{d}{dt}\left[G^{\text{BIII}}(x;\beta,\delta)\right]^{j+\alpha k}dx$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\alpha k}\vartheta_{jk}\int_0^{\infty}e^{sx}\frac{d}{dx}\left[G^{\text{BIII}}(x;\beta,\delta)\right]^{j+\alpha k}dx=\sum_{k=0}^{\infty}\sum_{j=0}^{\alpha k}\vartheta'_{jk}M_{EG}(s).$$

$M_{EG}(s)$ being the mgf of exponentiated BIII distribution that is exponentiated $G^{\text{BIII}}(x;\beta,\delta)$.

3.5. Distribution of order statistics

Consider a random sample $X_1,X_2,...,X_n$ from any $\text{NEBIII}(\alpha,\beta,\delta)$ distribution. Let $X_{r:n}$ denote the r^{th} order statistic. The pdf of $X_{r:n}$ can be expressed as

$$f_{r:n}(x)=\frac{n!}{(r-1)!(n-r)!}f^{\text{NEBIII}}(x)F^{\text{NEBIII}}(x)^{r-1}\{1-F^{\text{NEBIII}}(x)\}^{n-r}$$
$$=\frac{n!}{(r-1)!(n-r)!}\sum_{m=0}^{n-r}(-1)^m\binom{n-r}{m}f^{\text{NEBIII}}(x)\{F^{\text{NEBIII}}(x)\}^{m+r-1}.$$

The pdf of the r^{th} order statistic for of the $\text{NEBIII}(\alpha,\beta,\delta)$ can be derived on using the expansion of the pdf and cdf given in Section 2 as

$$f_{r:n}(x)=\frac{n!}{(r-1)!(n-r)!}\sum_{m=0}^{n-r}(-1)^m\binom{n-r}{m}\sum_{k=0}^{\infty}\sum_{j=0}^{\alpha k}\vartheta'_{jk}g^{\text{BIII}}(x;\beta,\delta)[G^{\text{BIII}}(x;\beta,\delta)]^{j+\alpha k-1}$$
$$\times\left[\sum_{p=0}^{\infty}\sum_{q=0}^{\alpha p}\vartheta_{pq}\left\{G^{\text{BIII}}(x;\beta,\delta)\right\}^{q+\alpha p}\right]^{m+r-1},$$

where ϑ'_{jk} and ϑ_{pq} defined above.

Using power series raised for positive integer $n\,(\geq 1)$, $\left(\sum_{i=0}^{\infty}a_iu^i\right)^n=\sum_{i=0}^{\infty}c_{n,i}u^i$, where the coefficient $c_{n,i}$ for $i=1,2,...$ are easily obtained from the recurrence equation $c_{n,i}=(ia_0)^{-1}\sum_{m=1}^{i}\left[m(n+1)-i\right]a_mc_{n,i-m}$, where $c_{n,0}=a_0^n$ (see Gradshteyn and Ryzhik 2000).

Now, $\left[\sum_{p=0}^{\infty}\sum_{q=0}^{\alpha p}\vartheta_{pq}\left\{G^{\text{BIII}}(x;\beta,\delta)\right\}^{q+\alpha p}\right]^{m+r-1}=\sum_{p=0}^{\infty}\sum_{q=0}^{\alpha p}d_{m+r-1,\,q+\alpha p}\left\{G^{\text{BIII}}(x;\beta,\delta)\right\}^{q+\alpha p}.$

