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Bayesian Estimation and Application of Shifted Exponential Mixture Distribution

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Abstract

In this study, Bayesian parameter estimation for the shifted exponential mixture model is conducted by using informative priors under squared error loss function (SELF), weighted loss function (WLF) and quadratic loss function (QLF). Properties of the proposed Bayes estimators (BEs) are highlighted through simulation study. One and two sample prediction bounds are obtained. Proposed mixture model is applied to a real life example.

Keywords: Heterogeneous data, informative prior, one sample prediction, two sample prediction.

1. Introduction

Exponential failure distributions occur frequently in practice and are widely used in mixed population problems where data arise from times to failure of units (e.g. patients, machines, components etc.). The exponential distribution finds applications in queuing theory and reliability studies. In queuing theory, it is used to model the time elapsed between customer arrivals at the terminal and in reliability it is being used to model the failure time of electronic components. In many life testing problems, shifted exponential (two parameters exponential) distribution is considered more appropriate for fitting life test data than one parameter distribution. In real life, failure does not always occur at its initial point. Hence such situations are best represented by shifted exponential distribution which contains a location parameter. So in life testing shifted exponential distribution is often required to provide an adequate representation of the observed results.

The shifted exponential distribution for a random variable X is formulated as

$$f(x;\lambda,\mu)=\lambda e^{-\lambda(x-\mu)},\quad x>\mu \quad \text{and} \quad \lambda>0, \tag{1}$$

where λ is the scale parameter and μ is the location parameter. The location parameter is referred as a threshold or guarantee time parameter before which no failure occurs and the scale parameter represents the mean life time of an object.

A number of authors have considered shifted exponential distribution in their work. Epstein (1960) has addressed the problem of estimation of shifted exponential distribution from censored samples. Wu and Yu (2008) proposed a simultaneous inferences of extreme populations for shifted exponential distribution based on doubly type-II censored samples. Krishnamoorthy et al. (2007) took into account the problem of hypothesis testing and interval estimation of the reliability parameter in stress-strength model for shifted exponential distribution. Balakrishnan and Basu (1995) had given excellent review on the properties, genesis and characterization of the distribution in their book “The exponential distribution: theory, methods and applications”. Sae-ung and Lertprapai (2010) made comparison of the scale parameter of a shifted exponential distribution based on multiple criteria decision making. Jewell (1982) provided maximum likelihood estimates of mixture of exponential distributions using EM algorithm. Raqab and Madi (2005) had taken into account two parameter generalized exponential distribution and importance sampling is used to estimate the parameters, as well as the reliability function, and the Gibbs and Metropolis samplers data sets are used to predict the behavior of further observations from the distribution. Mohamed et al. (2014) used heterogeneous population which is represented by a mixture of two generalized exponential distributions. Mohamed and Leonard (1996) had discussed the Bayesian estimation for the shifted exponential distributions.

In this paper Bayesian analysis of the shifted exponential mixture model is conducted using type-I censoring scheme assuming informative priors. The shifted exponential mixture model is described in Section 2. Likelihood function is constructed in Section 3. Section 4 consists of Bayes estimators and posterior risks under squared error loss function (SELF), weighted loss function (WLF) and quadratic loss function (QLF) using informative priors. Bayesian prediction is conducted in Section 5. Simulation analysis is provided in Section 6. A real data application is presented in Section 7. Finally in Section 8, conclusion is presented.

2. The Shifted Exponential Mixture Model

Suppose that a random variable X takes n independent values in a sample space Ω, and X is said to have a finite k component mixture distribution with known specified functional form but varying unknown parameters and mixing weights if its distribution can be represented by a following probability density function (pdf)

$$f(x)=\sum_{j=1}^{k} p_j f_j(x\mid\Theta),\quad p_j\in(0,1),\ j\in(1,k),\ \sum_{j=1}^{k} p_j=1, \tag{2}$$

where Θ is a complete set of parameters of the mixture model. The mixture model and its corresponding survival function, when shifted exponential distribution is assumed for the two-component of the mixture model are as under

$$f_m(x)=\sum_{j=1}^{2} p_j\lambda_j e^{-\lambda_j(x-\mu_j)}, \tag{3}$$

$$F_m(x) \ =\sum_{j=1}^{2} p_j\, e^{-\lambda_j(x-\mu_j)}. \tag{4}$$

3. Sampling and Likelihood Function

It is assumed that lifetime of an object is independently and identically distributed as shifted exponential random variable. It is common practice in the Bayesian analysis to consider location parameter known. Dozen of references exist on the subject. Some examples include Soland (1968), Zanakis (1979), Al-Hussaini et al. (2001), Panaitescu et al. (2010), Kundu and Howlader (2010) etc. For simplicity it is assumed here that location parameter μ_j is known and equal, thus $\mu_1 = \mu_2 = \mu.$

Experimenter frequently ends up experiments due to restrictions on available time or cost on reaching a predetermined length of time or predetermined number of failures. Censored sampling with fixed test termination time is known as type-I right censoring. The Likelihood function for mixture model under type-I censoring given by Mendenhall and Hader (1958), there are many practical situations in which the failed objects can be pointed out easily as subset of subpopulation-I or subpopulation-II. Out of ‘ r ’ units, suppose r_1 and r_2 units belong to subpopulation-I or subpopulation-II respectively and such that $r = r_1 + r_2$.

$$L\langle\Theta|\mathrm{x}\rangle \propto \left\{\prod_{j=1}^{r_1} p_1 f_1\left(x_{1j}\right)\right\}\left\{\prod_{j=1}^{r_2} p_2 f_2\left(x_{2j}\right)\right\}\left[R\left(t_0\right)\right]^{n-r}$$

$$\propto \left(\prod_{j=1}^{r_1} p_1\lambda_1 e^{-\lambda_1(x-\mu)}\right)\left(\prod_{j=1}^{r_2} p_2\lambda_2 e^{-\lambda_2(x-\mu)}\right)\left(p_1 e^{-\lambda_1(T-\mu)} + p_2 e^{-\lambda_2(T-\mu)}\right)^{n-r}$$

$$\propto \sum_{k=0}^{n-r}\binom{n-r}{k} p_1^{n-r_2-k} p_2^{r_2+k}\lambda_1^{r_1} e^{-\lambda_1\left\{\sum_{j=1}^{r_1}(x_{1j}-\mu)+(n-r-k)(T-\mu)\right\}}\lambda_2^{r_2} e^{-\lambda_2\left\{\sum(x_{2j}-\mu)+k(T-\mu)\right\}}, \quad (5)$$

where $X = (x_{11}, x_{12}, ..., x_{1r_1}, x_{21}, x_{22}, ..., x_{2r_2})$ are the observed failure times for the uncensored observations, $\Theta = (\lambda_1, \lambda_2, p_1)$, $R(t_0) = 1 - F(t)$, $p_1 + p_2 = 1$ and T is the fixed test termination time under type-1 right censoring.

4. Bayesian Estimation Assuming Informative Priors

Exponential prior which is also a conjugate prior and squared Rayleigh (SR) prior are taken as informative priors for the Bayesian estimation of the parameters of the mixture model.