Therefore, the density function of the r^{th} order statistics of $\text{NEBIII}(\alpha,\beta,\delta)$ distribution can be expressed as

$$f_{r:n}(x)=\frac{n!}{(r-1)!(n-r)!}\sum_{m=0}^{n-r}(-1)^m\binom{n-r}{m}\sum_{k=0}^{\infty}\sum_{j=0}^{\alpha k}\vartheta'_{jk}g^{\text{BIII}}(x;\beta,\delta)\left[G^{\text{BIII}}(x;\beta,\delta)\right]^{j+\alpha k-1}$$

$$\times\sum_{p=0}^{\infty}\sum_{q=0}^{\alpha p} d_{m+r-1,\ q+\alpha p}\left\{G^{\mathrm{BIII}}(x;\beta,\delta)\right\}^{q+\alpha p}$$

$$=\frac{n\ !}{(r-1)!(n-r)!}\sum_{m=0}^{n-r}(-1)^m\binom{n-r}{m}\sum_{k,p=0}^{\infty}\sum_{j=0}^{\alpha k}\sum_{q=0}^{\alpha p}\vartheta'_{jk}\, d_{m+r-1,q+\alpha p}\, g^{\mathrm{BIII}}(x;\beta,\delta)$$

$$\times\left[G^{\mathrm{BIII}}(x;\beta,\delta)\right]^{j+q+\alpha(p+k)-1}$$

$$=\frac{n!}{(r-1)!(n-r)!}\sum_{m=0}^{n-r}(-1)^m\binom{n-r}{m}\sum_{k,p=0}^{\infty}\sum_{j=0}^{\alpha k}\sum_{q=0}^{\alpha p}\vartheta'_{jk}\, d_{m+r-1,\ q+\alpha p}\, g^{\mathrm{BIII}}(x;\beta,\delta)$$

$$\times\left[G^{\mathrm{BIII}}(x;\beta,\delta)\right]^{j+q+\alpha(p+k)-1}$$

$$=g^{\mathrm{BIII}}(x;\beta,\delta)\sum_{k=p=0}^{\infty}\sum_{j=0}^{\alpha k}\sum_{q=0}^{\alpha p}\mu_{j,k,p,q}\left[G^{\mathrm{BIII}}(x;\beta,\delta)\right]^{j+q+\alpha(p+k)-1}$$

$$=g^{\mathrm{BIII}}(x;\beta,\delta)\sum_{k=p=0}^{\infty}\sum_{j=0}^{\alpha k}\sum_{q=0}^{\alpha p}\frac{\mu_{j,k,p,q}}{j+q+\alpha(p+k)}\frac{d}{dx}\left[G^{\mathrm{BIII}}(x;\beta,\delta)\right]^{j+q+\alpha(p+k)}, \tag{17}$$

where $\mu_{i,j,p,q}=\dfrac{n!}{(r-1)!(n-r)!}\sum_{m=0}^{n-r}(-1)^m\binom{n-r}{m}d_{m+r-1,\ q+\alpha p}\,\vartheta'_{jk}.$

3.6. Probability weighted moments

The probability weighted moments (PWMs), first proposed by Greenwood et al. (1979), are expectations of certain functions of a random variable whose mean exists. The $(p,q,r)^{\text{th}}$ PWM of X is defined by

$$\Gamma_{p,q,r}=\int_{-\infty}^{\infty}x^p\left[F(x)\right]^q\left[1-F(x)\right]^r f(x)\,dx.$$

The s^{th} moment of X can be written as

$$E(X^s)=\sum_{k=0}^{\infty}\sum_{j=0}^{\alpha k}\vartheta'_{jk}\int_0^{\infty}x^s[G^{\mathrm{BIII}}(x;\beta,\delta)]^{j+\alpha k-1}g^{\mathrm{BIII}}(x;\beta,\delta)\,dx=\sum_{k=0}^{\infty}\sum_{j=0}^{\alpha k}\vartheta'_{jk}\Gamma^{\mathrm{BIII}}_{s,j+\alpha k-1,\ 0},$$

where ϑ'_{jk} define in Section 2 and $\Gamma^{\mathrm{BIII}}_{p,q,r}=\int_0^{\infty}x^p\left\{G^{\mathrm{BIII}}(x;\beta,\delta)\right\}^q\left\{\bar{G}^{\mathrm{BIII}}(x;\beta,\delta)\right\}^r\left[g^{\mathrm{BIII}}(x;\beta,\delta)\right]dx$ is the PWM of $\mathrm{BIII}(\beta,\delta)$ distribution.