4.1. The posterior distribution assuming exponential prior

The parameters λ_i and p_1 are assumed to be independent. The parameters λ_i assume exponential distribution as a prior with hyperparameter κ_i and its pdf may be written as

$$g(\lambda_i) = \kappa_i\, e^{-\kappa_i\lambda_i}, \quad \lambda_i > 0, i = 1, 2, \quad (6)$$

and p_1 assumes an uniform prior i.e. $g(p_1) = 1$. The joint posterior distribution of the parameters λ_i and p_1 is

$$\pi(\Theta\,|\,\mathrm{x}) = \Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k} p_1^{(n-r_2-k)} p_2^{r_2+k}\prod_{i=1}^{2}\lambda_i^{r_i} e^{-\lambda_i\gamma_i}, \ 0 < \lambda_i < \infty, \ i = 1, 2, \ 0 < p_1 < 1, \quad (7)$$

where $\gamma_1 = \sum_{j=1}^{r_1}(x_{1j}-\mu)+(n-r-k)(T-\mu)+\kappa_1$, $\gamma_2 = \sum_{j=1}^{r_2}(x_{2j}-\mu)+k(T-\mu)+\kappa_2$, and

$\Lambda = \sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\dfrac{\Gamma(r_1+A)}{\gamma_1^{r_1+A}}\dfrac{\Gamma(r_2+A)}{\gamma_2^{r_2+A}}$, here $\mathrm{A}=1$.

Marginal posterior distributions of λ_i and p_1 can be derived by integrating the nuisance parameters, such that

$$p(\lambda_i \mid \mathrm{x}) = \Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-r_2-k+1, r_2+k+1)\left\{\prod_{j\neq i}\frac{\Gamma(r_j+A)}{\gamma_j^{r_j+A}}\right\}\lambda_i^{r_i}e^{-\lambda_i\gamma_i}\ ,\ 0<\lambda_i<\infty,\ i=1,2, \tag{8}$$

$$p(p_1 \mid x) = \Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}p_1^{n-r_2-k}p_2^{r_2+k}\left\{\prod_{i=1}^{2}\frac{\Gamma(r_j+A)}{\gamma_j^{r_j+A}}\right\},\quad 0<p_1<1\,. \tag{9}$$

1) Bayes estimators and their posterior risks under "SELF"

The squared error loss function (SELF) is suggested by Legendre (1806) and is defined as $L(\lambda,\hat{\lambda})=(\lambda-\hat{\lambda})^2$. The Bayes estimator and posterior risk under SELF are $\hat{\lambda}=E_{\lambda|x}(\lambda)$ and $\rho(\hat{\lambda})=E_{\lambda|x}(\lambda^2)-\left[E_{\lambda|x}(\lambda)\right]^2$, respectively.

Following BEs of λ_i and p_1 are obtained under SELF.

$$\hat{\lambda}_{i(S)} = \Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\frac{\Gamma(r_i+\mathrm{A}+1)}{\gamma_i^{r_i+\mathrm{A}+1}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+\mathrm{A})}{\gamma_j^{r_j+\mathrm{A}}}\right\},\ i=1,2, \tag{10}$$

$$\hat{p}_{1(S)} = \Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+2, r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+\mathrm{A})}{\gamma_i^{r_i+\mathrm{A}}}\right\}, \tag{11}$$

and the expressions for Bayes PRs for λ_i and p_1 are

$$\rho(\hat{\lambda}_{i(S)})=\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\frac{\Gamma(r_i+\mathrm{A}+2)}{\gamma_i^{r_i+\mathrm{A}+2}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+\mathrm{A})}{\gamma_j^{r_j+\mathrm{A}}}\right\}-\left[\hat{\lambda}_{i(S)}\right]^2, \tag{12}$$

$$\rho(\hat{p}_{1(S)}) = \Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+3, r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+\mathrm{A})}{\gamma_i^{r_i+\mathrm{A}}}\right\}-\left[\hat{p}_{1(S)}\right]^2. \tag{13}$$

2) Bayes estimators and their posterior risks under "WLF"

Bayes estimators under weighted loss function (WLF) can be evaluated from $\hat{\lambda}=\left[E(\lambda)^{-1}\right]^{-1}$ and posterior risk is the difference of BEs obtained under SELF and WLF

$$\hat{\lambda}_{i(W)} = \left[\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\frac{\Gamma(r_i+\phi)}{\gamma_i^{r_i+\phi}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+\mathrm{A})}{\gamma_j^{r_j+\mathrm{A}}}\right\}\right]^{-1},\ i=1,2, \tag{14}$$

$$\hat{p}_{1(W)}=\left[\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2,r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+\mathrm{A})}{\gamma_i^{r_i+\mathrm{A}}}\right\}\right]^{-1},\ \mathrm{A}=1,\ \phi=0. \quad (15)$$

Posterior risks can be attained from the following expressions

$$\rho(\hat{\lambda}_{i(W)})=\hat{\lambda}_{i(S)}-\hat{\lambda}_{i(W)},\ i=1,2 \text{ and } \rho(\hat{p}_{1(W)})=\hat{p}_{1(S)}-\hat{p}_{1(W)}.$$

3) Bayes estimators and their posterior risks under "QLF"

Under quadratic loss function (QLF), Bayes estimators can be attained from $\hat{\lambda}=\dfrac{E(\lambda^{-1})}{E(\lambda^{-2})}$ and posterior risks can be evaluated from $\rho(\hat{\lambda})=1-\dfrac{\left[E(\lambda^{(-1)})\right]^2}{E(\lambda^{(-2)})}$, so

$$\hat{\lambda}_{i(Q)}=\frac{\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2,r_2+k+1)\frac{\Gamma(r_i+\phi)}{\gamma_i^{r_i+\phi}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+\mathrm{A})}{\gamma_j^{r_j+\mathrm{A}}}\right\}}{\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2,r_2+k+1)\frac{\Gamma(r_i+\phi-1)}{\gamma_i^{r_i+\phi-1}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+\mathrm{A})}{\gamma_j^{r_j+\mathrm{A}}}\right\}},\ i=1,2, \quad (16)$$

$$\hat{p}_{1(Q)}=\frac{\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2,r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+\mathrm{A})}{\gamma_i^{r_i+\mathrm{A}}}\right\}}{\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2-1,r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+\mathrm{A})}{\gamma_i^{r_i+\mathrm{A}}}\right\}}. \quad (17)$$

The expressions for the Bayes posterior risks are given by

$$\rho(\hat{\lambda}_{i(Q)})=1-\frac{\left[\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2,r_2+k+1)\frac{\Gamma(r_i+\phi)}{\gamma_i^{r_i+\phi}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+\mathrm{A})}{\gamma_j^{r_j+\mathrm{A}}}\right\}\right]^2}{\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2,r_2+k+1)\frac{\Gamma(r_i+\phi-1)}{\gamma_i^{r_i+\phi-1}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+\mathrm{A})}{\gamma_j^{r_j+\mathrm{A}}}\right\}}, \quad (18)$$

$$\rho(\hat{p}_{1(Q)})=1-\frac{\left[\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2,r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+\mathrm{A})}{\gamma_i^{r_i+\mathrm{A}}}\right\}\right]^2}{\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2-1,r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+\mathrm{A})}{\gamma_i^{r_i+\mathrm{A}}}\right\}}. \quad (19)$$

4.2. The posterior distribution using squared Rayleigh (SR) prior

It is assumed that parameters λ_1,λ_2 follow a squared Rayleigh (SR) prior with pdf $g(\lambda_1,\lambda_2\,|\,m_1,m_2)\propto e^{-\left(\frac{\theta_1}{2m_1^2}+\frac{\theta_2}{2m_2^2}\right)}$ and p_1 follows a uniform prior. Assuming independence of the priors, posterior distribution of λ_1,λ_2 and p_1 is obtained as

$$\pi(\Theta\,|\,\mathrm{x})=\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}p_1^{(n-r_2-k)}p_2^{r_2+k}\prod_{i=1}^{2}\lambda_i^{r_i}e^{-\lambda_i\gamma_i},\quad 0<\lambda_i<\infty,\ i=1,2,\ 0<p_1<1, \quad (20)$$

where $\gamma_1 = \sum_{j=1}^{r_1}(x_{1j}-\mu)+(n-r-k)(T-\mu)+\frac{1}{2m_1^2}$, $\gamma_2 = \sum_{j=1}^{r_2}(x_{2j}-\mu)+k(T-\mu)+\frac{1}{2m_2^2}$ and

$\Lambda = \sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\frac{\Gamma(r_1+1)}{\gamma_1^{r_1+1}}\frac{\Gamma(r_2+1)}{\gamma_2^{r_2+1}}$.

Marginal posterior distributions of λ_i and p_1 can be obtained as

$$p(\lambda_i \mid x) = \Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-r_2-k+1, r_2+k+1)\left\{\prod_{j\neq i}\frac{\Gamma(r_j+1)}{\gamma_j^{r_j+1}}\right\}\lambda_i^{r_i}e^{-\lambda_i\gamma_i},\ 0<\lambda_i<\infty,\ i=1,2, \tag{21}$$

$$p(p_1 \mid x) = \Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}p_1^{n-r_2-k}p_2^{r_2+k}\left\{\prod_{i=1}^{2}\frac{\Gamma(r_j+1)}{\gamma_j^{r_j+1}}\right\},\ 0<p_1<1. \tag{22}$$

1) Bayes estimators and their posterior risks under "SELF"

Bayes estimators of λ_i and p_1 under squared error loss function (SELF) are

$$\hat{\lambda}_{i(S)} = \Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\frac{\Gamma(r_i+2)}{\gamma_i^{r_i+2}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+1)}{\gamma_j^{r_j+1}}\right\},\ i=1,2, \tag{23}$$

$$\hat{p}_{1(S)} = \Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+2, r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+1)}{\gamma_i^{r_i+1}}\right\}, \tag{24}$$

and the expressions for Bayes posterior risks for λ_i and p_1 are

$$\rho(\hat{\lambda}_{i(S)}) = \Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\frac{\Gamma(r_i+3)}{\gamma_i^{r_i+3}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+1)}{\gamma_j^{r_j+1}}\right\}-\left[\hat{\lambda}_{i(S)}\right]^2, \tag{25}$$

$$\rho(\hat{p}_{1(S)}) = \Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+3, r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+1)}{\gamma_i^{r_i+1}}\right\}-\left[\hat{p}_{1(S)}\right]^2. \tag{26}$$

2) Bayes estimators and their posterior risks under "WLF"

Bayes estimators under weighted loss function (WLF) can be obtained as

$$\hat{\lambda}_{i(W)} = \left[\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\frac{\Gamma(r_i)}{\gamma_i^{r_i}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+1)}{\gamma_j^{r_j+1}}\right\}\right]^{-1},\ i=1,2, \tag{27}$$

$$\hat{p}_{1(W)} = \left[\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2, r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+1)}{\gamma_i^{r_i+1}}\right\}\right]^{-1}, \tag{28}$$

and posterior risks can be attained from the following expressions

$$\rho(\hat{\lambda}_{i(W)}) = \hat{\lambda}_{i(S)} - \hat{\lambda}_{i(W)} \quad \text{and} \quad \rho(\hat{p}_{1(W)}) = \hat{p}_{1(S)} - \hat{p}_{1(W)}.$$

3) Bayes estimators and their posterior risks under "QLF"

Bayes estimators are obtained under the quadratic loss function (QLF) as

$$\hat{\lambda}_{i(Q)} = \frac{\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2, r_2+k+1)\frac{\Gamma(r_i)}{\gamma_i^{r_i}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+1)}{\gamma_j^{r_j+1}}\right\}}{\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2, r_2+k+1)\frac{\Gamma(r_i-1)}{\gamma_i^{r_i-1}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+1)}{\gamma_j^{r_j+1}}\right\}}, \ i=1,2, \tag{29}$$

$$\hat{p}_{1(Q)} = \frac{\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2, r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+1)}{\gamma_i^{r_i+1}}\right\}}{\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2-1, r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+1)}{\gamma_i^{r_i+1}}\right\}}. \tag{30}$$

The expressions for the Bayes posterior risks are given by

$$\rho(\hat{\lambda}_{i(Q)}) = 1 - \frac{\left[\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2, r_2+k+1)\frac{\Gamma(r_i)}{\gamma_i^{r_i}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+1)}{\gamma_j^{r_j+1}}\right\}\right]^2}{\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2, r_2+k+1)\frac{\Gamma(r_i-1)}{\gamma_i^{r_i-1}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+1)}{\gamma_j^{r_j+1}}\right\}}, \tag{31}$$

$$\rho(\hat{p}_{1(Q)}) = 1 - \frac{\left[\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2, r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+1)}{\gamma_i^{r_i+1}}\right\}\right]^2}{\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2-1, r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+1)}{\gamma_i^{r_i+1}}\right\}}. \tag{32}$$

5. Bayesian Prediction Using Exponential Prior

It is often desirable to predict future sample based on the current sample. Problem of prediction is being widely applied due to its increasing importance in analysis.

5.1. Bayesian one-sample predictions

Let $x_1 < ... < x_r$ be the informative sample and $x_{r+1} < ... < x_n$ represent the future sample. It is assumed that the both samples are from the mixture model given in (2). Let $Y_s = x_{r+s}$, $s = 1,...,n-r$. The conditional density function of the Y_s, $s = 1,...,n-r$, given by David (1981) is

$$\begin{aligned} f(y_s \mid \Theta) &= D(s)\left[R(x_r)-R(y_s)\right]^{s-1}\left[R(y_s)\right]^{n-r-s}\left[R(x_r)\right]^{-(n-r)} f(y_s) \\ &= D(s)\sum_{j=0}^{s-1}(-1)^j\binom{s-1}{j}\left[R(y_s)\right]^{\xi}\left[R(x_r)\right]^{\phi} f(y_s) \\ &= D(s)\sum_{j=0}^{s-1}\sum_{j_1=0}^{\xi}\sum_{j_2=0}^{\phi}(-1)^j\binom{s-1}{j}\binom{\xi}{j_1}\binom{\phi}{j_2} p_1^{\xi-j_1+\phi-j_2} p_2^{j_1+j_2} e^{-\lambda_1[(\xi-j_1)(y_s-\mu)+(\phi-j_2)(T-\mu)]} \\ &\quad e^{-\lambda_2[j_1(y_s-\mu)+j_2(T-\mu)]}\left[p_1\lambda_1 e^{-\lambda_1(y_s-\mu)} + p_2\lambda_2 e^{-\lambda_2(y_s-\mu)}\right], \end{aligned} \tag{33}$$

where $\xi = n-r-s+j$, $\phi = s-1-j-n+r$. The Bayesian predictive density function is defined by

$$f^*(y_s \mid x) = \int_0^1\int_0^\infty\int_0^\infty f(y_s \mid \Theta)\,\pi(\Theta \mid x)d\Theta,\ \ y_s > x_r, \tag{34}$$

where $\pi(\Theta \mid x),\ f(y_s \mid \Theta)$ are given in (7) and (32), respectively.