Proceeding similarly we can express s^{th} moment of the r^{th} order statistic $X_{r:n}$ in a random sample of size *n* from $\mathrm{NEBIII}(\alpha,\beta,\delta)$ on using (17) as

$$E(X^s_{r,m})=\sum_{k,p=0}^{\infty}\sum_{j=0}^{\alpha k}\sum_{q=0}^{\alpha p}\mu_{j,k,p,q}\Gamma_{s,j+q+\alpha(p+k)-1,0},$$

where $\mu_{j,k,p,q}$ is defined in Section 3.5.

3.7. Stress-strength reliability

If X_1 follows the strength of the system and X_2 follows stress on that system, then the $P(X_1>X_2)$ measure the chance that the system fails. Generally, the readers are referred to Kotz et al. (2003) for motivations and applications of stress-strength reliability analysis. Let

$X_1 = X_{strength}$~NEBIII$(\alpha_1, \beta_1, \delta)$ and $X_2 = X_{stress}$~NEBIII$(\alpha_2, \beta_2, \delta)$ are two be independent random variables. The stress-strength reliability is defined as

$$R = P(X_1 \geq X_2) = \int_{-\infty}^{\infty} F_2(x) f_1(x)\, dx. \tag{18}$$

Now the pdf of X_1 and cdf of X_2 can be obtained from (12) as

$$f_1(x) = \alpha_1 \beta_1 \delta \sum_{i=j=0}^{\infty} \varepsilon_{i,j} x^{-\delta-1} (1 + x^{-\delta})^{\alpha_1 \beta_1 (i+1) - \beta_1 j - 1}$$

and
$$F_2(x) = \sum_{i=j=0}^{\infty} \binom{i\alpha_2}{j} (1 + x^{-\delta})^{\alpha_2 \beta_2 i - \beta_2 j} \quad \text{for } i\alpha_2 \geq j,$$

where $\varepsilon_{i,j} = (-1)^{i+j} (i+1) \binom{\alpha_1 (i+1) - 1}{j}$. By inserting the values of $f_1(x)$ and $F_2(x)$ in (18), we get

$$R = \alpha_1 \beta_1 \delta \sum_{i=j=0}^{\infty} \rho_{i,j} \int_0^{\infty} x^{-\delta-1} (1 + x^{-\delta})^{\alpha_1 \beta_1 (i+1) + \alpha_2 \beta_2 i - \beta_1 j - \beta_2 j - 1},$$

where $\rho_{i,j} = \binom{i\alpha_2}{j} \varepsilon_{i,j}$.

4. Estimation of Parameters

Maximum Likelihood estimation is considered here to estimate the unknown parameters $(\alpha, \beta$ and $\delta)$ of the NEBIII(α, β, δ) distribution.

Let $X_1, X_2, ..., X_n$ be a random sample of size n from the distribution then the sample likelihood function is given as

$$\prod_{i=1}^{n} f(x; \alpha, \beta, \delta) = (\alpha \beta \delta)^n \prod_{i=1}^{n} \frac{x^{-\delta-1} (1 + x^{-\delta})^{-\alpha\beta - 1} \left\{1 - (1 + x^{-\delta})^{-\beta}\right\}^{\alpha - 1}}{\left[(1 + x^{-\delta})^{-\alpha\beta} + \left\{1 - (1 + x^{-\delta})^{-\beta}\right\}^{\alpha}\right]^2}.$$

The log-likelihood function is

$$L = n \log \alpha + n \log \beta + n \log \delta - (\delta + 1) \sum_{i=1}^{n} \log x - (\alpha\beta + 1) \sum_{i=1}^{n} \log(1 + x^{-\delta})$$
$$+ (\alpha - 1) \sum_{i=1}^{n} \left\{1 - (1 + x^{-\delta})^{-\beta}\right\} - 2 \sum_{i=1}^{n} \log\left[(1 + x^{-\delta})^{-\alpha\beta} + \{1 - (1 + x^{-\delta})^{-\beta}\}^{\alpha}\right]. \tag{19}$$