$$f^*(y_s \mid x) = \Lambda^{-1}D(s)\sum_{j=0}^{s-1}\sum_{j_1=0}^{\xi}\sum_{j_2=0}^{\phi}\sum_{k=0}^{n-r}(-1)^j\binom{s-1}{j}\binom{\xi}{j_1}\binom{\phi}{j_2}\binom{n-r}{k}\prod_{i=1}^{2}\lambda_i^{r_i}p_1^{n-r_2-k}p_2^{r_2+k}p_1^{\xi-j_1+\phi-j_2}p_2^{j_1+j_2}$$
$$e^{-\lambda_1[(\xi-j_1)(y_s-\mu)+(\phi-j_2)(T-\mu)+\gamma_1]}e^{-\lambda_2[j_1(y_s-\mu)+j_2(T-\mu)+\gamma_2]}\left[p_1\lambda_1e^{-\lambda_1(y_s-\mu)}+p_2\lambda_2e^{-\lambda_2(y_s-\mu)}\right]$$
$$= \Lambda^{-1}D(s)\sum_{j=0}^{s-1}\sum_{j_1=0}^{\xi}\sum_{j_2=0}^{\phi}\sum_{k=0}^{n-r}(-1)^j\binom{s-1}{j}\binom{\xi}{j_1}\binom{\phi}{j_2}\binom{n-r}{k}\left[B(\delta_1+\delta_2+\delta_3+2, j_1+j_2+\delta_4+1)\right.$$
$$\left.\times\frac{\Gamma(r_1+2)\Gamma(r_2+1)}{\xi_1^{*r_1+2}\xi_2^{r_2+1}}+\ B(\delta_1+\delta_2+\delta_3+1, j_1+j_2+\delta_4+2)\frac{\Gamma(r_1+1)\Gamma(r_2+2)}{\xi_1^{r_1+1}\xi_2^{*r_2+2}}\right]$$
$$= \Lambda^{-1}D(s)\sum^{*}k^*\left[\frac{(\delta_1+\delta_2+\delta_3+1)(r_1+1)}{\xi_1^{*r_1+2}\xi_2^{r_2+1}}+\frac{(j_1+j_2+\delta_4+1)(r_2+1)}{\xi_1^{r_1+1}\xi_2^{*r_2+2}}\right], \tag{35}$$

where Λ is the normalizing constant satisfying $\int_0^\infty f^*(y_s|x)dy_s = 1$,

$$\sum^{*}=\sum_{j=0}^{s-1}\sum_{j_1=0}^{\xi}\sum_{j_2=0}^{\phi}(-1)^j\binom{s-1}{j}\binom{\xi}{j_1}\binom{\phi}{j_2},\ \delta_1=\xi-j_1,\delta_2=\phi-j_2,\delta_3=n-r_2-k,\ \delta_4=r_2+k,$$

$$k^*=\left[\frac{\Gamma(\delta_1+\delta_2+\delta_3+1)\Gamma(\delta_4+j_1+j_2+1)\Gamma(r_1+1)\Gamma(r_2+1)}{\Gamma(\delta_1+\delta_2+\delta_3+\delta_4+j_1+j_2+3)}\right],$$

$$\xi_1=\delta_1(y_s-\mu)+\delta_2(T-\mu)+\gamma_1,\quad \xi_1^*=(1+\delta_1)(y_s-\mu)+(\phi-j_2)(T-\mu)+\gamma_1,$$
$$\xi_2=j_1(y_s-\mu)+j_2(T-\mu)+\gamma_2,\quad \xi_2^*=(1+j_1)(y_s-\mu)+j_2(T-\mu)+\gamma_2.$$

Predictive survival function thus may be obtained as

$$P(Y_s \ge v \mid x) = \int_v^\infty f^*(y_s \mid x)dy_s. \tag{36}$$

A $100\tau\%$ prediction interval for Y_s is then given by

$$P[L(x) < Y_s < U(x)] = \tau,$$

where $L(x)$ and $U(x)$ are obtained by solving the following two equations, respectively.

$$P(Y_s > L(x)) = \frac{1+\tau}{2}\ \text{ and }\ P(Y_s > U(x)) = \frac{1-\tau}{2}.$$

5.2. Bayesian two-sample prediction

Suppose a sample of size n is taken from a population whose density function is defined in (2) for life testing experiment. Now a future sample of size m, independent of past sample of size n is obtained from the same population. Let Y_s be the ordered lifetime of a s^{th} component to fail in a future sample of size m, $1 \le s \le m$. The s^{th} order statistic represents the life length of a $(m-s+1)$ out of m components. Then density function of Y_s is

$$h(y_s \mid \Theta) \propto \left[1-R(y_s)\right]^{s-1}\left[R(y_s)\right]^{m-s} f(y_s)$$

$$=\sum_{j2=0}^{s-1}(-1)^{j_2}\binom{s-1}{j_2}\left[R(y_s)\right]^{m-s+j_2} f(y_s)$$

$$=\sum_{j2=0}^{s-1}\sum_{j_3=0}^{m-s+j_2}(-1)^{j_2}\binom{s-1}{j_2}\binom{m-s+j_2}{j_3}p_1^{\delta_3}p_2^{j_3}e^{-\lambda_1[(y_s-\mu)\delta_3]}e^{-\lambda_2[(y_s-\mu)j_3]}f(y_s). \quad (37)$$

The Bayesian predictive density function of Y_s given x suggested by Aitchison and Dunsmore (1975) is defined by

$$h^*(y_s \mid x)=\int_0^1\int_0^\infty\int_0^\infty h(y_s \mid \Theta)\pi(\Theta \mid x)d\Theta, \quad (38)$$

where $\pi(\Theta \mid x)$ is the posterior density given by (7) and $h(y_s \mid \Theta)$ is the pdf of s^{th} component in a future sample given by (36)

$$h^*(y_s \mid x)=\Lambda^{-1}\sum^{*} kk^*\left[\frac{(\delta_1+\delta_3+1)(r_1+1)}{\xi_1^{*r_1+2}\xi_2^{r_2+1}}+\frac{(\delta_2+j_3+1)(r_2+1)}{\xi_1^{r_1+1}\xi_2^{*r_2+2}}\right], \quad (39)$$

where Λ is the normalizing constant satisfying $\int_0^\infty f^*(y_s \mid x)dy_s=1,$

$\xi_1=(y_s-\mu)\delta_3+\beta_1),\xi_1^*=(y_s-\mu)(1+\delta_3)+\beta_1,\ \xi_2=(y_s-\mu)j_3+\beta_2,$

$\xi_2^*=(y_s-\mu)(1+j_3)+\beta_2,\quad \sum^{*}=\sum_{j_1=0}^{n-r}\sum_{j_2=0}^{s-1}\sum_{j_3=0}^{m-s+j_2},\ k=(-1)^{j_2}\binom{n-r}{j_1}\binom{s-1}{j_2}\binom{m-s+j_2}{j_3},$

$k^*=\dfrac{\Gamma(\delta_1+\delta_3+1)\Gamma(\delta_2+j_3+1)\Gamma(r_1+1)\Gamma(r_2+1)}{\Gamma(\delta_1+\delta_3+\delta_2+j_3+3)},\ \delta_1=n-r_2-j_1,\ \delta_2=r_2+j_1,$ and