Now we have to maximize the log-likelihood function given in (19) to get the ML estimates of unknown parameters of the NEBIII(α, β, δ) distribution. For this purpose, we take the first derivative of the log-likelihood equation with respect to parameters.

$$L_\alpha = \frac{n}{\alpha} - \beta \sum_{i=1}^{n} \log(1 + x^{-\delta}) + \sum_{i=1}^{n} \left\{1 - (1 + x^{-\delta})^{-\beta}\right\}$$
$$+ 2 \sum_{i=1}^{n} \frac{\beta \log(1 + x^{-\delta})(1 + x^{-\delta})^{-\alpha\beta} - \log\left\{1 - (1 + x^{-\delta})^{-\beta}\right\}\left\{1 - (1 + x^{-\delta})^{-\beta}\right\}^{\alpha}}{(1 + x^{-\delta})^{-\alpha\beta} + \{1 - (1 + x^{-\delta})^{-\beta}\}^{\alpha}} \tag{20}$$

$$L_\beta = \frac{n}{\beta} - \alpha \sum_{i=1}^{n} \log(1 + x^{-\delta}) + (\alpha - 1) \sum_{i=1}^{n} \frac{\log(1 + x^{-\delta})(1 + x^{-\delta})^{-\beta}}{1 - (1 + x^{-\delta})^{-\beta}}$$

$$+2\sum_{i=1}^{n}\frac{\alpha\log(1+x^{-\delta})(1+x^{-\delta})^{-\alpha\beta}-\alpha\log(1+x^{-\delta})(1+x^{-\delta})^{-\beta}\left\{1-(1+x^{-\delta})^{-\beta}\right\}^{\alpha-1}}{(1+x^{-\delta})^{-\alpha\beta}+\left\{1-(1+x^{-\delta})^{-\beta}\right\}^{\alpha}} \tag{21}$$

$$L_{\delta}=\frac{n}{\delta}-\sum_{i=1}^{n}\log x+(\alpha\beta+1)\sum_{i=1}^{n}\frac{x^{-\delta}\log x^{-\delta}}{1+x^{-\delta}}-(\alpha-1)\sum_{i=1}^{n}\frac{\beta x^{-\delta}\log x(1+x^{-\delta})^{-\beta-1}}{1-(1+x^{-\delta})^{-\beta}}$$

$$-2\sum_{i=1}^{n}\frac{\alpha\beta x^{-\delta}\log x(1+x^{-\delta})^{-\alpha\beta-1}-\alpha\beta x^{-\delta}\log x(1+x^{-\delta})^{-\beta-1}\left\{1-(1+x^{-\delta})^{-\beta}\right\}^{\alpha-1}}{(1+x^{-\delta})^{-\alpha\beta}+\left\{1-(1+x^{-\delta})^{-\beta}\right\}^{\alpha}}. \tag{22}$$

The exact solution of the derived ML estimator for unknown parameters in (19)-(22) is derived equating the above first derivatives to zero which is genuinely not possible. Therefore, it is more appropriate to use the nonlinear optimization algorithms such as a Newton-Raphson algorithm for maximizing the likelihood function numerically. We can use R (optimal function or maxBFGS function) (R Core Team 2018), or MATHEMATICA (Maximize function) (Wolfram Research, Inc. 2010). After application of large sample property of the ML Estimates, MLE $\hat{\theta}$ can be treated as being approximately normal with the mean θ and the variance-covariance matrix equal to the inverse of the expected information matrix, i.e. $\sqrt{n}(\hat{\theta}-\theta)\sim N_k(0,V_n)$ where $V_n=(v_{ij})=I_n^{-1}(\theta),$ $I(\theta)$ is the information matrix then its inverse of matrix is $I^{-1}(\theta)$ provide the variances and covariance's. The variance-covariance matrix $I(\hat{\theta})$ is actually equal to the inverse of the expected information matrix $I^{-1}(\hat{\theta}).$ The elements of the variance-covariance information matrix can easily be obtained by taking the derivatives of the (19)-(22).

To inspect the performance of the new extended Burr-III distribution, we conduct a simulation study by using the Monte Carlos simulation. The simulation is done as follows:

- Data is generated from $F(x)=u,$ where u is uniformly distributed (0, 1).
- $(1.5, 0.5, 0.2)$ and $(3.5, 1.5, 2.5)$ are taken as the true parameter values α,β and $\delta.$
- Simulation is conducted for the sample sizes n = 10, 50, 100, 300 and 500.
- The repetition of the experiment is 10,000 times for each sample size.

Table 1 presents the outcomes of the Monte Carlos simulation study. We evaluate the average of estimated (AE) parameters, bias and mean square errors (MSE) of new developed model. These findings based on the first order asymptotic theory show that the bias and the MSE's decreases toward zero with the increase in sample size as expected.

The observations in Table 1 indicated that the MSE of the ML estimators of α,β and δ decreases and their biases decay towards zero as sample size increases. While the increase in shape parameters, MSE of estimated parameters increases.

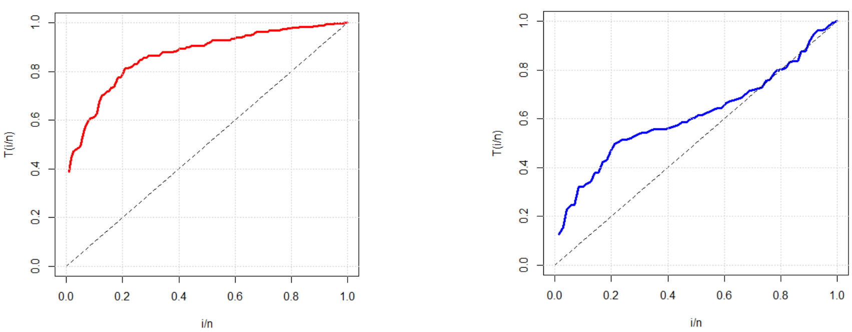
5. Applications

First data set is about fracture toughness from material Alumina (Al_2O_3). The observations of data are available online at http://www.ceramics.nist.gov/srd/summary/ftmain.htm. The second data is about survival time of guinea pigs injected with different doses of tubercle bacilli (Bjerkedal 1960). In this study, we used the data of animals in the same cage that under the same regimen; the data includes 72 observations.

To check the shape of the hazard function of the data sets we have used Aarset's (1987) TTT (Total time on test) plot in Figure 2 indicate that the first and second data sets has increasing hazard rate.

Table 1 Summaries of the estimates for the NEBIII(α,β,δ) distribution

True parameters			Sample size	Parameters	Summaries of Parameters		
α	β	δ			AE	Bias	MSE
1.50	0.50	0.20	10	α	1.76638	0.26637	0.78918
				β	0.70418	0.20418	13.9490
				δ	0.25387	0.05387	1.22024
			50	α	1.54674	0.04673	0.07661
				β	0.51900	0.01899	0.01150
				δ	0.20693	0.00694	0.00156
			100	α	1.52486	0.02485	0.03529
				β	0.51019	0.01019	0.00494
				δ	0.20369	0.00369	0.00069
			300	α	1.50707	0.00707	0.01100
				β	0.50293	0.00293	0.00152
				δ	0.20105	0.00105	0.00022
			500	α	1.50412	0.00412	0.00645
				β	0.50126	0.00126	0.00090
				δ	0.20051	0.00051	0.00012
0.50	1.50	0.90	10	α	0.58579	0.08579	0.08674
				β	1.57676	0.07676	0.12920
				δ	1.06533	0.16532	0.31634
			50	α	0.51377	0.01377	0.00812
				β	1.51619	0.01619	0.02902
				δ	0.92592	0.02592	0.02758
			100	α	0.50685	0.00685	0.00381
				β	1.50939	0.00938	0.01402
				δ	0.91270	0.01270	0.01291
			300	α	0.50279	0.00279	0.00123
				β	1.50344	0.00344	0.00473
				δ	0.90521	0.00521	0.00416
			500	α	0.50161	0.00161	0.00071
				β	1.50191	0.00191	0.00281
				δ	0.90309	0.00309	0.00243



(a) (b)

Figure 2 TTT-plots for the (a) Data set I and (b) Data set II

Descriptive statistics of the data sets are provided in Table 2.