$\delta_3=m-s+j_2-j_3.$

Predictive survival function thus may be obtained by evaluating the following expression.

$$P(Y_s \geq v \mid x)=\int_v^\infty h^*(y_s \mid x)dy_s.$$

A $100\tau\%$ prediction interval for Y_s is then given by

$$P[L(x)<Y_s<U(x)]=\tau,$$

where $L(x)$ and $U(x)$ are obtained by solving the following two equations, respectively.

$$P(Y_s>L(x))=\frac{1+\tau}{2} \quad \text{and} \quad P(Y_s>U(x))=\frac{1-\tau}{2}.$$

5.3. Bayesian reliability estimation

Bayes estimator of the reliability function under SELF is

$$\hat{R}(t_0)=E\left[R(t_0) \mid x\right]$$

$$=\Lambda^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}\left[B(n-r_2-k+2,r_2+k+1)\frac{\Gamma(r_1+1)\Gamma(r_2+1)}{\omega_1^{r_1+1}\gamma_2^{r_2+1}}+\right.$$

$$\left.B(n-r_2-k+1,r_2+k+2)\frac{\Gamma(r_1+1)\Gamma(r_2+1)}{\gamma_1^{r_1+1}\omega_2^{r_2+1}}\right], \quad (40)$$

where Λ is the normalizing constant, $\omega_1 = \gamma_1 + (T - \mu)$, $\omega_2 = \gamma_2 + (T - \mu)$ and γ_1, γ_2 are defined earlier.

5.4. Illustrative Example

The lower and upper prediction bounds for one and two sample scheme are obtained. It is assumed that these failure times follow a mixture model defined in (2). A sample of size $n = 15$ is generated from the said mixture model assuming parameters $\lambda_1, \lambda_2, \mu, p_1$ to be 0.2, 0.8, 0.5 and 0.40, respectively under type-I censoring. Censoring time i.e. T is taken to be 5. A uniform number u is generated n times by applying inverse transformation method. If $u < p_1$, r_1 observations are taken from subpopulation-1, and if $u > p_1$, r_2 observations are considered to belong to subpopulation-II. Thus the following sample is generated.

0.1780, 0.3067, 0.7904, 3.2148 so $r_1 = 4$,

0.3214, 0.4531, 0.4688, 0.6571, 0.6968, 0.8712, 1.8926 and 2.4649 so $r_2 = 8$.

Using this informative sample, Equation (9) are evaluated with $\tau = 0.95$ which provides probable values for the remaining "$n - r$" observations in the sample. Thus 95% lower and upper predictive bounds for x_{13}, x_{14} and x_{15} are (3.4251, 4.3751), (3.6073, 4.5573) and (4.2714, 5.2214), respectively.

Now on the basis of current sample Y_1 and Y_m, representing the first and last failure times in a future sample of size $m = 10$ are obtained. The lower and upper 95% prediction bounds for Y_1, the first failure time, are 0.1375, 1.0875 and the 95% prediction bounds for Y_{10}, the last failure time are given by 1.5173, 2.4673 respectively. The reliability estimate using (39) at $T = 5$ is obtained as 0.1907.

6. Simulation Study

To investigate the properties of derived Bayes estimators a large scale simulation study is conducted. Random samples of size n=50, 100, 200, 300 and 500 have been generated from the two-component shifted exponential mixture model. The data is generated in a Minitab (Minitab 1991) routine by generating pseudo-random uniform numbers. If $u < p_1$ then observations are taken randomly from first component of shifted exponential distribution, otherwise observations are taken from 2nd component of the same distribution. Mixture data is obtained by using probabilistic mixing. Different parametric points have been considered to have insight into characteristics of BEs. Threshold parameter, μ is considered as 0.5 and 1. Censoring rate is taken 10%. Estimates are computed using two informative Priors (Exponential and SR). For Bayesian analysis using informative priors, hyper-parameters are selected in such a way that the prior mean becomes the expected value of the corresponding parameter. Numerical results of a simulation study are presented in Tables 1-6.

Table 1 Bayes estimates of shifted exponential mixture parameters along with posterior risks (in parentheses) when $\lambda_1, \lambda_2, p_1$ (0.05, 0.5, 0.40) and $\mu = 0.5$. Hyperparameters taken are $k_1 = 20.5$, $k_2 = 2.5, a_1 = 3.54$, $a_2 = 1$

		Exponential Prior			Squared Rayleigh Prior		
λ	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
SELF	50	0.0400 (0.000062)	0.3024 (0.0040)	0.5576 (0.0047)	0.0414 (0.00076)	0.3127 (0.0043)	0.5576 (0.0047)
	100	0.0561 (0.000098)	0.4832 (0.0041)	0.4411 (0.0024)	0.0582 (0.00094)	0.4937 (0.0043)	0.4412 (0.0024)
	200	0.0454 (0.000027)	0.5244 (0.0026)	0.4802 (0.0012)	0.0460 (0.00025)	0.5310 (0.0027)	0.4802 (0.0012)
	300	0.0469 (0.000023)	0.6096 (0.0021)	0.4172 (0.0008)	0.0474 (0.00022)	0.6149 (0.0022)	0.4172 (0.0008)
	500	0.0578 (0.00002)	0.4994 (0.0008)	0.3864 (0.00051)	0.0583 (0.00019)	0.5015 (0.0008)	0.3864 (0.0005)
WLF	50	0.0382 (0.0018)	0.2891 (0.0133)	0.5490 (0.0086)	0.0396 (0.0018)	0.2990 (0.0137)	0.5490 (0.0086)
	100	0.0545 (0.0016)	0.4747 (0.0085)	0.4356 (0.0055)	0.0564 (0.0018)	0.4850 (0.0087)	0.4356 (0.0056)
	200	0.0448 (0.0006)	0.5194 (0.0050)	0.4776 (0.0026)	0.0454 (0.0006)	0.5260 (0.0050)	0.4776 (0.0026)
	300	0.0464 (0.0005)	0.6061 (0.0035)	0.4153 (0.0019)	0.0469 (0.0005)	0.6114 (0.0035)	0.4153 (0.0019)
	500	0.0574 (0.0004)	0.4978 (0.0016)	0.3852 (0.0012)	0.0579 (0.0004)	0.4998 (0.0017)	0.3852 (0.0012)
QLF	50	0.0364 (0.0455)	0.2758 (0.0461)	0.5400 (0.0164)	0.0378 (0.0455)	0.2853 (0.0460)	0.5399 (0.0164)
	100	0.0528 (0.0303)	0.4662 (0.0179)	0.4300 (0.0130)	0.0547 (0.0303)	0.4763 (0.0179)	0.4300 (0.0130)
	200	0.0442 (0.0133)	0.5144 (0.0096)	0.4750 (0.0055)	0.0448 (0.0133)	0.5209 (0.0096)	0.4750 (0.0055)
	300	0.0459 (0.0106)	0.6027 (0.0057)	0.4133 (0.0047)	0.0464 (0.0106)	0.6079 (0.0057)	0.4133 (0.0047)
	500	0.0569 (0.0070)	0.4962 (0.0033)	0.3840 (0.0032)	0.0575 (0.0070)	0.4982 (0.0033)	0.3840 (0.0032)

Table 2 Bayes estimates of shifted exponential mixture parameters along with posterior risks (in parentheses) when $\lambda_1, \lambda_2, p_1$ (0.2, 0.8, 0.40) and $\mu = 0.5.$ Hyperparameters taken as $k_1 = 5.5,\ k_2 = 1.75,\ a_1 = 1.58,\ a_2 = 0.79$