Table 2 Descriptive Statistics for Data I and Data II

	Min	Q_1	Median	Q_3	Mean	Max
Data I	1.680	3.850	4.380	5.000	4.325	6.810
Data II	12.000	54.500	70.000	109.500	96.420	376.000

By using these data sets, we have made comparison for the new extended Burr-III (NEBIII) distribution with the Kumaraswamy Burr-III (KBIII), transmuted modified Burr-III (TMBIII), Marshall Olkin Modified Burr-III (MOMBIII), exponentiated modified Burr-III (MBIII), modified Burr-III (MBIII) and Burr-III (BIII) distribution. We use a number of goodness of fit measures to compare the new model with other existing models such as the log likelihood function (-2ℓ), Akaike Information Criterion (AIC), corrected version of Akaike Information Criterion (AICc), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), Hanna-Quinn Information Criterion (HQIC), Kolmogrov-Smirnov (K-S), Anderson Darling (A^*) and Cramer-Von Mises (W^*). The pdf of other existing models are stated below.

- Kumaraswamy Burr-III distribution (Behairy et al. 2016)

  $g(x;\alpha,\beta,\gamma,\lambda)=\alpha\beta\gamma\lambda x^{-\gamma-1}(1+x^{-\gamma})^{-\alpha\lambda-1}(1-(1+x^{-\gamma})^{-\alpha\lambda})^{\beta-1}$ for $x>0$

- Transmuted modified Burr-III distribution (Ali and Ahmed 2015)

  $g(x;\alpha,\beta,\gamma,\lambda)=\alpha\beta x^{-\beta-1}(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}-1}(1+\lambda-2\lambda(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}})$ for $x>0$

- Marshall Olkin modified Burr-III distribution (Haq et al. 2020)

  $$g(x;\alpha,\beta,\gamma,\lambda)=\frac{\alpha\beta x^{-\beta-1}(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}-1}}{\left[\lambda+(1-\lambda)(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}}\right]^2} \text{ for } x>0$$

- Exponentiated modified Burr-III distribution

  $g(x;\alpha,\beta,\gamma,\lambda)=\alpha\beta\lambda x^{-\beta-1}(1+\gamma x^{-\beta})^{-\frac{\alpha\lambda}{\gamma}-1}$ for $x>0$

- Modified Burr-III distribution (Ali et al. 2015)

  $g(x;\alpha,\beta,\gamma)=\alpha\beta x^{-\beta-1}(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}-1}$ for $x>0$

Tables 3-6 provide the estimated parameters and goodness of fit measures of two real life data sets. It is obvious from the tables that the goodness of fit measures such as AIC, BIC, AICc, CAIC, HQIC, K-S, A^* and W^* of the new developed NEBIII distribution are less than that of the KBIII, TMBIII, MOMBIII, EMBIII, MBIII and BIII distribution, therefore, considered as a best fitted model. Figures 3 and 4 show the histogram of data and the estimated pdf curves and the estimated and the empirical cdf curves. It is evident from the tables and figures that the NEBIII distribution provides better fit as compared to other existing models considered here.

Table 3 ML estimates of the parameters for Data I

Models	Estimates (Std. Error)			
NEBIII(α,β,δ)	45.4010	1.1741	0.1485	-
	(0.0001)	(0.0148)	(0.0110)	
KBIII$(\alpha,\beta,\gamma,\lambda)$	6.5226	585.3100	0.8146	3.8814
	(0.1158)	(0.00002)	(0.0484)	(0.1947)
TMBIII$(\alpha,\beta,\gamma,\lambda)$	1030.500	5.2171	891.8400	−0.6347
	(4.6730)	(0.2934)	(4.7428)	(0.1894)
MOMBIII$(\alpha,\beta,\gamma,\lambda)$	115.0400	7.0535	475.7100	267.8600
	(1.6542)	(0.5064)	(3.0509)	(4.4859)
EMBIII$(\alpha,\beta,\gamma,\lambda)$	32.1110	4.8543	629.4200	30.1120
	(0.5824)	(0.2852)	(3.2430)	(1.3281)
MBIII(α,β,γ)	816.0100	4.4760	531.9200	-
	(3.6708)	(0.2750)	(3.1541)	
BIII(β,δ)	51.8940	3.0581	-	
	(1.1185)	(0.1799)		

Table 4 Some goodness of fit measures for Data I

Models	-2ℓ	AIC	BIC	AICc	CAIC	HQIC	K-S	A^*	W^*
NEBIII(α,β,δ)	356.8	362.8	371.1	362.9	366.9	366.2	0.081	1.912	0.185
KBIII$(\alpha,\beta,\gamma,\lambda)$	359.4	368.4	382.5	367.8	375.1	375.9	0.237	2.679	0.291
TMBIII$(\alpha,\beta,\gamma,\lambda)$	368.0	376.0	387.1	376.3	381.5	380.5	0.121	3.167	0.472
MOMBIII$(\alpha,\beta,\gamma,\lambda)$	359.2	368.2	382.3	367.5	374.7	375.7	0.230	2.666	0.283
EMBIII$(\alpha,\beta,\gamma,\lambda)$	374.7	382.7	389.1	382.9	385.9	384.1	0.133	3.916	0.607
MBIII(α,β,γ)	375.2	381.2	389.5	381.5	385.3	384.5	0.143	4.173	0.680
BIII(β,δ)	419.5	423.5	429.1	423.6	426.3	425.7	0.196	6.110	1.429

Table 5 ML estimates of the parameters for Data II

Models	Estimates (Std. Error)			
NEBIII(α,β,δ)	4.9190	4.5707	0.4210	-
	(5.8294)	(8.1284)	(0.4473)	
KBIII$(\alpha,\beta,\gamma,\lambda)$	12.3010	8.1828	0.6587	3.5953
	(0.4444)	(7.2531)	(0.1888)	(1.5205)
TMBIII$(\alpha,\beta,\gamma,\lambda)$	379.6700	1.7108	198.6500	−0.8171
	(6.1881)	(0.0481)	(1.2265)	(0.2166)
MOMBIII$(\alpha,\beta,\gamma,\lambda)$	364.7100	2.5819	3.7090	183.5900
	(1.0063)	(0.2743)	(2.0545)	(1.6860)
EMBIII$(\alpha,\beta,\gamma,\lambda)$	40.9750	1.7886	514.8100	39.8050
	(0.9458)	(0.0791)	(0.3059)	(0.9188)
MBIII(α,β,γ)	399.5600	1.5239	89.2210	-
	(2.5152)	(0.1188)	(0.9395)	
BIII(β,δ)	240.8500	1.3956	-	
	(1.0861)	(0.1109)		

Table 6 Some goodness of fit measures for Data II

Models	-2ℓ	AIC	BIC	AICc	CAIC	HQIC	K-S	A^*	W^*
NEBIII(α,β,δ)	761.9	767.9	774.7	768.3	771.3	770.6	0.088	0.501	0.085
KBIII$(\alpha,\beta,\gamma,\lambda)$	763.1	771.1	780.2	771.7	775.6	774.7	0.109	0.983	0.136
TMBIII$(\alpha,\beta,\gamma,\lambda)$	765.9	773.9	782.9	774.5	778.4	777.5	0.128	1.125	0.196
MOMBIII$(\alpha,\beta,\gamma,\lambda)$	763.9	768.9	781.0	770.6	775.5	775.2	0.101	0.562	0.097
EMBIII$(\alpha,\beta,\gamma,\lambda)$	765.6	773.6	782.6	774.2	778.1	777.2	0.108	0.835	0.146
MBIII(α,β,γ)	769.9	775.9	782.7	776.3	779.3	778.6	0.366	1.284	1.095
BIII(β,δ)	774.8	778.8	783.4	779.0	781.1	780.6	0.145	1.727	0.285