		Exponential Prior			Squared Rayleigh Prior		
λ	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
SELF	50	0.3231 (0.0130)	0.9557 (0.0429)	0.3241 (0.0048)	0.4303 (0.0422)	0.9326 (0.0525)	0.3127 (0.0053)
	100	0.1647 (0.0012)	0.8095 (0.0110)	0.3407 (0.0022)	0.1711 (0.0012)	0.8192 (0.0113)	0.3407 (0.0022)
	200	0.1962 (0.0005)	0.8413 (0.0068)	0.4497 (0.0012)	0.1991 (0.0006)	0.8476 (0.0069)	0.4497 (0.0013)
	300	0.2512 (0.0007)	0.8735 (0.0053)	0.4030 (0.0008)	0.2548 (0.0004)	0.8776 (0.0053)	0.4031 (0.0008)
	500	0.1861 (0.0002)	0.7849 (0.0023)	0.4048 (0.0005)	0.1873 (0.0002)	0.7869 (0.0023)	0.4048 (0.0005)
WLF	50	0.2881 (0.0350)	0.9094 (0.0463)	0.3082 (0.0159)	0.3609 (0.0694)	0.8745 (0.0581)	0.2942 (0.0185)
	100	0.1579 (0.0068)	0.7957 (0.0138)	0.3341 (0.0066)	0.1638 (0.0073)	0.8052 (0.0140)	0.3341 (0.0066)
	200	0.1933 (0.0029)	0.8332 (0.0081)	0.4470 (0.0027)	0.1962 (0.0029)	0.8395 (0.0081)	0.4470 (0.0027)
	300	0.2483 (0.0029)	0.8676 (0.0059)	0.4010 (0.0020)	0.2520 (0.0028)	0.8716 (0.0060)	0.4010 (0.0021)
	500	0.1848 (0.0013)	0.7820 (0.0029)	0.4036 (0.0012)	0.1860 (0.0013)	0.7840 (0.0029)	0.4036 (0.0012)
QLF	50	0.2567 (0.1094)	0.8621 (0.0520)	0.2911 (0.0556)	0.3096 (0.1422)	0.8162 (0.0667)	0.2740 (0.0690)
	100	0.1509 (0.0442)	0.7818 (0.0175)	0.3273 (0.0202)	0.1565 (0.0443)	0.7911 (0.0175)	0.3273 (0.0202)
	200	0.1905 (0.0146)	0.8251 (0.0098)	0.4442 (0.0062)	0.1934 (0.0146)	0.8313 (0.0098)	0.4442 (0.0062)
	300	0.2455 (0.0114)	0.8616 (0.0069)	0.3990 (0.0051)	0.2491 (0.0114)	0.8656 (0.0069)	0.3990 (0.0051)
	500	0.1836 (0.0067)	0.7791 (0.0037)	0.4024 (0.0030)	0.1848 (0.0067)	0.7811 (0.0037)	0.4024 (0.0030)

Table 3 Bayes estimates of shifted exponential mixture parameters along with posterior risks (in parentheses) when $\lambda_1, \lambda_2, p_1$ (2.0, 4.0, 0.40) and $\mu = 0.5$. Hyperparameters taken as $k_1 = 1.0,\ k_2 = 0.75,\ a_1 = 0.5,\ a_2 = 0.35$

		Exponential Prior			Squared Rayleigh Prior		
λ	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
SELF	50	1.9700	4.0443	0.3866	2.0591	4.1887	0.4005
		(0.0235)	(0.0037)	(0.0045)	(0.0210)	(0.0033)	(0.0052)
	100	1.9328	3.9168	0.3851	1.9822	3.9801	0.3857
		(0.0108)	(0.0014)	(0.0025)	(0.0103)	(0.0014)	(0.0025)
	200	1.8052	4.2100	0.3609	1.8309	4.2409	0.3611
		(0.0058)	(0.00055)	(0.0012)	(0.0056)	(0.0005)	(0.0012)
	300	1.6974	3.8720	0.3867	1.7298	3.7071	0.3877
		(0.0042)	(0.00045)	(0.0008)	(0.0042)	(0.0005)	(0.0008)
	500	2.2565	3.8954	0.3948	2.2663	3.9083	0.3949
		(0.0018)	(0.00031)	(0.0008)	(0.0018)	(0.0003)	(0.0005)
WLF	50	2.2373	4.2793	0.3876	2.2799	4.4169	0.3873
		(0.2673)	(0.2350)	(0.0010)	(0.2208)	(0.2282)	(0.0132)
	100	2.0357	4.0072	0.3789	2.0670	4.0667	0.3791
		(0.1029)	(0.0904)	(0.0062)	(0.0848)	(0.0866)	(0.0066)
	200	1.8499	4.2542	0.3577	1.8670	4.2824	0.3578
		(0.0447)	(0.0442)	(0.0032)	(0.0361)	(0.0415)	(0.0033)
	300	1.7419	3.7135	0.3855	1.7524	3.7331	0.3856
		(0.0445)	(0.1585)	(0.0012)	(0.0226)	(0.0260)	(0.0021)
	500	2.2813	3.9142	0.3935	2.2871	3.9264	0.3936
		(0.0248)	(0.0188)	(0.0013)	(0.0281)	(0.0181)	(0.0013)
QLF	50	2.5479	4.5132	0.3738	2.5744	4.6503	0.3737
		(0.1219)	(0.0518)	(0.0356)	(0.1292)	(0.0528)	(0.0353)
	100	2.1297	4.0953	0.3721	2.1614	4.1558	0.3724
		(0.0441)	(0.0215)	(0.0177)	(0.0437)	(0.0215)	(0.0177)
	200	1.8879	4.2962	0.3544	2.2938	4.3247	0.3545
		(0.0201)	(0.0098)	(0.0092)	(0.0206)	(0.0099)	(0.0093)
	300	1.7653	3.7396	0.3834	1.7760	3.7594	0.3835
		(0.0132)	(0.0070)	(0.0055)	(0.0133)	(0.0070)	(0.0055)
	500	2.3027	3.9323	0.3922	2.3085	3.9445	0.3922
		(0.0093)	(0.0046)	(0.0034)	(0.0093)	(0.0046)	(0.0034)

Table 4 Bayes estimates of shifted exponential mixture parameters along with posterior risks (in parentheses) when $\lambda_1, \lambda_2, p_1$ (0.05, 0.5, 0.40) and $\mu = 1.0$. Hyperparameters taken as $k_1 = 21,\ k_2 = 3,\ a_1 = 3.54,\ a_2 = 1$