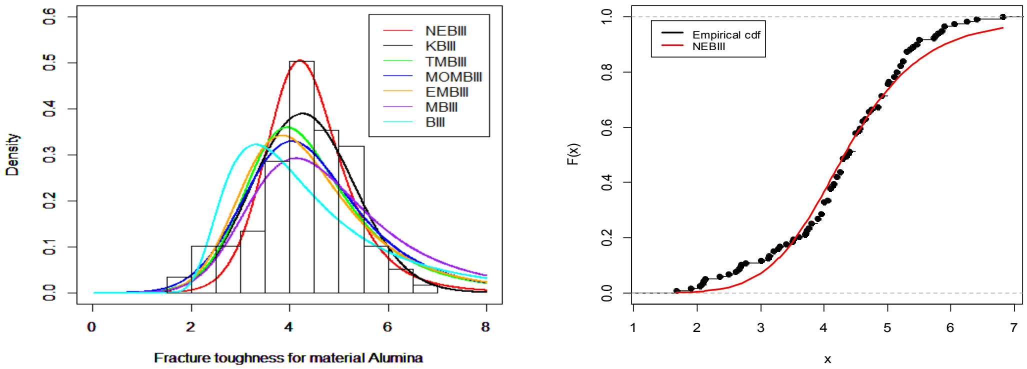


Figure 3 The fitted pdf of NEBIII model and other models and cdf of NEBIII model on Data I

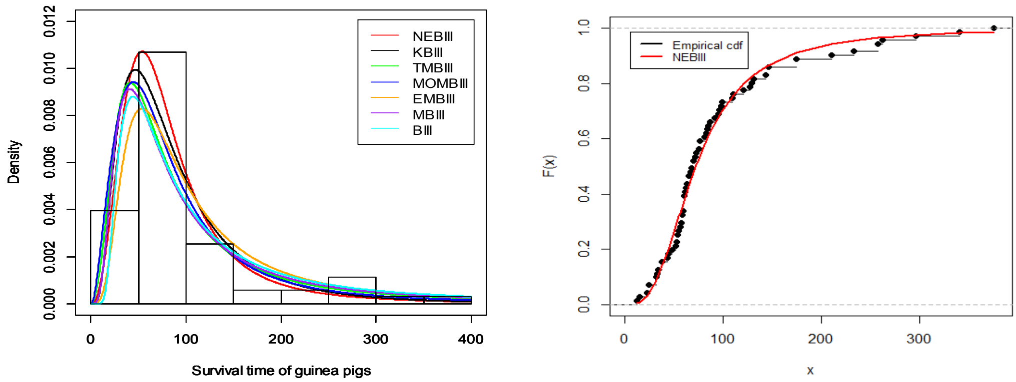


Figure 4 The fitted pdf of NEBIII model and other models and cdf of NEBIII model on Data II

6. Conclusions

A new lifetime distribution is introduced and provided some of its mathematical and statistical properties including the quantile function, entropy, moments, moment generating function, order statistics, power moments and stress-strength reliability. The maximum likelihood method is used to estimate the model parameters. For different parameter and sample sizes, a simulation study is performed to evaluate the performance of the MLEs of NEBIII parameters. Empirical results show that the two real data applications, that the NEBIII distribution can provide better fits than some other well-known other extended Burr-III models such as the BIII, KBIII, MBIII, EMBIII, MOMBIII and TMBIII distributions.

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