		Exponential Prior			Squared Rayleigh Prior		
λ	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
SELF	50	0.0458 (0.0002)	0.5180 (0.0082)	0.3653 (0.0044)	0.0854 (0.0003)	0.5473 (0.0090)	0.3654 (0.0043)
	100	0.0580 (0.000091)	0.5528 (0.0056)	0.4706 (0.0024)	0.0753 (0.0001)	0.5726 (0.0060)	0.4706 (0.0024)
	200	0.0636 (0.000077)	0.4196 (0.0015)	0.3761 (0.0012)	0.0764 (0.0001)	0.4247 (0.0014)	0.3761 (0.0011)
	300	0.0438 (0.000022)	0.5362 (0.0016)	0.3907 (0.00081)	0.0490 (0.0008)	0.5417 (0.0015)	0.3907 (0.0008)
	500	0.0547 (0.00002)	0.4670 (0.0008)	0.3924 (0.0005)	0.0587 (0.00002)	0.4696 (0.0007)	0.3924 (0.0005)
WLF	50	0.0423 (0.0035)	0.5023 (0.0157)	0.3529 (0.0124)	0.0816 (0.0038)	0.5309 (0.0164)	0.3529 (0.0125)
	100	0.0565 (0.0015)	0.5425 (0.0103)	0.4653 (0.0053)	0.0737 (0.0016)	0.5620 (0.0106)	0.4653 (0.0053)
	200	0.0625 (0.0011)	0.4163 (0.0033)	0.3730 (0.0031)	0.0753 (0.0011)	0.4213 (0.0034)	0.3730 (0.0031)
	300	0.0433 (0.0005)	0.5333 (0.0029)	0.3887 (0.0020)	0.0486 (0.0004)	0.5387 (0.0070)	0.3887 (0.0020)
	500	0.0543 (0.0004)	0.4655 (0.0015)	0.3912 (0.0012)	0.0583 (0.0004)	0.4681 (0.0015)	0.3912 (0.0012)
QLF	50	0.0387 (0.0833)	0.4866 (0.0313)	0.3400 (0.0367)	0.0778 (0.0465)	0.5145 (0.0310)	0.3400 (0.0367)
	100	0.0549 (0.0278)	0.5323 (0.0189)	0.4600 (0.0115)	0.0721 (0.0220)	0.5515 (0.0187)	0.4600 (0.0115)
	200	0.0613 (0.0185)	0.4129 (0.0081)	0.3699 (0.0084)	0.0741 (0.0158)	0.4178 (0.0081)	0.3698 (0.0084)
	300	0.0428 (0.0116)	0.5304 (0.0055)	0.3867 (0.0052)	0.0481 (0.0105)	0.5358 (0.0054)	0.3867 (0.0052)
	500	0.0539 (0.0069)	0.4640 (0.0033)	0.3900 (0.0031)	0.0579 (0.0065)	0.4665 (0.0033)	0.3900 (0.0031)

Table 5 Bayes estimates of shifted exponential mixture parameters along with posterior risks (in parentheses) when $\lambda_1, \lambda_2, p_1$ (0.2, 0.8, 0.40) and $\mu = 1.0$. Hyperparameters taken as $k_1 = 6.0, k_2 = 2.25,\ a_1 = 1.58,\ a_2 = 0.79$

		Exponential Prior			Squared Rayleigh Prior		
λ	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
SELF	50	0.2816 (0.0793)	0.7966 (0.0308)	0.3844 (0.0050)	0.3810 (0.0157)	0.7813 (0.0323)	0.3731 (0.0054)
	100	0.3025 (0.0049)	0.6203 (0.0122)	0.3968 (0.0028)	0.3728 (0.0112)	0.5889 (0.0146)	0.3838 (0.0032)
	200	0.1864 (0.0006)	0.6924 (0.0048)	0.4177 (0.0012)	0.1952 (0.0006)	0.6977 (0.0049)	0.4176 (0.0012)
	300	0.1789 (0.0003)	0.8437 (0.0043)	0.4230 (0.0008)	0.1843 (0.0003)	0.8494 (0.0045)	0.4230 (0.0008)
	500	0.1713 (0.0002)	0.8143 (0.0023)	0.4038 (0.0005)	0.1745 (0.0002)	0.8174 (0.0023)	0.4038 (0.0005)
WLF	50	0.2592 (0.0224)	0.7570 (0.0396)	0.3706 (0.0138)	0.3458 (0.0352)	0.7325 (0.0488)	0.3576 (0.0155)
	100	0.2879 (0.0146)	0.6000 (0.0203)	0.3894 (0.0074)	0.3474 (0.0254)	0.5637 (0.0252)	0.3753 (0.0085)
	200	0.1834 (0.0030)	0.6854 (0.0070)	0.4147 (0.0030)	0.1921 (0.0031)	0.6906 (0.0071)	0.4146 (0.0030)
	300	0.1771 (0.0018)	0.8385 (0.0052)	0.4211 (0.0019)	0.1824 (0.0019)	0.8442 (0.0052)	0.4211 (0.0019)
	500	0.1702 (0.0011)	0.8114 (0.0029)	0.4026 (0.0012)	0.1734 (0.0011)	0.8145 (0.0029)	0.4026 (0.0012)
QLF	50	0.2380 (0.0818)	0.7168 (0.0532)	0.3560 (0.0395)	0.3153 (0.0884)	0.6837 (0.0667)	0.3409 (0.0468)
	100	0.2747 (0.0462)	0.5791 (0.0347)	0.3818 (0.0195)	0.3260 (0.0617)	0.5387 (0.0444)	0.3664 (0.0238)
	200	0.1804 (0.0163)	0.6784 (0.0102)	0.4118 (0.0071)	0.1891 (0.0158)	0.6835 (0.0103)	0.4117 (0.0072)
	300	0.1752 (0.0105)	0.8333 (0.0062)	0.4191 (0.0046)	0.1806 (0.0103)	0.8390 (0.0062)	0.4191 (0.0046)
	500	0.1690 (0.0067)	0.8086 (0.0035)	0.4014 (0.0030)	0.1722 (0.0066)	0.8116 (0.0035)	0.4014 (0.0030)

Table 6 Bayes estimates of shifted exponential mixture parameters along with posterior risks (in parentheses) when $\lambda_1, \lambda_2, p_1$ (2.0, 4.0, 0.40) and $\mu = 1.0$. Hyperparameters taken as $k_1 = 1.5,\ k_2 = 1.25,\ a_1 = 0.5,\ a_2 = 0.35$

		Exponential Prior			Squared Rayleigh Prior		
λ	n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
SELF	50	2.0720	3.2333	0.4133	2.0982	3.2915	0.4133
		(0.0197)	(0.0050)	(0.0054)	(0.0193)	(0.0049)	(0.0053)
	100	1.8796	4.0772	0.4011	1.8932	4.1107	0.4012
		(0.0114)	(0.0602)	(0.0025)	(0.0113)	(0.0014)	(0.0025)
	200	1.9635	3.8352	0.4022	1.9613	3.8487	0.4021
		(0.0053)	(0.00078)	(0.0013)	(0.0054)	(0.0008)	(0.0013)
	300	1.9815	3.7935	0.4019	1.9993	3.8051	0.4022
		(0.0035)	(0.00052)	(0.0009)	(0.0034)	(0.0005)	(0.0009)
	500	2.0177	3.7134	0.4030	2.0207	3.7195	0.4030
		(0.0021)	(0.00033)	(0.0005)	(0.0021)	(0.0003)	(0.0005)
WLF	50	2.1360	3.2530	0.3895	2.2919	3.4652	0.3999
		(0.0640)	(0.0197)	(0.0238)	(0.1937)	(0.1737)	(0.0134)
	100	2.0036	4.1832	0.3953	1.9783	4.2094	0.3984
		(0.1240)	(0.1060)	(0.0058)	(0.0851)	(0.0987)	(0.0028)
	200	2.0254	3.8841	0.3994	2.0037	3.8931	0.3989
		(0.0619)	(0.0489)	(0.0028)	(0.0424)	(0.0444)	(0.0032)
	300	2.0226	3.8256	0.4000	2.0280	3.8342	0.4000
		(0.0411)	(0.0321)	(0.0019)	(0.0287)	(0.0291)	(0.0022)
	500	2.0426	3.7323	0.4018	2.0379	3.7365	0.4017
		(0.0249)	(0.0189)	(0.0012)	(0.0172)	(0.0170)	(0.0013)
QLF	50	2.3122	3.4000	0.3762	2.5193	3.6491	0.3859
		(0.0762)	(0.0432)	(0.0342)	(0.0903)	(0.0504)	(0.0349)
	100	2.1070	4.2833	0.3888	2.0790	4.3098	0.3883
		(0.0491)	(0.0234)	(0.0164)	(0.0484)	(0.0233)	(0.0164)
	200	2.0715	3.9291	0.3961	2.0490	3.9381	0.3957
		(0.0223)	(0.0114)	(0.0081)	(0.0221)	(0.0114)	(0.0081)
	300	2.0524	3.8548	0.3978	2.0580	3.8636	0.3979
		(0.0145)	(0.0076)	(0.0054)	(0.0146)	(0.0076)	(0.0054)
	500	2.0603	3.7494	0.4005	2.0556	3.7536	0.4004
		(0.0086)	(0.0046)	(0.0032)	(0.0086)	(0.0046)	(0.0032)

7. A Real Data Application

The life test data set given in Table 7 has been taken from the web site www.home.math.utwente.nl. This is survival data set of 49 patients with Dukes'C colorectal cancer. The data set is divided into two groups. Survival times (months) of two treatment groups are as follows. Suppose that the survival times, of a homogeneous group of n patients are represented by $t_1, t_2, ..., t_n$.

Table 7 Survival data of patients with Dukes'C colorectal cancer

Control ($n = 24$)	Treatment (γ linoleic acid , $n = 25$)
3, 6, 6, 6, 6, 8, 8, 12, 12, 12, 15, 16, 18, 18, 20, 22, 24, 28, 28, 28, 30, 30, 33, 42	1, 5, 6, 6, 9, 10, 10, 10,12, 12, 12, 12, 12, 13, 15, 16, 20, 24, 24, 27, 32, 34, 36, 36, 44

On the basis of the fact that life/failure times of patients are frequently modeled by exponential distribution, and since least survival time is known to be 1 month, considering it threshold value, the study can ideally be represented by a mixture model of shifted exponential distribution. In this example right censoring is considered, which indicates that some failure times are not known. For these unknown failure times, one only knows that the failure time exceeds some known value, the so-called censoring time. For the given situation, it is believed that study can be carried out only for twenty months and all patients who survive after 20 months are considered as censored. So from given data set the following information are extracted.

$n = 49, \quad n - r = 17, \quad r_1 = 15, \quad r_2 = 17, \quad r = r_1 + r_2 = 32, \; \mu = 1$

$T = 20, \; \sum(t_{1j} - \mu) = 158.5$ and $\sum(t_{2j} - \mu) = 172.5$.

Bayes estimates obtained by using real data are given in Table 8. To conduct Bayesian analysis under exponential and SR priors hyperparameters are selected by equating prior mean against sample mean.

Table 8 Bayes estimates obtained by using Real life data set with posterior risks in parentheses. Hyperparameters taken as: $k_1, k_2 = 0.56$ and $a_1, a_2 = 1$

SELF			WLF			QLF		
$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}$
Exponential Prior								
0.0528 (0.0004)	0.0559 (0.0004)	0.4783 (0.0107)	0.0465 (0.0063)	0.0499 (0.0060)	0.4539 (0.0244)	0.0410 (0.1173)	0.0446 (0.1048)	0.4278 (0.0575)
SR Prior								
0.0502 (0.0003)	0.0536 (0.0004)	0.4790 (0.0104)	0.0442 (0.0060)	0.0479 (0.0057)	0.4554 (0.0236)	0.0390 (0.1179)	0.0428 (0.1053)	0.4300 (0.0558)

Bayes estimates for real data set are obtained by using informative priors under three loss functions. Numerical results for real life data indicate that average survival rate for the two-treatment groups are almost same for both type of priors. However SELF produces lesser posterior risks than the other two loss functions used. Posterior risk for the two-component of mixture (two treatments) is also same under SELF. Graphs for marginal posterior densities depict symmetrical behavior. So Bayes estimates can be reported from these graphs. Now we consider the prediction problem. Suppose that 10 new patients arrive and put on a test. It is desired to construct 95% predictive interval for the smallest and largest predictive life-length. Based on the observed sample, it is found that smallest life length may occur between (22.5, 23.5) while largest life length can be (40.4, 41.3).

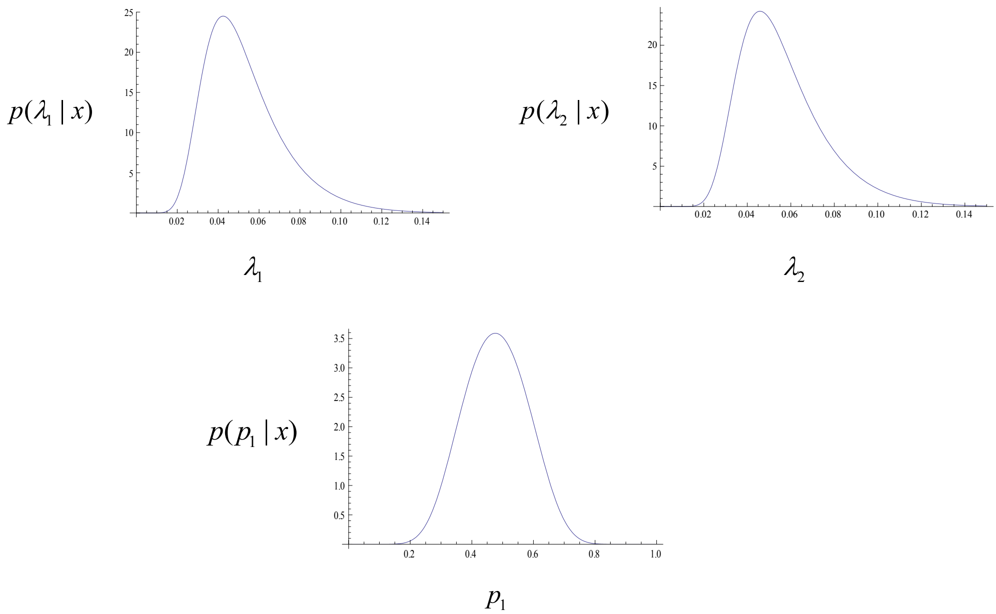

Figure 1 Graph of marginal posterior densities using real data set

8. Conclusions

In this paper Bayesian analysis of the shifted exponential mixture model is conducted using type-I censoring scheme assuming informative priors. A simulation study is carried out to scrutinize the convergence of Bayes estimators. Some important and interesting findings are observed from simulation results presented in Tables 1-6. No serious over/under estimation is observed. Posterior risks are pretty close to each other under both informative priors. Overall the variation among posterior risks is insignificant/negligible. But remarkable thing is that the posterior risks under WLF are the same for the both components of the population under both priors. Posterior risks for mixture weight are also almost same for both priors. However, SELF proves to be superior at every point by producing lesser posterior risks. Problem of prediction is considered and future sample is predicted for one sample and two sample cases. Reliability estimator is also considered. All the proposed estimators are applied to a real data set.

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